ODE (6324) — Fall 2012

LECTURE 1

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Goal: Study time evolution of a system (model)

Phase space = {states of the system}

$\boxed{\text{Assumption \#1}}$ (deterministic) Future completely determined by current position in phase space (& possibly time): $x(t) = f(x(t_0), t_0, t)$

$\boxed{\text{Assumption \#2}}$ (fin-dim) The phase space is a finite-dimensional smooth manifold (often $\mathbb{R}^n$)

$\boxed{\text{Assumption \#3}}$ (differentiable) Dependence of future on $x(t_0)$, t_0, t is differentiable

Remove #1, get SDEs, remove #2 & get PDEs (mention Sobolev space)

Examples: classical mechanics — solar system, pendula, springs, freshman physics
population dynamics, radioactive decay
– we will focus on theory, not examples

$\cancel{x}$ (higher order)

2 dichotomies: (1) linear vs. non-linear, (2) autonomous vs. non-autonomous

$$(*) \quad \dot{x} = F(x,t), \quad x \in M \qquad \left(\dot{x} = x' = \frac{dx}{dt}\right)$$

Linear: $M = \mathbb{R}^n$, $\quad F(x,t) = A(t)x \qquad A(t) \in M_{n\times n}$

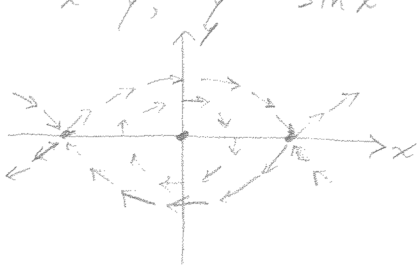
Autonomous: $F(x,t) = F(x)$, $\quad F: M \to TM$, $\quad F(x) \in T_xM$ (section)

autonomous ODE $\longleftrightarrow$ vector field

non-autonomous ODE $\longleftrightarrow$ changing vector field

Example| Pendulum specified by $(x,y) = (\text{angle}, \frac{d}{dt}\text{angle})$

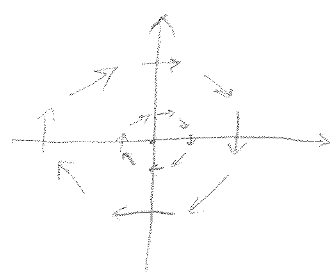
$$\dot{x} = y, \quad \dot{y} = -\sin x \qquad F(x,y) = (y, -\sin x)$$



$\sin x \approx x$ near 0

$$DF = \left(\frac{\partial f_i}{\partial x_j}\right)$$

$$\tilde{F}(x,y) = (y, -x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$F(\bar{x}) = 0$, then Taylor $\Rightarrow$ $F(x) = DF(\bar{x})\cdot(x-\bar{x}) + \underbrace{R(\bar{x},x)}_{= o(\|x-\bar{x}\|)}$

$y = x - \bar{x} \Rightarrow \dot{y} = DF(\bar{x})\cdot y$

Use linear eq'n (completely tractable) to understand non-linear case (hard).

* Linear often gives local picture

NB: Reduce higher-order to first order by enlarging phase space

Overall plan

$\boxed{\text{I}}$ Linear autonomous: $\dot{x} = Ax$, $A \in M_{n\times n}$, $x \in \mathbb{R}^n$

* Complete & explicit solutions, relying heavily on linear algebra, particularly matrix exponentials & Jordan normal form

$\boxed{\text{II}}$ Non-linear autonomous: $\dot{x} = F(x)$, $x \in \mathbb{R}^n$

* Existence & uniqueness of solutions
* Stability & local results from linear theory
* Global theory & qualitative results ← NB de-emphasis on "solving"

$\boxed{\text{III}}$ Other topics (as time & taste permit)

* Non-autonomous, Floquet theory ($\dot{x} = A(t)x$, A periodic)
* Bifurcation theory, changes in qualitative behaviour
* Normal forms & resonances
* Classical mechanics, Hamiltonian mechanics

LINEAR AUTONOMOUS HOMOGENEOUS ODEs

(first half of Hirsch-Smale, first chapter of Perko, midway through Arnold)

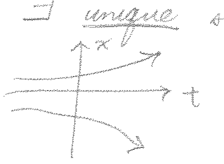
Central example: $\dot{x} = ax$, $a \in \mathbb{R}$, $x \in \mathbb{R}$ $(*)$

$\Rightarrow$ $x(t) = ce^{at}$, c determined by initial condition

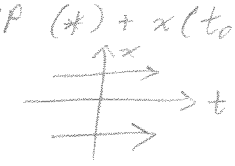
$\boxed{Q}$ Any other solutions? NO. Sps $x(t)$ is a sol'n, then

$\frac{d}{dt}\left(x(t)e^{-at}\right) = \dot{x}e^{-at} + x(-a)e^{-at} = axe^{-at} - axe^{-at} = 0$

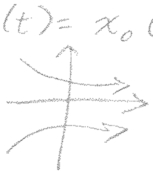
So $\exists$ unique sol'n to IVP $(*) + x(t_0) = x_0$: $x(t) = x_0 e^{a(t-t_0)}$



$a>0$



$a=0$



$a<0$

* Qualitative behaviour of solutions depends on parameter a, and there is a sudden change in this behaviour at a particular parameter value. (bifurcation point)

$\boxed{Q}$ How to make this statement precise?

Related $\boxed{Q}$ What sorts of qualitative behaviour are possible?
Which are typical & which are atypical?

Now replace (*) with higher-dim analogue

(**) $\dot{x} = Ax \qquad x \in \mathbb{R}^n \qquad A \in M_{n\times n}$

(will abuse notation on occasion by writing $(x_1, \ldots, x_n)$ for $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$)

Solution has form $x : \mathbb{R} \to \mathbb{R}^n$, $\begin{pmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = A \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ ← n coupled ODEs

Geometrically, tangent vector to curve lies in vector field $x \mapsto Ax$

$\boxed{\text{FACT}}$

Solution of (**) is $x(t) = e^{At} x_0$. (NOTE ORDER... $x_0 e^{At}$ meaningless)

But what does e^{At} mean?? "always remember what you are"

LECTURE 2

$\boxed{8/30}$

Preliminaries from linear analysis

V a vec sp, a norm on V is a fn $N : V \to \mathbb{R}$ s.t.

1. $N(x) \geq 0 \;\; \forall x \in V, \quad N(x) = 0 \Leftrightarrow x = 0$
2. $N(x+y) \leq N(x) + N(y)$
3. $N(\alpha x) = |\alpha| \cdot N(x) \;\; \forall \alpha \in \mathbb{R}$ (or whatever the scalar field is)

$\boxed{\text{Examples}}$

$V = \mathbb{R}^n$: $\quad \|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \|x\|_1 = \sum_i |x_i|, \quad \|x\|_\infty = \max_i |x_i|$

$V = L^\infty(X, \mu)$: $\quad \|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad p \geq 1. \qquad \|f\|_\infty = \operatorname{ess\,sup} \{ |f(x)| : x \in X \}$

Norm $\rightsquigarrow$ metric $\rightsquigarrow$ topology
("size") ("distance") ("open set")

So a norm gives us notions of "smallness", "proximity", & "convergence".

Two norms N_1, N_2 are equivalent if $\exists\ A, B > 0$ s.t.

$$A N_1(x) \le N_2(x) \le B N_1(x) \qquad \forall\ x \in V.$$

Equivalent norms give the same topology (& more...)

$\boxed{\text{Prop}}$ If V is finite-dimensional, then any 2 norms are equivalent.

$\boxed{\text{Pf}}$ suffices to take $V = \mathbb{R}^n$.

① Any norm N is cts wrt. L^1-norm

- $N(x) = N(\Sigma_i x_i e_i) \le \Sigma_i N(x_i e_i) = \Sigma_i |x_i| \cdot N(e_i) \le \underset{:= \max_i N(e_i)}{M} \|x\|_1$
- $N(x) = N(x - y + y) \le N(y) + N(x-y) \Rightarrow |N(x) - N(y)| \le N(x-y) \le M \|x-y\|_1$

② Given N_1, N_2, $\varphi(x) = \frac{N_1(x)}{N_2(x)}$ is L^1-cts on $\mathbb{R}^n \setminus \{0\}$

- Cts on $S^{n-1} = \partial B(0,1) = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$, which is cpt

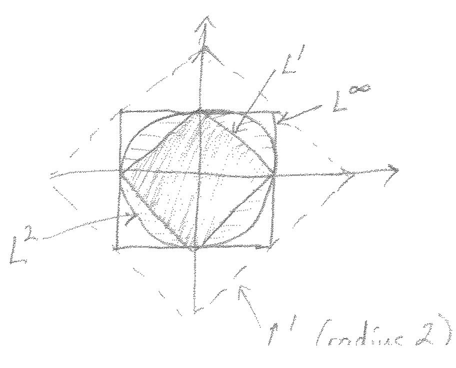
$\therefore$ achieves max & min, say B & A. $\therefore\ 0 < A \le B < \infty$

③ $S^{n-1} \rightsquigarrow \mathbb{R}^n$ by normalising ■

(Rmk) Fails utterly in infinite dimensions – for $V = L^\infty(X, \mu)$, the various L^p-norms induce different topologies & life becomes quite complicated.

(Rmk) In pf, could have compared N to $\|\cdot\|_1$ & then shown that equivalence is transitive.

Example | All L^p-norms on $\mathbb{R}^n$ – compare unit balls



$$B_{L^1}(1) \subset B_{L^\infty}(1) \subset B_{L^1}(n)$$

↑ as $n \to \infty$, lose equivalence

$\mathcal{L}(V) = \{\text{linear maps } V \to V\}$ vector space

norm on V $\rightsquigarrow$ operator norm on $\mathcal{L}(V)$

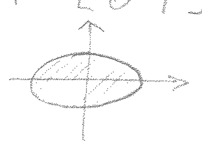
defined by $\|A\| = \max\{|Ax| : |x| \le 1\}$

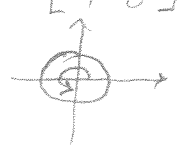
(often write $|\cdot|$ for norm on V & $\|\cdot\|$ for operator norm)

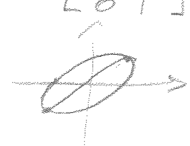
Properties
① $\|Ax\| \le \|A\| \cdot |x| \quad \forall\, x \in V,\ A \in \mathcal{L}(V)$
② $\|AB\| \le \|A\| \cdot \|B\| \quad \forall\, A, B \in \mathcal{L}(V)$
③ $\|A^m\| \le \|A\|^m \quad \forall\, m \in \mathbb{N}$

- Exercise: prove $\|\cdot\|$ is a norm on $\mathcal{L}(V)$.
- Exercise: $\mathcal{L}(V)$ is complete wrt $\|\cdot\|$
- Exercise: operator norm does not come from an inner product on $\mathcal{L}(V)$ - use parallelogram law
$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$
- Exercise: $\langle A, B\rangle = \mathrm{Tr}(A^T B)$ defines an inner product, & hence norm, on $\mathcal{L}(V)$.

Example $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$







$\|A\| = 2$ $\|B\| = 1$ $\|C\| = ?$

$\max\{|(x+y, y)| : |(x,y)| = 1\} = \frac{1+\sqrt{5}}{2}$ in Euclidean norm, or 2 in $\|\cdot\|_1$ & $\|\cdot\|_\infty$.

Def'n The matrix exponential e^A of $A \in M_{n\times n}$ is
$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Q Does this make sense? Does it have expected properties?

FOLLOWING HIRSCH-SMALE

recall distinction between absolute & conditional convergence $\longrightarrow$ (series converges absolutely)

Thm For all $A \in M_{n\times n}$, the limit $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!} = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{A^k}{k!}$ exists in $M_{n\times n}$

Pf Let $S_m = \sum_{k=0}^{m} \frac{A^k}{k!}$, then for $m > l$,

$$\|S_m - S_l\| = \left\|\sum_{k=l+1}^{m} \frac{A^k}{k!}\right\| \le \sum_{k=l+1}^{m} \frac{1}{k!}\|A^k\|$$
$$\le \sum_{k=l+1}^{m} \frac{1}{k!}\|A\|^k$$

Given $\|A\|$, $\varepsilon > 0$, $\exists M$ s.t. $m > l \ge M \Rightarrow \sum_{k=l+1}^{m} \frac{1}{k!}\|A\|^k < \varepsilon$

$\therefore \|S_m - S_l\| < \varepsilon$ $\therefore$ Cauchy. $M_{n\times n}$ is complete ▨

Rmk This also shows that $\|e^A\| \le e^{\|A\|}$

Rmk Same procedure works for any Taylor series as long as $\|A\| <$ radius of convergence.

Exercise: Show that $e^A = \lim_{m\to\infty}\left(I + \frac{A}{m}\right)^m$

LECTURE 3

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What about multiplying?

Lemma If $A = \sum_{j\ge0} A_j$ & $B = \sum_{k\ge0} B_k$ are absolutely convergent in $M_{n\times n}$, then $AB = C = \sum_{l\ge0} C_l$, where $C_l = \sum_{j+k=l} A_j B_k$

Pf Write $\alpha_m = \sum_{j=0}^{m} A_j$, $\beta_m = \sum_{k=0}^{m} B_k$, $\gamma_m = \sum_{l=0}^{m} C_l$

Then $AB = \lim \alpha_m\beta_m$ & $C = \lim \gamma_{2m}$, so suffices to estimate $\gamma_{2m} - \alpha_m\beta_m = \sum' A_j B_k + \sum'' A_j B_k$

$\sum'$: $j+k \le 2m$, $0 \le j \le m$, $m < k \le 2m$

$\sum''$: $j+k \le 2m$, $m < j \le 2m$, $0 \le k \le m$

$$\left\|\sum_m' A_j B_k\right\| \le \sum_m' \|A_j B_k\| \le \sum_m' \|A_j\|\cdot\|B_k\| \le \left(\sum_{j=0}^{m}\|A_j\|\right)\left(\sum_{k=m+1}^{2m}\|B_k\|\right) \xrightarrow{m\to\infty} 0$$

& similarly for $\sum''$. ▨

Prop $A, B, P \in M_{n\times n}$, then ① $C = PAP^{-1} \Rightarrow e^C = Pe^AP^{-1}$

② $AB = BA \Rightarrow e^{A+B} = e^Ae^B$ ③ $e^{-A} = (e^A)^{-1}$

④ $n=2$ & $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \Rightarrow e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$

Rmk $e^{\mathrm{diag}(\lambda_1,\dots,\lambda_n)} = \mathrm{diag}(e^{\lambda_1},\dots,e^{\lambda_n})$

Pf ① $C_m = P A_m P^{-1}$, ~~take~~ limit

② binomial ~~thm~~ + ~~commuting~~ $\Rightarrow (A+B)^n = n! \sum_{j+k=n} \frac{A^j}{j!}\frac{B^k}{k!}$

$\therefore\ e^{A+B} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{A^j}{j!}\frac{B^k}{k!}$, apply lemma.

③ A & $-A$ commute

④ $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $J = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$ $IJ = JI \Rightarrow e^{aI+bJ} = e^{aI}e^{bJ}$

$e^{aI} = e^a I$ $e^{bJ} = \sum \frac{b^k}{k!} J^k$ $\boxed{J^2 = -I}$

cf. ¢ $\Rightarrow e^{bJ} = I + bJ - \frac{b^2}{2} I - \frac{b^3}{6} J + \cdots$

$= (\cos b) I + (\sin b) J = \begin{bmatrix}\cos b & -\sin b\\ \sin b & \cos b\end{bmatrix}$

NB e^A always invertible, regardless of A.

NB In general $e^{A+B} \neq e^A e^B$ (cf. Campbell-Hausdorff)

$\left(e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]-[B,[A,B]])+\cdots}\right)$

* So e^{tA} makes sense. Does it help us solve the ODE $\dot{x} = Ax$?

Prop $\frac{d}{dt} e^{tA} = Ae^{tA}$ (in $M_{n\times n}$)

Pf $\frac{d}{dt} e^{tA} = \lim_{h\to 0} \frac{1}{h}\left(e^{(t+h)A} - e^{tA}\right) = \left[\lim_{h\to 0} \frac{1}{h}\left(e^{hA} - I\right)\right]\cdot e^{tA}$

$= \left[\lim_{h\to 0} \sum_{k=1}^{\infty} \frac{1}{h}\cdot\frac{h^k A^k}{k!}\right] e^{tA} = A e^{tA}$ ∎

Thm $A \in M_{n\times n} \Rightarrow x(t) = e^{tA} x_0$ is unique soln to $\dot{x} = Ax$, $x(0) = x_0$.

Pf $\frac{d}{dt}(e^{tA} x_0) = Ae^{tA} x_0$ & $e^{0\cdot A} = I$. For uniqueness,

let $x(t)$ be any soln, ~~then~~ let $y(t) = e^{-tA} x(t)$, ~~so~~

$\dot{y} = -Ae^{-tA} x(t) + e^{-tA}\dot{x}(t) = -e^{-tA}Ax + e^{-tA}Ax = 0$ ∎

Lem If $v \in ¢^n$ is an eigvec of A w/ eigval λ, ~~then~~ v is an eigvec of e^A w/ e.v. e^λ

Pf $e^A v = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} A^k v = \lim \sum_{k=0}^{m} \frac{1}{k!} \lambda^k v = (e^\lambda) v$ ∎

Example $\dot{x} = Ax, \quad A = \begin{pmatrix} -3 & 2 \\ -5 & 4 \end{pmatrix}$

If we can write $A = PDP^{-1}$, then $e^{At} = Pe^{Dt}P^{-1}$ (D diag)

So diagonalise A: c.p. is $\lambda^2 - (\mathrm{Tr}\,A)\lambda + \det A = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda +$

$\therefore \lambda = -1, 2$ are eigvals. $A - \lambda I = \begin{pmatrix} -2 & 2 \\ -5 & 5 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} -5 & 2 \\ -5 & 2 \end{pmatrix} \to \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ } eigvecs

$\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}: e_i \mapsto v_i \qquad \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix} \frac{1}{3} \qquad P = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$

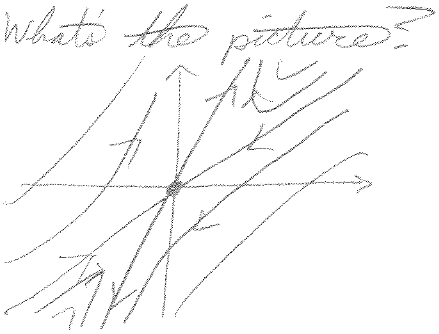
$$e^{At} = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 5e^{-t} & -2e^{-t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5e^{-t} - 2e^{2t} & -2e^{-t} + 2e^{2t} \\ 5e^{-t} - 5e^{2t} & -2e^{-t} + 5e^{2t} \end{pmatrix}$$

Thus soln to $\dot{x} = Ax$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$x(t) = \left(\tfrac{5}{3} e^{-t} - \tfrac{2}{3} e^{2t}, \ \tfrac{5}{3} e^{-t} - \tfrac{5}{3} e^{2t} \right)$$

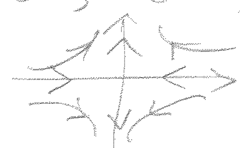
What's the picture? $\lambda_1 = -1$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\lambda_2 = 2$, $v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$



$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x(t) = (e^{-t}, e^{-t})$

$x(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow x(t) = (2e^{2t}, 5e^{2t})$

NB $\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \longrightarrow$



$\boxed{\text{FACT}}$ $\forall A \in M_{2\times 2}\ \exists P \in GL(2, \mathbb{R})$ s.t. $PAP^{-1} = J$ has one of the following forms:

$\boxed{\mathrm{I}}$ $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$ $\boxed{\mathrm{II}}$ $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$ $\boxed{\mathrm{III}}$ $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $\lambda \in \mathbb{R}$

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LECTURE 4

$\boxed{\mathrm{I}}$ eigvals $\lambda, \mu \in \mathbb{R}$, 2 eigvecs

$w, \bar{w} \in \mathbb{C}^2$ eigvecs, let $v_1 = w + \bar{w}$, $v_2 = i(w - \bar{w})$

$\boxed{\mathrm{II}}$ eigvals $a \pm ib \in \mathbb{C}$

$Av_1 = Aw + A\bar{w} = \lambda w + \bar{\lambda}\bar{w} = (a+ib)w + (a-ib)\bar{w} = a v_1 + b v_2 \ \ldots$

$\boxed{\mathrm{III}}$ only 1 eigvec: ~~then~~ $A - \lambda I$ has rank 1, Range$(A - \lambda I)$ is A-inv

$\therefore$ ~~contains~~ eigvec $\therefore \exists\, w$ s.t. $(A - \lambda I)w = v$. $P = [v\ w]$

How does ODE behave in each case? Change vars to $y = Px$, so that $\dot{y} = P\dot{x} = PAx = PAP^{-1}y = Jy$

I $\quad J = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix} \Rightarrow e^{Jt} = \begin{pmatrix} e^{\lambda t} & \\ & e^{\mu t} \end{pmatrix}$

$y(t) = (e^{\lambda t} y_1(0),\ e^{\mu t} y_2(0))$

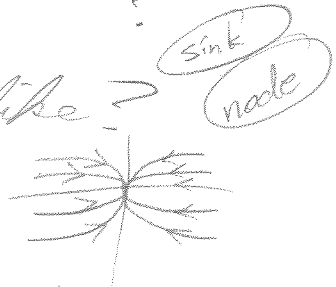
($\lambda, \mu = 0$ is quite boring)

① $\lambda < 0 < \mu$ ② $\lambda = \mu < 0$ ③ ($\lambda = \mu > 0$) ④ $\lambda < \mu < 0$

?

In ④, solns $\to 0$, but what do they look like?

$\lambda < \mu \Rightarrow$ faster decay in horizontal direction



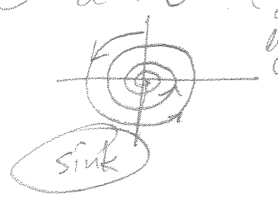
$*$ $x = e^{\lambda t} x_0,\ y = e^{\mu t} y_0 \Rightarrow x^{\mu} y^{-\lambda} = x_0^{\mu} y_0^{-\lambda}$ is constant.

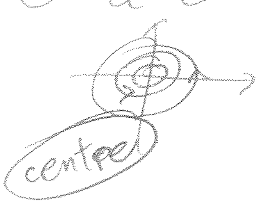
In particular, solutions lie on curves $\boxed{y = C x^{\mu/\lambda}}$

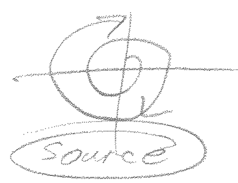
Note difference between $|\mu| < |\lambda|$, $\mu = \lambda$, $|\mu| > |\lambda|$.

II $\quad J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Rightarrow e^{Jt} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$ $\quad (\lambda = a \pm ib)$

① $a < 0$ (can reverse direction by swapping basis vectors) ② $a = 0$ ③ $a > 0$







III $\quad J = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} = S + N \qquad S = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \qquad N = \begin{pmatrix} & 1 \\ & \end{pmatrix}$

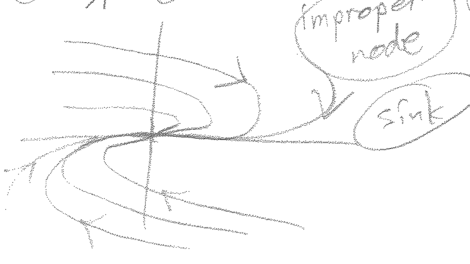
$SN = NS, \quad N^2 = 0$

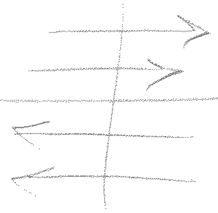
$e^{Jt} = e^{St} e^{Nt} = \begin{pmatrix} e^{\lambda t} & \\ & e^{\lambda t} \end{pmatrix} \left(I + Nt + (Nt)^2 + \cdots \right)$

$= \begin{pmatrix} e^{\lambda t} & \\ & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ & e^{\lambda t} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$\boxed{x = Cy + \frac{1}{\lambda} y \log|y|}$

① $\lambda < 0$ ② $\lambda = 0$ ③ $\lambda > 0$





In original coords, axes move:

 etc.

$\boxed{\text{Rmk}}$ $\text{Re}\,\lambda < 0 \Rightarrow$ sink $\quad \text{Re}\,\lambda > 0 \Rightarrow$ source

$\lambda < 0 < \mu \Rightarrow$ saddle $\quad \text{Re}\,\lambda = 0 \Rightarrow$ centre

Define a 1-parameter family of maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ by $\varphi_t(x) = e^{tA}x$. This is a flow since $\varphi_{s+t} = \varphi_s \circ \varphi_t \ \forall s,t$ (in fact, a linear flow) - the flow generated by $\dot{x} = Ax$.

$\boxed{\text{GOAL}}$ Describe φ_t qualitatively $\forall n$, $A \in M_{n\times n}$, by generalising $\boxed{\text{I}}, \boxed{\text{II}}, \boxed{\text{III}}$ for $\mathbb{R}^2$ above. The key ~~tool~~ will be Jordan normal form.

Brief digression: Higher order ODEs

$\ddot{x} + a\dot{x} + bx = 0$, let $y_1 = x$, $y_2 = \dot{x}$

$\therefore \dot{y}_1 = y_2$, $\dot{y}_2 = -ay_2 - by_1 \Rightarrow \dot{y} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} y$

More generally, if $x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0$, let $y_j = x^{(j-1)}$, $1 \le j \le n$, & get $\dot{y} = Ay$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$\leftarrow$ companion matrix, diagonalised by Vandermonde

$\boxed{\text{Lemma}}$ $\text{c.p.}(A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$

Pf induction, or directly by examining permutations

$\boxed{\text{Fact}}$ If roots are $\underbrace{\lambda_1, \ldots, \lambda_n}_{\text{distinct}}$ then $P = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$ has $\det P = \prod_{i<j} (\lambda_j - \lambda_i)$ &

$PAP^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_n)$

2nd order linear homogeneous & constant coefficients:

$\ddot{x} + a\dot{x} + b \rightsquigarrow \lambda^2 + a\lambda + b = 0$

$\boxed{\text{I}}$ $\lambda_1 \neq \lambda_2$, real, $\begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix} = P\begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}P^{-1}$

$\Rightarrow y(t) = P\begin{bmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{bmatrix}P^{-1}\begin{bmatrix}y_1(0)\\ y_2(0)\end{bmatrix}$

$\Rightarrow$ all solutions $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ for some $c_1, c_2 \in \mathbb{R}$.

$\boxed{\text{II}}$ $\lambda = \alpha \pm i\beta$ $\begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix} = P\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}P^{-1}$

$\Rightarrow y(t) = P e^{\alpha t}\begin{bmatrix}\cos\beta t & -\sin\beta t\\ \sin\beta t & \cos\beta t\end{bmatrix}P^{-1}\begin{bmatrix}y_1(0)\\ y_2(0)\end{bmatrix}$

$\Rightarrow x(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t$

$\boxed{\text{III}}$ $\lambda_1 = \lambda_2 = \lambda \Rightarrow \begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\lambda^2 & 2\lambda\end{bmatrix} = \begin{bmatrix}1 & 0\\ \lambda & 1\end{bmatrix}\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}\begin{bmatrix}1 & 0\\ -\lambda & 1\end{bmatrix}$

$\Rightarrow y(t) = \begin{bmatrix}1 & 0\\ \lambda & 1\end{bmatrix} e^{\lambda t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\lambda & 1\end{bmatrix} y(0)$

$\Rightarrow x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

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So solution space spanned by $e^{\lambda t}$, $e^{\alpha t}\cos\beta t$, $e^{\alpha t}\sin\beta t$, $te^{\lambda t}$ & constants determined by initial conditions

$\boxed{\text{Example}}$ $\ddot{x} - 3\dot{x} + 2x = 0 \Rightarrow (\lambda - 2)(\lambda - 1) = 0$

$\Rightarrow x(t) = c_1 e^{t} + c_2 e^{2t}$

$x(0) = 1$, $\dot{x}(0) = -1 \Rightarrow c_1 + c_2 = 1$, $c_1 + 2c_2 = -1$

$\Rightarrow c_2 = -2$, $c_1 = 3$, so $x(t) = 3e^{t} - 2e^{2t}$

Complexification

$A \in M_{n\times n}$ acts on $\mathbb{C}^n$ in the natural way, and every eigval has an eigvec. Not true over $\mathbb{R}$ since $A - \lambda I$ does not act on $\mathbb{R}^n$. So how to put in a nice form over $\mathbb{R}^n$?