PROBLEM SET 4

Due *5pm Thursday Dec. 6*. You must give complete justification for all answers in order to receive full credit.

1. *(Perko, exercise 3.5.2, page 213)* Consider the nonlinear system

$$\begin{aligned}\dot{x} &= x - 4y - \frac{x^3}{4} - xy^2 \\ \dot{y} &= x + y - \frac{x^2 y}{4} - y^3 \\ \dot{z} &= z.\end{aligned}$$

(a) Show that $\gamma(t) = (2\cos 2t, \sin 2t, 0)$ is a periodic solution with period $T = \pi$.
(b) Determine the linearisation of this sytem around the periodic orbit γ by finding $A(t) = Df(\gamma(t))$ such that the linearisation is $\dot{v} = A(t)v$.
(c) Show that the following is a fundamental matrix for $\dot{v} = A(t)v$:

$$\Phi(t) = \begin{pmatrix} e^{-2t}\cos 2t & -2\sin 2t & 0 \\ \frac{1}{2}e^{-2t}\sin 2t & \cos 2t & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

(d) Write $\Phi(t)$ as $\Phi(t) = Q(t)e^{Bt}$, where Q is a T-periodic 3×3 matrix and $B \in \mathbb{M}_{3\times 3}$ is constant.
(e) Use the previous part to determine the characteristic exponents and the characteristic multipliers of the periodic orbit $\gamma(t)$. What can you say about the stability of the periodic orbit?

2. *(Hirsch–Smale exercise 11.1.5, page 241)*
Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field and suppose that the flow φ_t of $\dot{x} = f(x)$ is defined for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. Suppose that $X \subset \mathbb{R}^n$ is a nonempty compact invariant set for φ_t, and suppose in addition that X is *minimal*: that is, X does not contain any compact invariant nonempty proper subset. Prove the following:
(a) Every trajectory in X is dense in X.
(b) $\alpha(x) = \omega(x) = X$ for every $x \in X$.
(c) For every $x_0 \in X$ and $\epsilon > 0$, there is a number $T > 0$ such that for every $x \in X$ and $t_0 \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that $|t - t_0| < T$ and $|\varphi_t(x) - x_0| < \epsilon$. *(This is a statement about how quickly orbits become dense.)*

3. *(Perko, exercise 3.2.4, page 183)*
Sketch the phase portrait of a single planar system (an ODE in $\mathbb{R}^2$) containing points $x_1, \ldots, x_5$ with the following properties:
(a) $\alpha(x_1) = \omega(x_1) = \{x_0\}$, but $x_0 \neq x_1$;
(b) $\omega(x_2)$ is a single orbit;
(c) $\omega(x_3)$ is the union of an orbit and an equilibrium point;
(d) $\omega(x_4)$ is the union of two orbits and a single equilibrium point;
(e) $\omega(x_5)$ is the union of two orbits and two equilibrium points.
Identify the points $x_1, \ldots, x_5$ in your picture.

4. Consider the following ODE in $\mathbb{R}^2$:

$$\dot{x} = y,$$
$$\dot{y} = -x + y(4 - 2x^2 - 3y^2).$$

(a) For the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$, compute $\dot{V}$ and deduce that the annulus $A = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$ is forward-invariant.
(b) Use the Poincaré–Bendixson Theorem to show that A contains a non-trivial periodic orbit.