Math 4377/6308 Advanced Linear Algebra I Fall 2013
Dr. Vaughn Climenhaga, PGH 651A

HOMEWORK 1

Due *4pm Wednesday, August 28.*

1. Determine whether each of the following statements is true or false. Justify your answers.

(a) $\forall a \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that we have $a + n = 10$.

Solution. *[5 points]* True. Given any integer $a \in \mathbb{Z}$, we can choose $n \in \mathbb{Z}$ to be the integer $n = 10 - a$. Then $a + n = a + (10 - a) = 10$ and so our choice of n makes the equation hold.

(b) $\exists n \in \mathbb{Z}$ such that $\forall a \in \mathbb{Z}$ we have $a + n = 10$.

Solution. *[5 points]* False. To prove that the statement is false, we first write the logical negation: "$\forall n \in \mathbb{Z}$, $\exists a \in \mathbb{Z}$ such that $a + n \neq 10$". Proving that the original statement is false is equivalent to proving that this statement is true. Given any $n \in \mathbb{Z}$, choose a as follows: if $n = 10$ then choose $a = 1$, otherwise if $n \neq 10$ choose $a = 1$. In either case we have $a + n \neq 10$, so the logical negation of the original statement is true, hence the original statement is false.

(c) $\forall a \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ such that we have $a + n = 10$.

Solution. *[5 points]* False. As in the previous question, we prove the logical negation: "$\exists a \in \mathbb{R}$ such that $\forall n \in \mathbb{Z}$ we have $a + n \neq 10$". It is enough to exhibit a real number a such that for every $n \in \mathbb{Z}$, we have $a + n \neq 10$. To this end, let $a = 1/2$ and observe that for every integer n, the sum $a + n$ is not an integer, and in particular is not equal to 10.

(d) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, $\exists a, b, c \in \mathbb{R}$ with $(a, b, c) \neq (0, 0, 0)$ such that $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$. *Hint: the last equation can be turned into a system of two equations in three variables.*

Solution. *[5 points]* True. Given any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, we write $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, and $\mathbf{w} = (w_1, w_2)$. A triple of real numbers (a, b, c) satisfies the given equation if and only if it solves the following system of two equations in three variables:

$$u_1 a + v_1 b + w_1 c = 0,$$
$$u_2 a + v_2 b + w_2 c = 0.$$

This is an underdetermined system of homogeneous linear equations, and we recall from the basic theory of such systems that it has infinitely many solutions (a, b, c). In particular, it has a non-zero solution. Choosing these values of a, b, c satisfies the equation in the original statement, hence the original statement is true.

---

2. Write the logical negation of the sentence in **1.(d)**.

Solution. *[5 points]* Negating each level of the statement, one step at a time, the logical negation is

- "$\exists \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\forall a, b, c \in \mathbb{R}$ with $(a, b, c) \neq (0, 0, 0)$, we have $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \neq \mathbf{0}$."

---

3. Consider the complex numbers $z = 2 - i$ and $w = 1 + 3i$. Write the complex numbers $zw, \bar{w}, \bar{z} + w, |w|$, and $\frac{1}{z}$ in the form $a + bi$, where $a, b \in \mathbb{R}$.

Solution. *[10 points]*

$$\begin{aligned}
zw &= (2 - i)(1 + 3i) = 2 - i + 6i + 3 = 5 + 5i, \\
\bar{w} &= 1 - 3i, \\
\bar{z} + w &= (2 + i) + (1 + 3i) = 3 + 4i, \\
|w| &= \sqrt{1^2 + 3^2} = \sqrt{10}, \\
\frac{1}{z} &= \frac{1}{2 - i} = \frac{2 + i}{(2 - i)(2 + i)} = \frac{2 + i}{4 - i^2} = \frac{2}{5} + \frac{1}{5}i.
\end{aligned}$$

---

4. List all of the elements in the following sets.

(a) $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq a < 3,\ b^2 \leq 1\}$

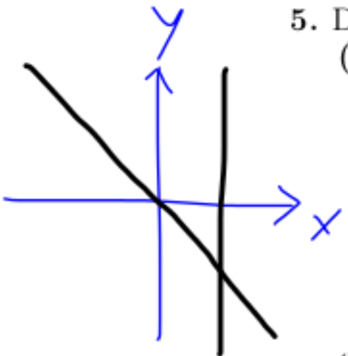
Solution. *[5 points]* a can be either 1 or 2, and b can be any of $-1, 0, 1$, so the set is $\{(1, -1), (1, 0), (1, 1), (2, -1), (2, 0), (2, 1)\}$.

(b) $\{(c, d) \in \mathbb{Z} \times \mathbb{N} \mid d^3 \leq 8,\ |c| \leq d\}$

Solution. *[5 points]* d can be either 1 or 2. If $d = 1$ then c can be any of $-1, 0, 1$. If $d = 2$ then c can be any of $-2, -1, 0, 1, 2$. Thus the set is $\{(-1, 1), (0, 1), (1, 1), (-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2)\}$.

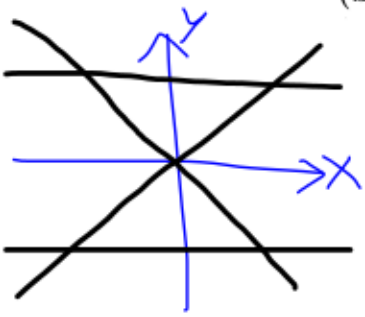
---

5. Draw a sketch of the following subsets of $\mathbb{R}^2$.



(a) $\{(x, y) \in \mathbb{R}^2 \mid (x + y)(x - 1) = 0\}$

Solution. *[5 points]* If $(x + y)(x - 1) = 0$ then either $x + y = 0$ or $x - 1 = 0$. Thus the set is the union of $\{(x, y) \mid x + y = 0\}$ and $\{(x, y) \mid x - 1 = 0\}$. The former is the line through the origin with slope -1, the latter is the vertical line through $(1, 0)$.



(b) $\{(x, y) \in \mathbb{R}^2 \mid (y^2 - 4)(x^2 - y^2) = 0\}$

Solution. *[5 points]* First note that $(y^2 - 4)(x^2 - y^2) = (y - 2)(y + 2)(x - y)(x + y)$, and so the set is the union of the graphs of $y = 2$, $y = -2$, $x = y$, and $x = -y$. The first two sets are horizontal lines through $(0, \pm 2)$ and the last two are diagonal lines through the origin with slopes ± 1.

---

6. For each of the following maps, determine whether it is 1-1, and whether it is onto.

(a) $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2$

Solution. *[5 points]* T is neither 1-1 nor onto. Indeed, $T(-1) = 1 = T(1)$, so T is not 1-1. Moreover, $T(x) \geq 0$ for all x, and so there is no x for which $T(x) = -1$, so T is not onto.

(b) $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^3$

Solution. *[5 points]* T is both 1-1 and onto. If $T(x) = T(y)$ then $x^3 = y^3$, and taking the cube root of both sides yields $x = y$. Conversely, given $y \in \mathbb{R}$ we can take $x = \sqrt[3]{y}$ and get $T(x) = (\sqrt[3]{y})^3 = y$, so T is onto.

(c) $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = \begin{cases} x-1 & x \leq 0 \\ x+1 & x > 0 \end{cases}$

Solution. *[5 points]* T is 1-1, but not onto. Notice that if $T(x) \leq 0$, then $x \leq 0$, while if $T(x) > 0$, then $x > 0$. Thus if $T(x) = T(y)$, we must have that either x, y are both positive, or they are both ≤ 0. If they are both positive then $T(x) = x+1$ and $T(y) = y+1$, so $x + 1 = y + 1$ and hence $x = y$. On the other hand, if they are both ≤ 0, then $T(x) = x - 1$ and $T(y) = y - 1$, so $x - 1 = y - 1$ and once again $x = y$. This shows that T is one-to-one. To see that T is not onto, observe that $T(x) \leq -1$ whenever $x \leq 0$ and $T(x) > 1$ whenever $x > 0$. In particular, there is no x for which $T(x) = 0$.