Thermodynamic formalism for dynamical systems

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Deterministic systems can exhibit stochastic behaviour.

Driven by expansion + recurrence in phase space.

Treat as stochastic process; choose invariant measure. Given by equilibrium state in thermodynamic formalism.

Mechanisms driving stochasticity may not be uniform.
Predictions in dynamical systems

Key objects:

- $X =$ phase space for a dynamical system.
  
  *Points in $X$ correspond to configurations of the system.*

- $f : X \looparrowright$ describes evolution of the state of the system over a single time step. $f^n = f \circ \cdots \circ f$ (*$n$ times*)

Standing assumptions: $X$ is a compact metric space, $f$ is continuous

*Often $X$ a smooth manifold, $f$ a diffeomorphism*
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Initial measurement gives neighbourhood \(U \subset X\). To make a prediction based on this measurement, we must describe \(f^n(U)\).

Common phenomenon: diam \(f^n(U)\) becomes large relatively quickly no matter how small \(U\) is.
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Initial measurement gives neighbourhood $U \subset X$. To make a prediction based on this measurement, we must describe $f^n(U)$.

Common phenomenon: diam $f^n(U)$ becomes large relatively quickly no matter how small $U$ is. Stronger phenomenon:
  - iterates $f^n(U)$ become dense in $X$  ← mechanism for rigorous results
Examples

Lorenz equations (1963) – atmospheric dynamics

\[
\begin{align*}
\dot{x} &= \sigma(y - x) & \sigma &= 10 \\
\dot{y} &= x(\rho - z) - y & \rho &= 28 \\
\dot{z} &= xy - \beta z & \beta &= 8/3
\end{align*}
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Hénon map (1976) – models stretching and folding

\[f(x, y) = (y + 1 - ax^2, bx)\quad a = 1.4, b = .3\]
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Hénon map \textit{(1976)} – models stretching and folding

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f(x, y) = (y + 1 - ax^2, bx) \quad a = 1.4, b = 0.3
\]

Lorenz and Hénon systems are \textit{non-uniformly hyperbolic}.

Situation simplifies for (less realistic) \textit{uniformly hyperbolic} systems, exemplified by the

\textbf{Doubling map} \( f : S^1 \to S^1, x \mapsto 2x \) (mod 1)
Invariant and ergodic measures

Given $\varphi \in C(X)$, view $\varphi \circ f^n : X \to \mathbb{R}$ as sequence of random variables

- Pick $\mu \in \mathcal{M} = \{\text{Borel probability measures on } X\}$
- $(X, \mu, \varphi \circ f^n)$ defines a stochastic process

Does this process satisfy any limit laws? It is not usually i.i.d.
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\[ \mu \in \mathcal{M} \text{ is invariant if } \int \varphi \, d\mu = \int \varphi \circ f \, d\mu \text{ for all } \varphi \in C(X) \]

- Equivalent to the RVs \((X, \mu, \varphi \circ f^n)\) being identically distributed
- \( \mathcal{M}_f = \{ \text{invariant measures} \} \subset \mathcal{M} \) \hspace{1cm} (convex, weak*-compact)
- \( \mathcal{M}_f^e = \{ \text{extreme points of } \mathcal{M}_f \} = \{ \text{ergodic measures} \} \)

Each \( \mu \in \mathcal{M}_f \) is a convex combination of ergodic measures (uniquely)
Theorem (G.D. Birkhoff, 1931)

If $\mu \in \mathcal{M}_f^e$ then $\frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k(x) \to \int \varphi \, d\mu$ for $\mu$-a.e. $x$

The stochastic process $(X, \mu, \varphi \circ f^n)$ obeys the law of large numbers.
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Other limit laws? CLT? Large deviations? Iterated logarithm?

- Identically distributed (by invariance) but generally not independent.

What ergodic measure should we use?

- Natural measure for diffeos is ‘physical’: volume. Often not invariant.
An abundance of measures

\( \mathcal{M}_f^e \) is often very large.

- Example: \( X = \Sigma_2^+ = \{0, 1 \}^\mathbb{N} \), \( f = \sigma: x_0 x_1 x_2 \ldots \mapsto x_1 x_2 x_3 \ldots \)

Periodic measures: \( f^p(x) = x \leadsto \mu = \frac{1}{p} (\delta_x + \delta_{fx} + \cdots + \delta_{f^{p-1}x}) \in \mathcal{M}_f^e \)

- Periodic orbits are dense. \( (f^p(x) = x \ has \ 2^p \ solutions) \)
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\( \alpha, \beta > 0, \alpha + \beta = 1 \leadsto (\alpha, \beta)\text{-Bernoulli measure:} \)

- \( w \in \{0, 1\}^n \leadsto \text{cylinder set } [w] = \{x \in X \mid x_1 \cdots x_n = w\} \)
- \( k = \# \text{ of 0's in } w \Rightarrow \mu([w]) = \alpha^k \beta^{n-k} \)
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- Periodic orbits are dense. ($f^p(x) = x$ has $2^p$ solutions)

$\alpha, \beta > 0$, $\alpha + \beta = 1 \sim (\alpha, \beta)$-Bernoulli measure:

- $w \in \{0, 1\}^n \sim$ cylinder set $[w] = \{x \in X | x_1 \cdots x_n = w\}$

- $k = \#$ of 0's in $w \Rightarrow \mu([w]) = \alpha^k \beta^{n-k}$

Also have Markov measures, Gibbs measures, etc.

How do we pick a good ergodic measure?

- (and what statistical properties does it have?)
Coding by symbolic systems

Doubling map $f : S^1 \to S^1$, $x \mapsto 2x \pmod{1}$

Full shift $\Sigma_2^+ = \{0, 1\}^\mathbb{N}$, $f = \sigma : x_0x_1x_2 \ldots \mapsto x_1x_2x_3 \ldots$

General procedure for symbolic description of dynamics:

1. Partition $X$ as a disjoint union $A_1 \cup \cdots \cup A_d$
2. $f^n(x) \in A_{y_n}$ defines $y = \pi(x) \in \{1, \ldots, d\}^\mathbb{N}$
3. $\pi : X \to \{1, \ldots, d\}^\mathbb{N}$ is the coding map
4. $Y = \pi(X)$ is the coding space

If $y_1 \ldots y_n = y'_1 \ldots y'_n$ but $y_{n+1} \neq y'_{n+1}$, then $d(y, y') = 2^{-n}$
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Coding space is closed and $\sigma$-invariant: $\sigma(Y) \subseteq Y$.

Typically many “forbidden” sequences. When is $Y$ “good”?
Entropy for shift spaces

Topological entropy of a shift space $X$:

- $\mathcal{L} = \{\text{words that appear in some } x \in X\} =$ language of $X$
- $h_{\text{top}}(X) = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{L}_n$

Example

$X = \Sigma_2^+ \Rightarrow \# \mathcal{L}_n = 2^n \Rightarrow h_{\text{top}}(X) = \log 2$
Entropy for shift spaces

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  \[ \mathcal{L}_n = \{\text{words of length } n\} \subset \mathcal{L} \]

Example

$X = \Sigma_2^+ \Rightarrow \# \mathcal{L}_n = 2^n \Rightarrow h_{\text{top}}(X) = \log 2$

Measure-theoretic entropy for $\mu \in \mathcal{M}_f$:
- $h(\mu) := \lim_{n \to \infty} \frac{1}{n} \sum_{w \in \mathcal{L}_n} H(\mu[w])$
  \[ H(p) = -p \log p \]

Example

Entropy of $(\alpha, \beta)$-Bernoulli measure is $h(\mu) = -\alpha \log \alpha - \beta \log \beta$. 
Variational principles

Variational principle: \( h_{\text{top}}(X) = \sup \{ h(\mu) \mid \mu \in \mathcal{M}_f \} \)

- \( h(\mu) = h_{\text{top}}(X) \leadsto \mu \) is a measure of maximal entropy (MME)
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Generalises to topological pressure of a potential function \( \varphi \in C(X) \):

- \( \Lambda_n(\varphi) = \sum_{w \in \mathcal{L}_n} e^{S_n \varphi(w)} \quad S_n \varphi(w) = \sup_{x \in [w]} \sum_{k=0}^{n-1} \varphi(\sigma^k x) \)

- Topological pressure of \( \varphi \) is \( P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(\varphi) \)

- \( P(\varphi) = \sup \{ h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f \} \)

- A measure achieving the supremum is an equilibrium state

Example: \( X = \Sigma^+_2, \varphi(x) = s\chi[0] + t\chi[1] \)

- \( P(\varphi) = \log(e^s + e^t) \), unique eq. state is \((e^{s-P(\varphi)}, e^{t-P(\varphi)})\)-Bernoulli
Unique equilibrium states

Unique equilibrium states often have strong statistical properties: central limit theorem, decay of correlations, large deviations, etc.

- The sequence of observations \((X, \mu, \varphi \circ f^n)\) has many properties in common with i.i.d. sequence of random variables.

### Decay of correlations:

- \(\varphi, \psi \in C^\alpha + \int \varphi \, d\mu = 0 \Rightarrow C_n(\varphi, \psi) = \int (\varphi \circ f^n) \psi \, d\mu \to 0\)
- Often: unique \(\Rightarrow\) exponential, non-unique \(\Rightarrow\) polynomial.

### Central limit theorem:

- \(\psi \in C^\alpha + \int \psi \, d\mu = 0 \Rightarrow \exists \xi \geq 0\) such that for all \(r \in \mathbb{R}\),

\[
\mu \left\{ x \mid \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi(f^k x) < r \right\} \xrightarrow{n \to \infty} \frac{1}{\xi \sqrt{2\pi}} \int_{-\infty}^{r} e^{-x^2 / 2\xi^2} \, dx
\]
SRB measures

**Key example:** \( f \) a diffeo, \( TM = E^u \oplus E^s \) a \( Df \)-invariant splitting, \[
\| Df^n(v^u) \| \to \infty \text{ and } \| Df^n(v^s) \| \to 0 \text{ exponentially in } n.
\]

Equilibrium states for \( - \log | \det (Df|_{E^u}) | \) are ‘physical’ measures.
- Not smooth on \( M \), but smooth along unstable manifolds

Existence, exponential decay of correlations, CLT known in many cases
- Uniformly hyperbolic systems: (Ya. Sinai, D. Ruelle, R. Bowen)
A (brief) digression: some applications

- **Hausdorff dimension:** If $f : M \circlearrowleft$ is conformal and $J$ is a uniformly expanding repeller for $f$, then $\dim_H J = t$ solves $P_J(-t \log \|Df\|) = 0$ (R. Bowen 1979, D. Ruelle 1982). Also holds in more general settings (Gatzouras–Peres 1997, Rugh 2008, C. 2011).
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- **Multifractal analysis:** Let $K^\varphi_\alpha = \{x \mid \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) \to \alpha\}$. If $T_\varphi : t \mapsto P(t \varphi)$ is differentiable, then the multifractal spectrum $\alpha \mapsto h_{\text{top}} K^\varphi_\alpha$ is the Legendre transform of $T_\varphi$. 
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- **Biology:** pressure can be used to distinguish between coding and non-coding DNA sequences (D. Koslicki, D. Thompson)
Unique MME for full shift is \((\frac{1}{2}, \frac{1}{2})\)-Bernoulli

- Has exponential decay of correlations, CLT, large deviations

More general: \(X \subset \{1, \ldots, d\}^\mathbb{N}\) is a subshift of finite type (SFT)

- Set of walks on a directed graph with vertices labeled \(1, \ldots, d\).

\[ X = \{\text{words on } \{1, 2\} \text{ such that 2 never follows 2}\} \]

Given by \(d \times d\) transition matrix \(A\)

- \(A_{ij} = 1\) if \(j\) can follow \(i\), and 0 otherwise
- \(\lambda = \text{largest eigenvalue of } A \Rightarrow h_{\text{top}} (X, f) = \log \lambda\)
- Unique MME given in terms of left and right eigenvectors for \(\lambda\)
Uniformly hyperbolic systems

Results generalise to equilibrium states for Hölder potentials \( \varphi \)

- \( \varphi = 0 \): transition matrix \( A : \mathbb{R}^d \rightarrow \mathbb{R}^d \) contracts positive cone
- More generally: Perron–Frobenius operator \( L_\varphi : C^\alpha(X) \rightarrow C^\alpha(X) \)

A diffeomorphism \( f : M \rightarrow M \) is uniformly hyperbolic if there is a \( Df \)-invariant splitting \( T_x M = E^u(x) \oplus E^s(x) \) and \( \chi > 1 \) such that

- \( \| Df(v^u) \| > \chi \| v^u \| \)
- \( \| Df(v^s) \| < \chi^{-1} \| v^s \| \)

Uniformly hyperbolic systems have Markov partitions

- Can be coded using SFTs
- Unique equilibrium states with strong statistical properties
Non-uniform hyperbolicity

Many (most) natural “chaotic” systems are not uniformly hyperbolic...

Hénon map

- $E^u(x)$ and $E^s(x)$ depend only measurably on $x$, and may become arbitrarily close together
- $\|DF^n(v^s)\| \leq C_x \chi^{-n} \|v^s\|$ and $\|DF^n(v^u)\| \geq C_x^{-1} \chi^n \|v^u\|$, but $C_x$ depends only measurably on $x$, and may become arbitrarily large

Cannot be coded with SFTs. Need to consider broader classes of symbolic systems in order to study non-uniform hyperbolicity.

- One possibility: use a countable alphabet
- Another option: finite alphabet, but more general language
Beyond SFTs, what classes of symbolic systems have unique MMEs?

- Should be transitive (any two words can eventually be joined): otherwise consider \( \{1, 2\}^\mathbb{N} \cup \{1, 2\}^\mathbb{N} \). Has \( h_{\text{top}} = \log 2 \) and two MMEs: \( \nu \) on \( \{1, 2\}^\mathbb{N} \) and \( \mu \) on \( \{1, 2\}^\mathbb{N} \), both \( (\frac{1}{2}, \frac{1}{2}) \)-Bernoulli.
Multiple MMEs

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Need more than transitivity: \( X \subset \Sigma_5 = \{0, 1, 2, 1, 2\}^\mathbb{N} \). Define the language \( \mathcal{L} \) by \( v0^n w, w0^n v \in \mathcal{L} \) if and only if \( n \geq |v| + |w| \).
- Transitive and \( h_{\text{top}} (X, \sigma) = \log 2 \)
- Same two measures of maximal entropy as above
Uniform transitivity

Full shift: words can be freely concatenated: \( v, w \in \mathcal{L} \Rightarrow vw \in \mathcal{L} \)

Transitive \( \Rightarrow \forall v, w \in \mathcal{L} \) there exists \( u \in \mathcal{L} \) such that \( vuw \in \mathcal{L} \)

- Length of \( u \) may vary depending on \( v, w \)
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- Specification: \( \exists \tau \) such that \( |u| \leq \tau \) for all \( v, w \)

Transitive SFTs have specification
Uniform transitivity

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Transitive SFTs have specification

Theorem (R. Bowen, 1974)

*Specification \( \Rightarrow \) unique equilibrium state \( \mu_{\varphi} \) for Hölder potential \( \varphi \)*

Theorem (C., 2013)

\( \mu_{\varphi} \) has exponential decay of correlations and satisfies the CLT
For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta : [0, 1] \to [0, 1], \quad x \mapsto \beta x \pmod{1}$$

where $1_\beta = a_1 a_2 \cdots$, with $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$
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where $1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$.

$x \in \Sigma_\beta \iff x$ labels a walk starting at $B \iff \sigma^n x \leq 1_\beta$ for all $n$.

(Here $1_\beta = 2100201 \ldots$)
Towers

Specification fails if $1_\beta$ contains arbitrarily long strings of 0’s

$\Sigma_\beta$ given by a countable graph $\Rightarrow$ use countable state analogue of SFTs

Still get unique ES for Lipschitz $\varphi$
(P. Walters 1978, F. Hofbauer 1979)

Vaughn Climenhaga (University of Houston)
Towers

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This leads to \textit{tower approach} to non-uniform hyperbolicity

- **Idea:** Find $Z \subset X$ and a countable partition $Z = \bigsqcup_i Z_i$ such that $f^{\tau_i}(Z_i) = Z$ for some inducing time $\tau_i$

- $Z$ “big enough” + $\tau_i$ “small enough” $\Rightarrow$ unique ES, stat. properties

Used for Hénon maps and billiard systems (Young 1998)
Decompositions

When is it possible to build a tower? Or to get results via other means?

For symbolic systems, can use decompositions of the language $\mathcal{L}$.

$$\mathcal{L} = SGS \iff \mathcal{G}, S \subseteq \mathcal{L} \text{ are such that every } w \in \mathcal{L} \text{ can be written as } w = v^p uv^s \text{ for some } u \in \mathcal{G} \text{ and } v^p, v^s \in S$$

**Example**

$$X = \Sigma_2^+ = \{0, 1\}^\mathbb{N}$$

$$\mathcal{G} = \{1w1 \mid w \in \mathcal{L}\}$$

$$S = \{0^n \mid n \geq 0\}$$

- The entropy of $S$ is $h(S) = \lim_{n \to \infty} \frac{1}{n} \log \#S_n$

**Key observation:** If $\mathcal{G}$ has specification and $h(S) < h_{\text{top}}(X)$, then many ideas from Bowen’s proof can be adapted.

*For the full shift, this is unnecessary, since $\mathcal{L}$ already has specification, but the above decomposition is useful for other reasons.*
Non-uniform specification for $\Sigma_\beta$

The only obstruction to specification is the tail of the sequence $1_\beta$.

Let $G = \{\text{words whose path begins and ends at } B\}$
- $G$ has specification

Let $S = \{\text{words whose path never returns to } B\}$ (cusp excursions)
- $L = GS$ and $h(S) = 0$
Obstructions to specification

\( \mathcal{G} \subset \mathcal{L} \leadsto \mathcal{G}^M := \{uvw \mid v \in \mathcal{G}, \|u\|, \|w\| \leq M \} \leadsto \) filtration \( \mathcal{L} = \bigcup_M \mathcal{G}^M \)

For the \( \beta \)-shift, \( \mathcal{G}^M \) corresponds to walks ending on one of the first \( M \) vertices. Can return from these vertices to the base vertex in uniform time, so each \( \mathcal{G}^M \) has specification.

“Every \( \mathcal{G}^M \) has specification” means we can glue words together, provided we are allowed to remove an obstructing piece from the end of each word.
Equilibrium states with non-uniform specification

Theorem (C.–Thompson, 2013)

Let $X$ be a symbolic system with language $\mathcal{L}$. Suppose $\mathcal{L}$ has a decomposition $SGS$ such that every $G^M$ has specification. If $\varphi$ is Hölder and $P(S, \varphi) < P(X, \varphi)$, then $\varphi$ has a unique equilibrium state $\mu_\varphi$.

$$P(S, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log \Lambda_n(S_n, \varphi)$$

Theorem (C., 2013)

Under the above conditions, there is a tower such that $\mu_\varphi \{ x \mid \tau(x) \geq n \}$ decays exponentially in $n$. In particular, $\mu_\varphi$ has exponential decay of correlations and satisfies the CLT.
Large deviations

Given $\mu$ and $\varphi$, let $LD_n(\epsilon) = \{ x \in X \mid \left| \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k x) - \int \varphi \, d\mu \right| > \epsilon \}$

Birkhoff ergodic theorem $\Rightarrow \mu(LD_n(\epsilon)) \to 0$ as $n \to \infty$

Question: how quickly does $\mu(LD_n(\epsilon))$ decay?

$\mu$ satisfies large deviations principle (LDP) with rate function $q(\epsilon)$ if

$$\lim_{n \to \infty} \frac{1}{n} \log(LD_n(\epsilon)) = q(\epsilon) < 0$$

- Specification $\Rightarrow \mu_\varphi$ has LDP (Young, 1990)
- Non-uniform (SGS) specification $\Rightarrow \mu_\varphi$ has LDP if $L$ is edit approachable by $G$ (C.–Thompson–Yamamoto, 2013)

Edit approachable: $w \in \mathcal{L}_n$ can be turned into $\tilde{w} \in \mathcal{G}$ by making $o(n)$ edits
Non-symbolic applications

All the results quoted using specification are in the symbolic setting.

This is a playground motivating results for smooth systems.

Uniqueness results have been extended to smooth systems assuming non-uniform version of expansivity.

Currently being developed: Applications to partially hyperbolic systems, geodesic flows on manifolds of non-positive curvature.