Math 6324 — Ordinary Differential Equations I — Fall 2012

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PROBLEM SET 1

Due *5pm Thursday Sept. 20.* You must give complete justification for all answers in order to receive full credit.

1. Let $\|\cdot\|$ be the operator norm on $\mathcal{L}(\mathbb{R}^n)$ induced by some norm on $\mathbb{R}^n$.

(a) Let $T \in \mathcal{L}(\mathbb{R}^n)$ be such that $\|T - I\| < 1$. Show that T is invertible and that the series $\sum_{k=0}^{\infty}(I - T)^k$ converges absolutely to T^{-1}. Find an upper bound for $\|T^{-1}\|$.

(b) Let $T \in \mathcal{L}(\mathbb{R}^n)$ be invertible. Find $\epsilon > 0$ such that every S with $\|S - T\| < \epsilon$ is invertible as well. *(Hint: apply the above result to $T^{-1}S$.)*

(c) Show that if $\|T - I\|$ is sufficiently small, then there is $S \in \mathcal{L}(\mathbb{R}^n)$ such that $e^S = T$. To what extent is S unique? *(Hint: consider the power series of* $\log(1 + x)$*.)*

2. Given $T \in \mathcal{L}(\mathbb{R}^n)$, let $\sigma(T)$ be the set of eigenvalues for T. Let $\lambda_{\min} = \min\{|\lambda| \mid \lambda \in \sigma(T)\}$ and $\lambda_{\max} = \max\{|\lambda| \mid \lambda \in \sigma(T)\}$. Suppose that T is invertible, and let $\|\cdot\|$ be the operator norm on $\mathcal{L}(\mathbb{R}^n)$ associated to some norm $|\cdot|$ on $\mathbb{R}^n$.

(a) Show that $\|T^{-1}\|^{-1} = \min\{\|Tx\| \mid |x| = 1\}$ and that:

$$\|T^{-1}\|^{-1} \leq \lambda_{\min} \leq \lambda_{\max} \leq \|T\| \tag{1}$$

$$\lim_{k\to\infty} \|T^k\|^{1/k} = \lambda_{\max} \qquad \text{and} \qquad \lim_{k\to\infty} \|T^{-k}\|^{-1/k} = \lambda_{\min}. \tag{2}$$

(b) Let B be an $m \times m$ upper Jordan block, so that $B = \lambda I + N$, where N is nilpotent of order m, with ones immediately above the main diagonal and zeros elsewhere. Let $Q = \mathrm{diag}(1, \epsilon, \epsilon^2, \dots, \epsilon^{m-1})$ for some fixed $\epsilon > 0$, and show that $Q^{-1}BQ = \lambda I + \epsilon N$. Thus the ones above the main diagonal in a Jordan block can be replaced by any $\epsilon > 0$.

(c) Show that for every $\epsilon > 0$ there exists a norm $|\cdot|$ on $\mathbb{R}^n$ such that the associated operator norm on $\mathcal{L}(\mathbb{R}^n)$ has

$$\lambda_{\min} - \epsilon \leq \|T^{-1}\|^{-1} \leq \lambda_{\min} \leq \lambda_{\max} \leq \|T\| \leq \lambda_{\max} + \epsilon. \tag{3}$$

3. Prove that $T \in \mathcal{L}(\mathbb{R}^n)$ is nilpotent if and only if all its eigenvalues are zero.

4. Solve the following initial value problems by writing an equivalent first order system and using matrix exponentials:

(a) $\ddot{x} + 9x = 0, \qquad x(0) = 0, \quad \dot{x}(0) = 1.$

(b) $\ddot{x} - 4\dot{x} + 3x = 0, \qquad x(1) = -4, \quad \dot{x}(1) = 0.$

5. Give necessary and sufficient conditions on $a, b \in \mathbb{R}$ for $\ddot{x} + a\dot{x} + bx = 0$ to have a non-trivial periodic solution.

6. Solve $\dot{x} = Ax$, $x(0) = (1, -1, 1, 0)^t$ where

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

7. Let $A \in \mathbb{M}_{n\times n}$ and suppose that all eigenvalues of A have nonpositive real parts.

(a) If A is semisimple, show that every solution of $\dot{x} = Ax$ is bounded in forward time, in the following sense: for every $x(0)$ there exists $M \in \mathbb{R}$ such that $|x(t)| \leq M$ for all $t \geq 0$.

(b) Give an example of an invertible matrix A such that all eigenvalues have nonpositive real parts but $\dot{x} = Ax$ has a solution such that $|x(t)| \to \infty$ as $t \to \infty$. What is the smallest dimension in which such an example can exist?