Math 4377/6308 Advanced Linear Algebra I Spring 2013

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MIDTERM TEST #1

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You must give complete justification for all answers in order to receive full credit.

Name: SOLUTIONS

	Points	Possible
Problem 1		/10
Problem 2		/25
Problem 3		/20
Problem 4		/20
Problem 5		/10
Problem 6		/15
Total		/100

1. (a) Let $x \sim y$ be a relation on a set X. Define what it means for $\sim$ to be an *equivalence relation*. [*5 points*]

$\sim$ is an equivalence relation if the following axioms are ~~true~~:

① Reflexivity: $x \sim x \quad \forall\, x \in X$

② Symmetry: If $x, y \in X$ and $x \sim y$, then $y \sim x$

③ Transitivity: If $x, y, z \in X$, and $x \sim y$, and $y \sim z$, then $x \sim z$.

(b) Define a relation on $\mathbb{R}$ by $x \sim y$ if and only if $xy \geq 0$. Is this an equivalence relation? Prove your answer. [*5 points*]

No, it is not transitive. For any $x \in \mathbb{R}$ we have $x \cdot 0 = 0 \geq 0$ $\therefore$ $x \sim 0$, and so $-1 \sim 0$ and $0 \sim 1$. However, $-1 \cdot 1 = -1 < 0$ $\therefore$ $-1 \nsim 1$.

2. (a) Let $W = \{(x, y) \in \mathbb{R}^2 \mid xy + x = 0\}$. Is W a subspace of $\mathbb{R}^2$? Prove your answer. *[10 points]*

No. Subspaces are closed under addition, but W is not. Indeed, for $(0,1)$ and $(1,-1)$ we have:

$(0,1) \Rightarrow x=0,\ y=1 \Rightarrow xy+x = 0\cdot 1 + 0 = 0 \Rightarrow (0,1) \in W$

$(1,-1) \Rightarrow x=1,\ y=-1 \Rightarrow xy+x = 1(-1)+1 = -1+1=0 \Rightarrow (1,-1) \in W$

However, $(0,1)+(1,-1) = (1,0)$, and $1\cdot 0 + 1 = 1 \neq 0$, so $(1,0) \notin W$.

(b) Let V and W be vector spaces and let $T\colon V \to W$ be a linear transformation. Let W_1 be a subspace of W and define a set $V_1 \subset V$ by $V_1 = \{x \in V \mid T(x) \in W_1\}$. Show that V_1 is a subspace of V. *[15 points]*

Check conditions of Theorem 1.3:

① $x = \vec{0}_V \Rightarrow T(x) = \vec{0}_W$, and since W_1 is a subspace of W we have $\vec{0}_W \in W_1$. Thus $\vec{0}_V \in V_1$

② If $x, y \in V_1$, then $T(x), T(y) \in W_1$, so

$$\underbrace{T(x+y) = T(x)+T(y)}_{T \text{ is linear}} \underbrace{\in W_1}_{W_1 \text{ a subspace}}, \text{ thus } x+y \in V_1$$

③ If $x \in V_1$ and $c \in F$, then $T(x) \in W_1$, so

$$\underbrace{T(cx) = cT(x)}_{T \text{ is linear}} \underbrace{\in W_1}_{W_1 \text{ a subspace}}, \text{ thus } cx \in V_1.$$

By Theorem 1.3, V_1 is a subspace of V.

3. (a) Define what it means for a vector space V to be the direct sum of two subspaces $W_1, W_2 \subset V$. [*5 points*]

V is the direct sum of W_1 and W_2 if

① $V = W_1 + W_2$ ($\forall\, v \in V\ \exists\, w_1 \in W_1$ and $w_2 \in W_2$ ~~such that~~ such that $v = w_1 + w_2$)

and ② $W_1 \cap W_2 = \{\vec{0}_V\}$

(b) In $\mathbb{R}^3$, consider the subspaces $W_1 = \{(0, -b, b) \mid b \in \mathbb{R}\}$ and $W_2 = \{(a_1, a_2 + a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. Is $\mathbb{R}^3 = W_1 \oplus W_2$? Justify your answer. [*15 points*]

Yes. From the HW, $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.
Here $\dim W_1 = 1$, $\dim W_2 = 2$, and $\dim \mathbb{R}^3 = 3$.
From a result in class, $W_1 + W_2 = \mathbb{R}^3$ iff $\dim(W_1 + W_2) = 3$.
Thus $W_1 + W_2 = \mathbb{R}^3$ iff $\dim(W_1 \cap W_2) = 0$, that is, $W_1 \cap W_2 = \{\vec{0}\}$, and in this case $\mathbb{R}^3 = W_1 \oplus W_2$.

To compute $W_1 \cap W_2$, let $(x, y, z) \in W_1 \cap W_2$, so

$$(x, y, z) = (0, -b, b) = (a_1, a_2 + a_1, a_2)$$

for some $a_1, a_2, b \in \mathbb{R}$. The first coordinate gives $a_1 = 0$, so $a_2 = -b = b$. Thus $b = a_2 = 0 \therefore (x, y, z) = \vec{0}$.
We conclude ~~that~~ $W_1 \cap W_2 = \{\vec{0}\}$, hence $\mathbb{R}^3 = W_1 \oplus W_2$.

4. (a) Let V be a vector space over a field F, and let $S \subset V$. Define what it means for S to be linearly independent. [*5 points*]

S is linearly independent if for every $u_1, u_2, \ldots, u_n \in S$, the only solution to $\sum_{i=1}^{n} a_i u_i$ $(a_i \in F)$ is $a_1 = a_2 = \cdots = a_n = 0$.

(b) Consider the vector space $\mathbb{P}_3(\mathbb{R})$ consisting of polynomials with degree 3 or less, and the subset

$$S = \{1 + x^3,\ x + x^2,\ x^2 - x^3,\ 1 - x\} \subset \mathbb{P}_3(\mathbb{R}).$$

Does S span $\mathbb{P}_3(\mathbb{R})$? [*15 points*]

No. $\#S = 4 = \dim \mathbb{P}_3(\mathbb{R})$, so S spans $\mathbb{P}_3(\mathbb{R})$ if and only if it is linearly independent. However, we have

$$(1+x^3) - (x+x^2) + (x^2-x^3) - (1-x) = 0,$$

so S is linearly dependent and hence does not span $\mathbb{P}_3(\mathbb{R})$.

5. Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$. Is $\{v_1 + v_2, v_1 - v_2\}$ necessarily a basis for $\mathbb{R}^2$? *[10 points]*

Yes. Because $\{v_1 + v_2, v_1 - v_2\}$ has 2 elements and $\dim \mathbb{R}^2 = 2$, it is a basis if and only if it is linearly independent. Suppose $a_1, a_2 \in \mathbb{R}$ are such ~~that~~

$$a_1(v_1 + v_2) + a_2(v_1 - v_2) = \vec{0}. \qquad (*)$$

Then $(a_1 + a_2)v_1 + (a_1 - a_2)v_2 = \vec{0}$, and linear independence of the basis $\{v_1, v_2\}$ implies ~~that~~ $a_1 + a_2 = 0$ & $a_1 - a_2 = 0$. The only solution is $a_1 = a_2 = 0$, thus $\{v_1 + v_2, v_1 - v_2\}$ is linearly independent, hence it ~~spans $\mathbb{R}^2$~~ is (because of the theorem on dimension from class), so it is a basis.

6. (a) Let V be a vector space. State what it means for V to be finite-dimensional. Assuming V is finite-dimensional, state the definition of the dimension of V. [*5 points*]

V is finite-dimensional if it has a finite basis. The dimension of V is the number of elements in any finite basis for V.

(b) Let V and W be finite-dimensional vector spaces and $T\colon V \to W$ a linear transformation. Suppose that $\dim W > \dim V$, and prove that T is not onto. [*10 points*]

By the theorem on rank and nullity,

$$\text{rank}(T) + \text{nullity}(T) = \dim V < \dim W$$

Because $\text{nullity}(T) \geq 0$, this implies

$$\text{rank}(T) < \dim W.$$

T is onto iff $\text{rank}(T) = \dim W$, so this implies that T is not onto.

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