Thermodynamics for non-uniformly mixing systems: factors of $\beta$-shifts are intrinsically ergodic

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November 11, 2010

Joint work with Daniel Thompson
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Basic thermodynamic concepts

Topological dynamical system:
- $X$ a compact metric space, $f: X \to X$ continuous
- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle: $h_{\text{top}}(X, f) = \sup_{\mu \in \mathcal{M}} h_\mu(f)$
- If $h_\mu(f) = h_{\text{top}}(X, f)$, then $\mu$ is a measure of maximal entropy (MME)
- $(X, f)$ is intrinsically ergodic if there exists a unique MME
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When is a transitive dynamical system intrinsically ergodic?
More general variational principle for topological pressure $P(\varphi)$ of a continuous potential function $\varphi : X \to \mathbb{R}$

$$P(\varphi) = \sup_{\mu \in \mathcal{M}} \left( h_\mu(f) + \int \varphi \, d\mu \right)$$

If $h_\mu(f) + \int \varphi \, d\mu = P(\varphi)$, then $\mu$ is an equilibrium state.
Motivation and context

More general variational principle for topological pressure $P(\varphi)$ of a continuous potential function $\varphi : X \to \mathbb{R}$

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If $h_\mu(f) + \int \varphi \, d\mu = P(\varphi)$, then $\mu$ is an equilibrium state.

- Existence of a unique equilibrium state is connected to statistical properties, large deviations, multifractal analysis, phase transitions, etc.
- $\varphi \equiv 0$: reduces to intrinsic ergodicity. Techniques for showing intrinsic ergodicity usually generalise to help prove other thermodynamic results.
Focus on shift spaces (subshifts):
- $X \subset \Sigma_p$ or $X \subset \Sigma^+_p$, $X$ closed and $\sigma$-invariant
- $\mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\}$ is the language of $X$

When is a transitive shift space intrinsically ergodic?
Intrinsic ergodicity for shift spaces

Focus on shift spaces (subshifts):
- \( X \subset \Sigma_p \) or \( X \subset \Sigma_p^+ \), \( X \) closed and \( \sigma \)-invariant
- \( \mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\} \) is the language of \( X \)

When is a transitive shift space intrinsically ergodic? *Not always.*

**Example:** \( X \subset \Sigma_5 = \{0, 1, 2, 1, 2\}^\mathbb{Z} \). Define the language \( \mathcal{L} \) by

\[ v0^nw, \ w0^n v \in \mathcal{L} \] if and only if \( n \geq 2 \max(|v|, |w|) \).

- \((X, \sigma)\) is topologically transitive
- \( h_{top} (X, \sigma) = \log 2 \)
- 2 measures of maximal entropy:
  \[ \nu = (\frac{1}{2}, \frac{1}{2})\text{-Bernoulli on } \{1, 2\}^\mathbb{Z}, \]
  \[ \mu = (\frac{1}{2}, \frac{1}{2})\text{-Bernoulli on } \{1, 2\}^\mathbb{Z}. \]
Classes of intrinsically ergodic shifts

The following classes of shift spaces are intrinsically ergodic:

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- Shifts with specification \textit{(Bowen 1974)}
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- Irreducible subshifts of finite type \textit{(Parry 1964)}
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- Shifts with specification \textit{(Bowen 1974)}
- $\beta$-shifts \textit{(Walters 1978, Hofbauer 1979)}
$\beta > 1$, $b = \lceil \beta \rceil$. The $\beta$-shift $\Sigma_\beta \subset \Sigma_b^+$ is the natural coding space for the map

$$f_\beta : [0, 1] \rightarrow [0, 1], \quad x \mapsto \beta x \pmod{1}$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$
\(\beta > 1, \ b = \lceil \beta \rceil\). The \(\beta\)-shift \(\Sigma_\beta \subset \Sigma_+^b\) is the natural coding space for the map

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\(1_\beta = a_1 a_2 \cdots\), where \(1 = \sum_{n=1}^{\infty} a_n \beta^{-n}\)

**Fact:** Sequences \(x \in \Sigma_\beta\) are precisely those sequences in \(\Sigma_+^b\) that label trajectories of the following graph beginning at the vertex \(B\).

(Here \(1_\beta = 2100201\ldots\))

![Graph of \(f_\beta(x)\)](attachment:graph.png)
Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^\mathbb{Z}$ as before
- $Y \subset \Sigma_6 = \{0, 1, 2, 1, 2, 3\}^\mathbb{Z}$ by similar rule
- $X$ is a factor of $Y$; $Y$ is intrinsically ergodic; $X$ is not
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What intrinsically ergodic classes are closed under factors?

- Closure of SFTs is class of sofic systems
- Specification preserved by factors
- Factors of $\beta$-shifts = ?????
An open problem

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Theorem (C.–Thompson 2010)

Yes.
An open problem

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Are factors of \( \beta \)-shifts intrinsically ergodic? (Klaus Thomsen)

Theorem (C.–Thompson 2010)

*Every subshift factor of a \( \beta \)-shift is intrinsically ergodic.*
The classical specification property

- $\mathcal{L} =$ language for a shift space $X$
- $\mathcal{L} \leftrightarrow \{\text{cylinders in } X\}$
- $|w| =$ length of $w$, $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$

$X$ has specification if there exists $t \in \mathbb{N}$ such that for every $w_1, \ldots, w_m \in \mathcal{L}$, there exist $z_1, \ldots, z_{m-1} \in \mathcal{L}_t$ for which the concatenated word $w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

(Arbitrary orbit segments can be connected by a single orbit)
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(Arbitrary orbit segments can be connected by a single orbit)

Topological transitivity guarantees the existence of such words $z_i \in \mathcal{L}$. Specification demands that the words $z_i$ can be chosen to have uniformly bounded length $t$, where $t$ is independent of the words $w_i$ and their lengths.
Shifts with and without specification

The following shifts have the specification property:

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$\Sigma_\beta$ does not have the specification property if $1_\beta$ contains arbitrarily long strings of 0’s.
Shifts with and without specification

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- Mixing subshifts of finite type
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$\Sigma_\beta$ does not have the specification property if $1_\beta$ contains arbitrarily long strings of 0’s.

$\Sigma_\beta$ does not have specification for Lebesgue-a.e. $\beta > 1$.

We must replace specification with a property that

- holds for every $\beta$-shift;
- implies intrinsic ergodicity;
- is preserved by factors.
A restricted version of the specification property

Fix a subset $G \subset \mathcal{L}$. We say that $G$ has specification if there exists $t \in \mathbb{N}$ such that for every $w_1, \ldots, w_m \in G$, there exist $z_1, \ldots, z_{m-1} \in \mathcal{L}_t$ for which the concatenated word

$$x := w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$$

is in $\mathcal{L}$.

Only difference from classical property is that we take $w_i \in G$. 
A restricted version of the specification property

Fix a subset \( G \subset \mathcal{L} \). We say that \( G \) has *specification* if there exists \( t \in \mathbb{N} \) such that for every \( w_1, \ldots, w_m \in G \), there exist \( z_1, \ldots, z_{m-1} \in \mathcal{L}_t \) for which the concatenated word 
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Only difference from classical property is that we take \( w_i \in G \).

Say that \( G \) has *(Per)-specification* if in addition to the above condition, the cylinder \( [x] \) contains a periodic point of period \( |x| + t \).


A restricted version of the specification property

Fix a subset $G \subset L$. We say that $G$ has specification if there exists $t \in \mathbb{N}$ such that for every $w_1, \ldots, w_m \in G$, there exist $z_1, \ldots, z_{m-1} \in L_t$ for which the concatenated word $x := w_1z_1w_2z_2\cdots z_{m-1}w_m$ is in $L$.

Only difference from classical property is that we take $w_i \in G$.

Say that $G$ has (Per)-specification if in addition to the above condition, the cylinder $[x]$ contains a periodic point of period $|x| + t$.

**Example:** For $X = \Sigma_\beta$, let $G$ be the set of words corresponding to paths that begin and end at B. Then $G$ has (Per)-specification with $t = 0$. 
Decomposing the language

A CGC-decomposition of the language $\mathcal{L}$ is a collection of words $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ with the following properties.

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A **CGC-decomposition** of the language $\mathcal{L}$ is a collection of words $C^p, G, C^s \subset \mathcal{L}$ with the following properties.

1. $G$ has specification.
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3. For every $uvw \in \mathcal{L}$ of the above form, there exist $x, y \in \mathcal{L}$ such that $xuvwy \in G$. 
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A CGC-decomposition is **uniform** if the lengths of $x$ and $y$ in the last condition depend only on the lengths of $u$ and $w$. (And not on $u, v, w$ themselves.)
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Example: For $X = \Sigma_\beta$, let $C^p = \emptyset$ and let $C^s$ be the set of words corresponding to paths that begin at $B$ and never return. Then $(C^p, G, C^s)$ is a uniform CGC-decomposition.
Intrinsic ergodicity for shifts with CGC-decompositions

Given a collection of words $\mathcal{D} \subset \mathcal{L}$, let $h(\mathcal{D}) = \lim_{n \to \infty} \frac{1}{n} \log \# \mathcal{D}_n$. Observe that $h_{\text{top}}(X, \sigma) = h(\mathcal{L})$. 

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**Theorem (C.–Thompson 2010)**

Let $X$ be a shift space admitting a uniform CGC-decomposition.

If $h(C^p \cup C^s) < h_{\text{top}}(X, \sigma)$, then $(X, \sigma)$ is intrinsically ergodic.

If $\mathcal{G}$ has (Per)-specification, then the unique MME is the limit of the periodic orbit measures $\mu_n = \frac{1}{\# \{x \mid f^n(x) = x \}} \sum_{f^n(x) = x} \delta_x$. 

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If $\mathcal{G}$ has (Per)-specification, then the unique MME is the limit of the periodic orbit measures $\mu_n = \frac{1}{\#\{x | f^n(x) = x\}} \sum_{f^n(x) = x} \delta_x$.

**Example:** For $X = \Sigma_\beta$, let $x = 1_\beta$. Then $(\mathcal{C}_p \cup \mathcal{C}_s)_n = \{x_1 \cdots x_n\}$, and so $h(\mathcal{C}_p \cup \mathcal{C}_s) = 0$. Thus $(\Sigma_\beta, \sigma)$ is intrinsically ergodic.
Behaviour under factors

Let $(\tilde{X}, \sigma)$ be a factor of $(X, \sigma)$, and let $\mathcal{L}, \tilde{\mathcal{L}}$ be the languages.

- If $\mathcal{L}$ has a uniform CGC-decomposition, then so does $\tilde{\mathcal{L}}$.
  Furthermore, $h(\tilde{\mathcal{C}}^p \cup \tilde{\mathcal{C}}^s) \leq h(\mathcal{C}^p \cup \mathcal{C}^s)$.

Every factor with $h_{top}(\tilde{X}, \sigma) > h(\mathcal{C}^p \cup \mathcal{C}^s)$ is intrinsically ergodic.
Behaviour under factors

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Dichotomy for shifts with uniform CGC-decompositions:
Either \(h_{top}(X, \sigma) > 0\), or \(X\) comprises a single periodic orbit.
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Dichotomy for shifts with uniform CGC-decompositions:
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**Theorem (C.–Thompson 2010)**

Let \(X\) be a shift space admitting a uniform CGC-decomposition.

If \(h(C^p \cup C^s) = 0\), then every subshift factor of \((X, \sigma)\) is intrinsically ergodic.
**S-gap shifts**

Fix $S \subset \mathbb{N}$ and suppose $S$ is infinite. The associated *S-gap shift* is the subshift $\Sigma_S \subset \{0, 1\}^\mathbb{Z}$ with language

$$\mathcal{L} = \{0^k 10^{n_1} 10^{n_2} 1 \cdots 10^{n_j} 10^\ell \mid n_i \in S, k, \ell \in \mathbb{N}\}.$$
Fix $S \subset \mathbb{N}$ and suppose $S$ is infinite. The associated $S$-gap shift is the subshift $\Sigma_S \subset \{0, 1\}^\mathbb{Z}$ with language

$$\mathcal{L} = \{0^k 10^{n_1} 10^{n_2} 1 \cdots 10^{n_j} 10^\ell \mid n_i \in S, k, \ell \in \mathbb{N}\}.$$ 

A uniform CGC-decomposition for $\Sigma_S$ is given by

$$\mathcal{G} = \{0^n 1 \mid n \in S\}$$
$$\mathcal{C}^p = \{0^k 1 \mid k \geq 0\}$$
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Then $\#(C^p \cup C^s)_n = 2$ for all $n \geq 1$, and so $h(C^p \cup C^s) = 0$. It follows that every subshift factor of an $S$-gap shift is intrinsically ergodic.
Coded systems

A shift space $X$ is **coded** if its language $\mathcal{L}$ is freely generated by a countable set of **generators** $\{w_n\}_{n \in \mathbb{N}} \subset \mathcal{L}$.

\[
\mathcal{L} = \{ \text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N} \}
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Let $\hat{h} = h(\{ \text{prefixes and suffixes of generators} \})$.

- $\hat{h} < h_{\text{top}}(X, \sigma) \Rightarrow (X, \sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of $(X, \sigma)$ is intrinsically ergodic