Counting closed geodesics

Vaughn Climenhaga

University of Houston

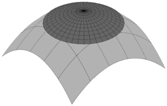
February 15, 2022

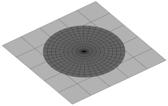
Joint work with Gerhard Knieper (Bochum) and Khadim War (IMPA)

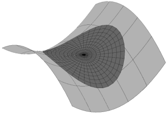
Curvature and growth

Consider a surface with (constant) Gaussian curvature K.

- How do circles/discs behave? (Growth of length/area)
- How do nearby geodesics behave? (Growth of distance)
- How many geodesics are there? (Growth of cardinality)







$K > 0$ $K = 0$ $K < 0$

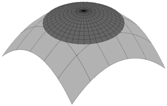
How many geodesics?? Infinitely many!

More precisely, count geodesic **segments** of length r that start at x and separate by at least ϵ ("distinguishable")

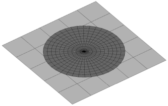
Curvature and growth

Consider a surface with (constant) Gaussian curvature K.

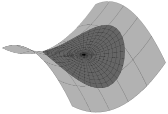
- How do circles/discs behave? (Growth of length/area)
- How do nearby geodesics behave? (Growth of distance)
- How many geodesics are there? (Growth of cardinality)



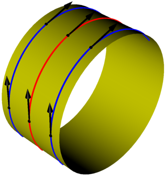
$K > 0$



$K = 0$



$K < 0$



Circumference $= 2\pi r$, area $= \pi r^2$

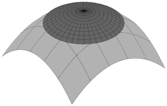
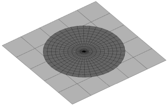
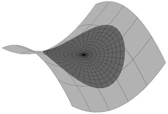
Distance constant (if parallel) or linear

Number $= 2\pi r/\epsilon$

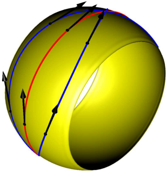
Curvature and growth

Consider a surface with (constant) Gaussian curvature K.

- How do circles/discs behave? (Growth of length/area)
- How do nearby geodesics behave? (Growth of distance)
- How many geodesics are there? (Growth of cardinality)

$K > 0$ $\quad$ $K = 0$ $\quad$ $K < 0$



Circumference $< 2\pi r$, area $< \pi r^2$

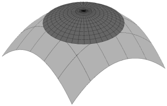
Distance bounded, conjugate points exist

Number of distinguishable geodesics bounded

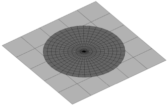
Curvature and growth

Consider a surface with (constant) Gaussian curvature K.

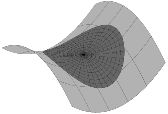
- How do circles/discs behave? (Growth of length/area)
- How do nearby geodesics behave? (Growth of distance)
- How many geodesics are there? (Growth of cardinality)



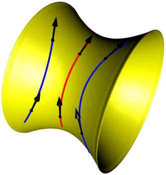
$K > 0$



$K = 0$



$K < 0$



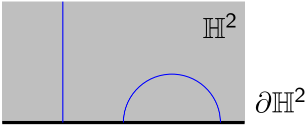
Circumference $> 2\pi r$, area $> \pi r^2$

Distance grows. . . how fast?

Number grows. . . how fast?

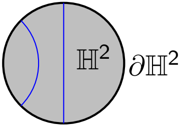
Hyperbolic geometry ($K \equiv -1$) and exponential growth

Upper half-plane model ($y > 0$)



$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

Disc model ($x^2 + y^2 < 1$)



$$ds^2 = 4\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$$

Geodesics = circles/lines orthogonal to $\partial\mathbb{H}^2$

Exercise: radius r circle has

- circumference $= \pi(e^r - e^{-r})$
- area $= \pi(e^r - 2 + e^{-r})$

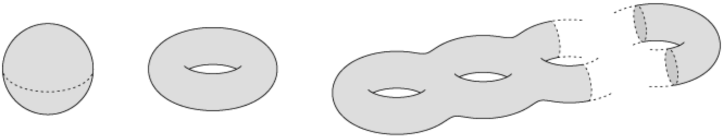
Large scale: send $r \to \infty$ and write $f(r) \sim g(r)$ if $\frac{f(r)}{g(r)} \to 1$

$\mathrm{area}(B(z,r)) \sim \pi e^r$ $\quad$ $\#\{\epsilon\text{-separated } r\text{-geod. from } z\} \sim \frac{\pi}{\epsilon}e^r$

Topology and geometry – surfaces as quotients

Closed surface: compact, connected, boundaryless, orientable

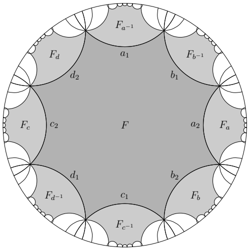
Every such surface admits a metric of constant curvature.



S^2 $(K = 1)$ $\mathbb{R}^2/\mathbb{Z}^2$ $(K = 0)$ $\mathbb{H}^2/\Gamma$ $(K = -1)$

All octagons shown are isometric; tile $\mathbb{H}^2$.

- $\gamma_a \in \mathrm{Isom}^+(\mathbb{H}^2)$ takes $a_1 \mapsto a_2$
- $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c, \gamma_d \rangle$ discrete
- $M = \mathbb{H}^2/\Gamma$ surface of genus 2
- $\pi_1(M) \cong \Gamma$



Fundamental group and closed geodesics

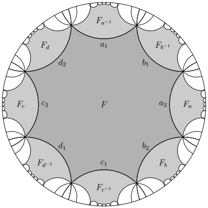
$M = \mathbb{H}^2/\Gamma$ surface of genus 2, with $\Gamma = \langle \gamma_a, \gamma_b, \gamma_c, \gamma_d \rangle \cong \pi_1(M)$.

Fundamental group produces closed geodesics:

$\boxed{\gamma \in \pi_1(M)}$ $\rightsquigarrow$ $\boxed{\text{free homotopy class of closed curves}}$ $\rightsquigarrow$ $\boxed{\text{shortest } c_\gamma \text{ is closed geod.}}$

Fix $p \in F$. Recall $\text{area}(B(p,r)) \sim \pi e^r$

- Let $G_r = \{\gamma \in \Gamma : \gamma F \subset B(p,r)\}$
- Area estimate $\Rightarrow \#G_r \geq Ce^r$
- For all $\gamma \in G_r$, get $|c_\gamma| \leq d(p, \gamma p) \leq r$.



Suggests $\#\{\text{closed geodesics with length} \leq r\}$ grows exponentially.

Warning: conjugate elements of $\pi_1(M)$ give same closed geodesic.

Exponential growth associated to $M = \mathbb{H}^2/\Gamma$

Volume growth: $\text{area}(B(x,t)) \sim \pi e^t$ (Same for all Γ, M)

Geodesic growth on M: $\#\{\epsilon\text{-sep. } t\text{-geodesics on } M\} \sim C_{M,\epsilon} e^t$

Closed geodesics on M:
$\#\{\text{closed geodesics with length} \leq t\}$ grows exponentially in t

1. How precise can we make "grows exponentially in t"?
2. What if M has *variable* negative curvature?

Also get exponential "word growth" in fundamental group $\pi_1(M)$

The first result for closed geodesics

Discrete $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^2)$ is cofinite if $M = \mathbb{H}^2/\Gamma$ has finite area.

Theorem (Huber, 1959)

Given M, Γ as above, let $P(t)$ denote the set of closed geodesics on M with length $\leq t$. Then $\#P(t) \sim \frac{e^t}{t}$.

Huber's proof relies on Selberg trace formula, which relates lengths of closed geodesics to spectrum of the Laplacian.

The first result for closed geodesics

Discrete $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^2)$ is cofinite if $M = \mathbb{H}^2/\Gamma$ has finite area.

Theorem (Huber, 1959)

Given M, Γ as above, let $P(t)$ denote the set of closed geodesics on M with length $\leq t$. Then $\#P(t) \sim \frac{e^t}{t}$.

Huber's proof relies on Selberg trace formula, which relates lengths of closed geodesics to spectrum of the Laplacian.

Analogies to prime number theory and Riemann zeta function.

$\pi(N) \sim \frac{N}{\log N} \quad \xleftrightarrow{T=\log N} \quad \#\{p \text{ prime}: \log p \leq T\} \sim \frac{e^T}{T}$

I am the wrong person to tell you about all this...

Beyond constant curvature

Let M be a surface of genus ≥ 2 with *variable* curvature

- Still get $M = X/\Gamma$ where universal cover X is homeomorphic to disc and $\Gamma \cong \pi_1(M)$ acts discretely and isometrically on X

Two Riemannian metrics: g (variable curvature), g_0 (constant)

Compact $\Rightarrow g = C^{\pm 1} g_0 \Rightarrow B_0(x, C^{-1}r) \subset B(x, r) \subset B_0(x, Cr)$

Still get exponential volume growth, but lose precise formula

Topological entropy of the "geodesic flow" on M is the number h such that (# of ϵ-distinguishable t-geodesic segments) $\approx e^{ht}$, where "$\approx$" is used quite loosely and is weaker than $\sim$. Formally,

$$h := \lim_{\epsilon \to 0} \overline{\lim_{t \to \infty}} \frac{1}{t} \log(\#\text{ of } \epsilon\text{-distinguishable } t\text{-geodesics})$$

Margulis asymptotic estimates

Theorem (Margulis, 1970 thesis, published 2004)

Let M be a closed Riemannian manifold with negative sectional curvatures, and $P(t)$ the set of closed geodesics with length $\leq t$. Let $h > 0$ be the topological entropy of geodesic flow on M. Then

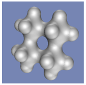
- $\#P(t) \sim \frac{e^{ht}}{ht}$, *and*
- *there is a continuous function c on the universal cover X such that for every $x \in X$ we have* $\operatorname{vol}(B(x, r)) \sim c(x)e^{hr}$.

Margulis's approach was publicized by Anatole Katok via the thesis of Charles Toll (1984) and the book with Boris Hasselblatt (1995).

An alternate proof was given by Parry and Pollicott (1983).

Beyond negative curvature

Margulis asymptotics for closed geodesics now proved for:



- surfaces with $K < 0$ outside radially symmetric "caps" (Bryce Weaver, *J. Mod. Dyn.* 2014)
- rank 1 manifolds of nonpositive curvature – in fact CAT(0) (Russell Ricks, arXiv:1903.07635)[1] (Count homotopy classes)
- rank 1 manifolds without focal points (Weisheng Wu, arXiv:2105.01841)
- **surfaces of genus ≥ 2 without conjugate points** (C., Knieper, War, *Comm. Cont. Math.*, to appear)

In last 3 settings, volume asymptotics proved by Weisheng Wu (arXiv:2106.07493)

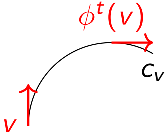
All these results follow the dynamical approach of Margulis

[1] Also prior unpublished work in 2002 thesis of Roland Gunesch

Geodesic flow and horocycles

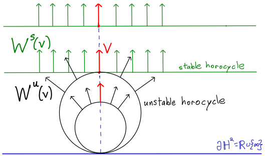
Study geodesic flow ϕ^t on unit tangent bundle $SM = \{v \in TM : \|v\| = 1\}$

$v \rightsquigarrow$ geodesic c_v with $\dot{c}_v(0) = v \;\rightsquigarrow\; \phi^t(v) := \dot{c}_v(t)$



Closed geodesics $\leftrightarrow$ periodic orbits for geodesic flow

For the time being, consider constant negative curvature



Each $v \in S\mathbb{H}^2$ is normal to two horocycles (horizontal lines or circles tangent to $\partial\mathbb{H}^2$)

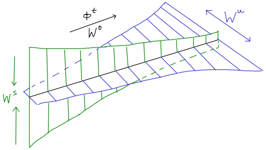
Normal vector fields $W^s(v), W^u(v) \subset S\mathbb{H}^2$ give stable/unstable foliations of $S\mathbb{H}^2$

Given $w \in W^s(v)$, we have $d(\phi^t(v), \phi^t(w)) = e^{-t} d(v, w)$

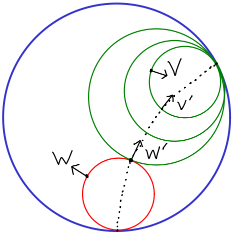
Given $w \in W^u(v)$, we have $d(\phi^t(v), \phi^t(w)) = e^{t} d(v, w)$

Product structure on $S\mathbb{H}^2$

Local product structure using W^u, W^s, and orbit foliation W^0



Important idea in hyperbolic dynamics:
"Any past can be joined to any future"



Can get a global picture too:

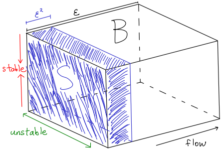
- Identify each leaf of $W^{s,u}$ with $\partial\mathbb{H}^2$.
- For all $(\xi, \eta) \in \partial^2\mathbb{H}^2 := (\partial\mathbb{H}^2)^2 \setminus \mathrm{diag}$ there is a unique geodesic from ξ to η.
- Parametrizing gives homeomorphism $S\mathbb{H}^2 \to \partial^2\mathbb{H}^2 \times \mathbb{R}$ (Hopf map).

Setting up the Margulis argument

$C(t) = \{\text{closed geod. with } |c| \in (t-\epsilon, t]\}$ $\qquad P(T) = \bigsqcup_k C(t_k)$

Estimate $\#C(t)$ and sum (becomes integral as $\epsilon \to 0$).

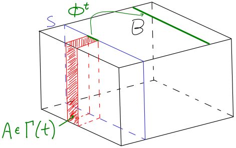
Use probability measure $\nu_t = \frac{1}{\#C(t)} \sum_{c \in C(t)} \frac{1}{t} \mathrm{Leb}_{\dot c}$



B = flow box $\subset SM$

S = slab/slice

$\nu_t(B) = \frac{\epsilon \cdot (\#\text{ transits})}{t \cdot \#C(t)}$



$\{\text{transits of } B \text{ by some } c \in C(t)\}$

immediate $\downarrow$ $\quad\uparrow$ Closing Lemma

$\Gamma(t) = \{\text{conn. components of } S \cap \phi^{-t} B\}$

$$\boxed{\nu_t(B) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\#C(t)}}$$

Completing the argument using ergodic theory

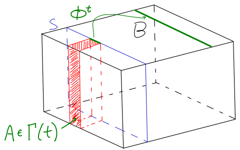
$\Gamma(t) = \{\text{conn. components of } S \cap \phi^{-t} B\}$ $\qquad$ $\nu_t(B) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\#C(t)}$

Liouville measure m on SM given by normalizing $m^s \times m^u \times \text{Leb}$, where $m^{s,u}$ are Lebesgue measure along $W^{s,u}$, and satisfy:

$$m^u(\phi^t A) = e^t m^u(A) \quad \text{and} \quad m^s(\phi^t A) = e^{-t} m^s(A)$$

Scaling: $m(A) \approx e^{-t} m(S)$ for all $A \in \Gamma(t)$

- $m(S \cap \phi^{-t} B) \approx e^{-t} m(S) \#\Gamma(t)$



Mixing: $\frac{m(S \cap \phi^{-t} B)}{m(S)} \to m(B)$

- $m(B) \approx e^{-t} \#\Gamma(t)$

Equidistribution: $\nu_t \xrightarrow{\text{wk}^*} m$, so $m(B) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\#C(t)} \Rightarrow \#C(t) \approx \frac{\epsilon}{t} e^t$

Ingredients needed for the Margulis argument

Product structure (for flow and measure)

- Used for flow box, closing lemma, mixing property

Scaling properties of leaf measure m^u

- Relied on fact that contraction rate along $W^{s,u}$ is constant

Equidistribution property $\nu_t(B) \to m(B)$

- Can prove it directly, or use the fact that m is the unique measure of maximal entropy

Entropy (as analogue of dimension)

d-dimensional measure

$m(B(x,\epsilon)) \approx \epsilon^d$

$$d = \lim_{\epsilon\to 0} \frac{\log m(B(x,\epsilon))}{\log \epsilon}$$

d-dimensional set $[0,1]^d$

$N(\epsilon) \approx \epsilon^{-d}$ balls to cover

$$d = \lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{-\log \epsilon}$$

For **entropy** of geodesic flow, refine *dynamically* via Bowen balls

$$B_t(v,\epsilon) = \{w \in SM : d(c_v(s), c_w(s)) < \epsilon \text{ for all } s \in [0,t]\}$$

Topological entropy: $h = \lim_{t\to\infty} \frac{1}{t} \log \Lambda_t(\epsilon)$ (ϵ fixed small)

$\Lambda_t(\epsilon) = \min\{\#E : \bigcup_{v\in E} B_t(v,\epsilon) = SM\}$ $\Lambda_t \approx e^{ht}$

Measure-theoretic entropy: μ flow-invariant prob. measure,

$h_\mu = \int \lim_{t\to\infty} -\frac{1}{t} \log \mu(B_t(v,\epsilon))\, d\mu(v)$ $\mu(B_t) \approx e^{-h_\mu t}$

Variational principle

Topological entropy: Value of h such that
(# of ϵ-distinguishable t-geodesic segments) $\approx e^{ht}$

Now consider a flow-invariant probability measure μ.

Measure-theoretic entropy: Value of h_μ such that
$\mu\{w : c_w \text{ } \epsilon\text{-indistinguishable from } c_v \text{ through time } t\} \approx e^{-h_\mu t}$

Variational principle: $h = \sup\{h_\mu : \mu \text{ flow-inv. prob. meas.}\}$

If $h_\mu = h$ then μ is a measure of maximal entropy (MME)

- When $K \equiv -1$, Liouville measure m has $h_m = 1 = h$
- In fact, m is the *unique* MME: Adler, Weiss, Bowen (1970s)

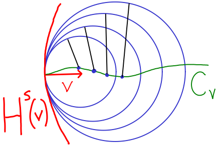
Anosov flows

Now move to setting of variable negative curvature, so $M = X/\Gamma$, where universal cover X is still homeomorphic to disc.

Still get stable horocycle for all $v \in SX$ by

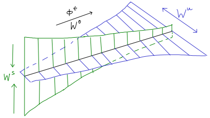
$$H^s(v) = \lim_{r\to\infty} \partial B_X(c_v(r), r)$$

Also unstable horocycle $H^u(v) = H^s(-v)$.



Normal vec. fields give foliations $W^{s,u}$ with uniform hyperbolicity:

$$w \in W^s(v) \quad \Rightarrow \quad d(\phi^t v, \phi^t w) \le Ce^{-\lambda t} d(v,w)$$
$$w \in W^u(v) \quad \Rightarrow \quad d(\phi^{-t} v, \phi^{-t} w) \le Ce^{-\lambda t} d(v,w)$$



Here $\lambda > 0$, and inequality is for all $t \ge 0$.

$(\phi^t : SM \to SM)_{t\in\mathbb{R}}$ is an Anosov flow

Anosov flows have local product structure

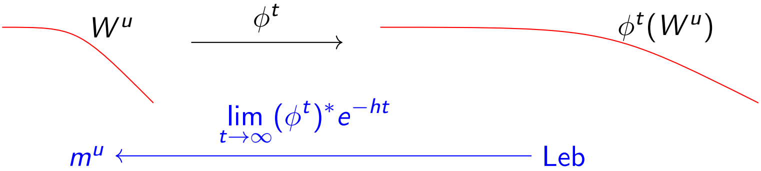
Margulis leaf measures in variable negative curvature

Surface of genus $\geq 2 \;\Rightarrow\; h > 0$ (by exponential volume growth), but Lebesgue measure on leaves may not scale by

$$m^u(\phi^t A) = e^{ht} m^u(A) \quad \text{and} \quad m^s(\phi^t A) = e^{-ht} m^s(A) \qquad (\star)$$

For **any** Anosov flow, Margulis built m^u, m^s satisfying $(\star)$

Idea: pull back Leb from $\phi^t(W^u)$, scale by e^{-ht}, take a limit



$m = m^u \times m^s \times$ Leb is flow-invariant Bowen–Margulis measure

- Unique MME, $\neq$ Liouville unless $K \equiv$ constant
- Allows to run the Margulis proof and get $\#P(t) \sim \frac{e^{ht}}{ht}$

Many constructions of Margulis leaf measures

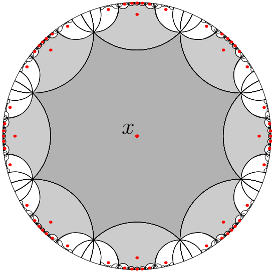
Various ways to formalize the details of the construction

- Fixed point argument on an appropriate space (Margulis 1970)
- Can also use Hausdorff measure in appropriate metric (Hamenstädt 1989, Hasselblatt 1989, ETDS)
- Interpretation via Bowen's alternate definition of entropy (C.–Pesin–Zelerowicz BAMS 2019, also C. arXiv:2009.09260)
- For geodesic flow can also use Patterson–Sullivan approach

Identify leaves of $W^{s,u}$ with ∂X. Build family $\{\nu_p : p \in X\}$ of measures on ∂X:

$$\nu_p = \lim_{s \searrow h} \left[\text{normalize}\left(\sum_{\gamma\in\Gamma} e^{-sd(p,\gamma x)}\delta_{\gamma x}\right)\right]$$

Weights give scaling properties (w.r.t. p) corresponding to Margulis measure.



(Patterson and Sullivan 1970s, Kaimanovich 1990)

No conjugate points

A manifold M has no conjugate points if any two points in the universal cover are joined by a unique geodesic.

$P(t) = \{\text{free homotopy classes of closed geod. with length} \leq t\}$

Theorem (C., Knieper, War, to appear in *Comm. Contemp. Math.*)

Let M be a surface of genus ≥ 2 with no conjugate points. Then $\#P(t) \sim \frac{e^{ht}}{ht}$.

Margulis's proof works for any closed manifold with negative sectional curvatures, in any dimension. Our proof covers **some** higher-dimensional examples, but a detailed description is rather technical.

Foliations via horospheres are troublesome

M a manifold without conjugate points, X universal cover

Horospheres $H^{s,u}$ and foliations $W^{s,u}$ as in negative curvature.

- $W^{s,u}(v)$ may not contract under $\phi^{\pm t}$ or be transverse (e.g. $\mathbb{R}^2$)
- Dependence on v might even be discontinuous (Ballmann, Brin, Burns "dinosaur" example)

How to define the flow box B? Requires product structure...

Define boundary at infinity ∂X as set of equivalence classes of geodesics, where $c_1 \sim c_2$ when $\sup_{t>0} d(c_1(t), c_2(t)) < \infty$

- "Set of possible futures/pasts"
- Can we join every past to every future?
  In general, **no**. For surfaces of genus ≥ 2, **yes**.

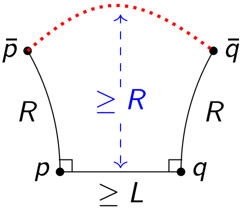
The Morse Lemma (not the one about critical points)

(M, g) surface, genus ≥ 2, no conjugate points; X universal cover

g_0 constant negative curvature metric $\Rightarrow$ $g = C^{\pm 1} g_0$

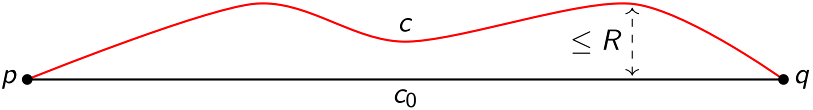
Exercise (Hyperbolic geometry for g_0)

$\exists\ L, R$ such that if $\bar{p}p$, pq, $q\bar{q}$ in picture are g_0-geodesics, then g_0-length of $\bar{p}\bar{q}$ (red dotted curve) is $> C^2 d_0(\bar{p}, \bar{q})$



Consequence: $\bar{p}\bar{q}$ not a g-geodesic

Morse Lemma: If $d_0(p, q) \geq L$ and c_0, c are g_0, g-geodesics from p to q, then Hausdorff distance from c_0 to c is $\leq R$.



A coarse kind of product structure

(M, g) surface, genus ≥ 2, no conjugate points; X universal cover

Morse Lemma: Every g_0-geodesic is R-shadowed by a g-geodesic (may not be unique), and vice versa.

Can join every past and future. (To within $2R$)

- $(\xi, \eta) \in \partial^2 X$ represented by g-geodesics c_ξ, c_η
- R-shadow c_ξ, c_η by g_0-geodesics c_ξ^0 and c_η^0
- Join $c_\xi^0(\infty)$ and $c_\eta^0(\infty)$ by g_0-geodesic c^0
- R-shadow c^0 by a g-geodesic c, which joins (ξ, η)

Hopf map $H\colon SX \to \partial^2 X \times \mathbb{R}$ is onto and continuous.

- Not 1-1, which causes technical headaches.
- Define flow box following Ricks: $B = H^{-1}(\mathbf{P} \times \mathbf{F} \times [0, \epsilon])$ where $\mathbf{P}, \mathbf{F}$ are disjoint neighborhoods in ∂X

New challenges for manifolds with no conjugate points

Desired ingredients for the Margulis argument:

- Product structure for flow (Provided by ∂X and Hopf map)
- Leaf measures m^s, m^u that scale by $e^{\pm ht}$ (Patterson–Sullivan)
- $m = m^s \times m^u \times \text{Leb}$ is mixing and is the unique MME (???)

Still get MME, but no proof of mixing or uniqueness

Theorem (C.–Knieper–War 2021, Adv. Math.)

For surfaces of genus ≥ 2 without conjugate points, a "coarse specification" argument establishes uniqueness of the MME.

With this in hand, Margulis argument (via Ricks) goes through.

Uniqueness using coarse specification

Joining past to future involves shadowing at some scale δ

- Formally, talk about "specification property at scale δ"

Argument due to Rufus Bowen (1970s) gives unique MME if

- δ small w.r.t. injectivity radius of M, say $\operatorname{inj} M > 120\delta$, and
- every pair $(\xi, \eta) \in \partial^2 X$ joined by **unique** geodesic.

Second condition guarantees an "expansivity" property.

- For surfaces with no conjugate points, this condition can fail, but only on a set of zero entropy.
- C.–Thompson (Adv. Math. 2016): unique MME if "obstructions to specification and expansivity" have small entropy, with $\operatorname{inj} M > 120\delta$.

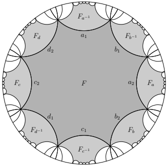
Morse Lemma gives specification at **large** scale δ (think $3R$), but this can easily be large compared to $\operatorname{inj} M$.

Salvation via residual finiteness

Specification scale δ depends on R from Morse Lemma, likely large.

Get uniqueness if $\operatorname{inj} M > 120\delta$. Probably false.

Solution: Replace M with a finite cover N with $\operatorname{inj} N$ big enough.



- Entropy-preserving bijection between flow-invariant measures on SM and SN.
- Theorem gives unique MME on SN
- Thus there is a unique MME on SM

Why possible? $\dim M = 2$ implies $\pi_1(M)$ is *residually finite*.

Higher dimensions

Method works for higher-dim M with no conjugate points **if**

1. $\exists$ Riemannian metric g_0 on M with negative curvature;
2. divergence property: $c_1(0) = c_2(0) \Rightarrow d(c_1(t), c_2(t)) \to \infty$;
3. $\pi_1(M)$ is residually finite;
4. $\exists h^* < h_{\text{top}}$ such that if μ-a.e. v has non-trivially overlapping horospheres, then $h_\mu \leq h^*$.

First is a real topological restriction: rules out Gromov example.

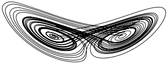
Second and third might be redundant? No example satisfying (1) where they are known to fail

Fourth is true if $\{v : H_v^s \cap H_v^u \text{ trivial}\}$ contains an open set. Unclear if this is always true.

Some examples where Margulis asymptotics remain open

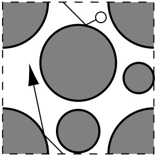
Lorenz flow (the famous “butterfly attractor”)

- Unique MME: Leplaideur (arXiv:1905.06202), Pacifico, Fan Yang, Jiagang Yang (arXiv:2201.06622)



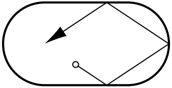
Sinai billiard flow on torus with finite number of convex scatterers

- Unique MME: Baladi, Demers (JAMS, 2020)



Bunimovich stadium billiard

- No results on MME yet



Geodesic flows in positive curvature (?)

- “Biscuit surface” approximates stadium
- Kourganoff relates geodesic flow, billiard

Thank you!