$\boxed{\text{Lemma}}$ If $v \in \mathbb{C}^n$ is an eigvec of $A \in M_{n\times n}(\mathbb{R})$ with eigval λ, then $\bar{v}$ is an eigvec with eigval $\bar{\lambda}$.

Pf $Av = \lambda v \Rightarrow \overline{Av} = \overline{\lambda v}$, & $\bar{A} = A$. ∎

Let $w_1 = Re(v)$, $w_2 = Im(v)$, so $v = w_1 + iw_2$ & $\bar{v} = w_1 - iw_2$. Then $w_1 = \frac{1}{2}(v + \bar{v})$

$$w_2 = \frac{1}{2i}(v - \bar{v})$$

Write $\lambda = \alpha + i\beta$, so $\bar{\lambda} = \alpha - i\beta$. Then

$$\begin{aligned} Aw_1 &= \tfrac{1}{2}(\lambda v + \bar{\lambda}\bar{v}) = \tfrac{1}{2}((\alpha + i\beta)(w_1 + iw_2) + (\alpha - i\beta)(w_1 - iw_2)) \\ &= \tfrac{1}{2}(\alpha w_1 - \beta w_2 + i(\beta w_1 + \alpha w_2) + \alpha w_1 - \beta w_2 - i(\beta w_1 + \alpha w_2)) \\ &= \alpha w_1 - \beta w_2 \end{aligned}$$

& similarly $Aw_2 = \beta w_1 + \alpha w_2$. Thus with $P = [w_2\ w_1 \ldots]$ invertible, we get $P^{-1}AP = \left[\begin{array}{c|c} \begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix} & * \\ \hline 0 & * \end{array}\right]$

$\boxed{\text{Rmk}}$ $w_1 = w_2 \Rightarrow Aw_1 = (\alpha - \beta)w_1 = (\alpha + \beta)w_1 \Rightarrow \beta = 0$ so $w_1 \neq w_2$ as long as $\lambda \notin \mathbb{R}$.

With $Q = [v\ \bar{v} \ldots]$, $Q^{-1}AQ = \left[\begin{array}{c|c} \begin{smallmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{smallmatrix} & * \\ \hline 0 & * \end{array}\right]$

$\boxed{\text{Def'n}}$ $A \in M_{n\times n}$ is <u>semisimple</u> if it can be diagonalised <u>over $\mathbb{C}^n$</u>, and <u>nilpotent</u> if $A^k = 0$ for some $k \in \mathbb{N}$. $T \in \mathcal{L}(\mathbb{R}^n)$ is semisimple (nilpotent) if its matrix is.

$\boxed{\text{Exercise}}$ T semisimple $\Leftrightarrow$ $\forall$ subspaces $V \subset \mathbb{R}^n$ $(\mathbb{C}^n)$ s.t. $TV \subset V$ $\exists$ subspace $W \subset \mathbb{R}^n$ s.t. $TW \subset W$ & $\mathbb{R}^n = V \oplus W$

$\boxed{\text{Thm}}$ If c.p.(T) has n distinct roots $(T \in \mathcal{L}(\mathbb{R}^n))$, then T is semisimple.

Pf $\lambda_1, \dots, \lambda_n \longrightarrow v_1, \dots, v_n \in \mathbb{C}^n \rightarrow P = [v_1 \cdots v_n]$ ∎

Rmk $\{\text{semisimple}\} \cap \{\text{nilpotent}\} = \{0\}$

Example $\begin{bmatrix} 0 & 1 & \\ & \ddots & \\ & & 0 \end{bmatrix}$ nilpotent. Is this all?

Example $\dot{x} = Ax, \quad A = \left[\begin{array}{cc|cc} 3 & -2 & & \\ 1 & 1 & & \\ \hline & & 1 & -5 \\ & & 1 & -1 \end{array}\right]$

save for Thursday

c.p.: $(\lambda^2 - 4\lambda + 5)(\lambda^2 + 4) = 0 \quad \Longrightarrow \quad \lambda = 2 \pm i, \ \pm 2i$

$\lambda_1 = 2+i, \quad \bar{\lambda}_1 = 2-i \qquad \lambda_2 = 2i \quad \bar{\lambda}_2 = -2i$

$A - \lambda_1 I = \left[\begin{array}{cc|c} 1-i & -2 & \\ 1 & -1-i & \\ \hline & & * \end{array}\right] \qquad v_1 = \begin{bmatrix} 1+i \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \bar{v}_1 = \begin{bmatrix} 1-i \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$A - \lambda_2 I = \left[\begin{array}{c|cc} * & & \\ \hline & 1-2i & -5 \\ & 1 & -1-2i \end{array}\right] \qquad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1+2i \\ 1 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1-2i \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1+i & 1-i & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1+2i & 1-2i \\ 0 & 0 & 1 & 1 \end{bmatrix} \Longrightarrow P^{-1}AP = \begin{bmatrix} 2+i & & & \\ & 2-i & & \\ & & 2i & \\ & & & -2i \end{bmatrix}$

What about $\mathbb{R}$?

$w_1 = Im(v_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad w_2 = Re(v_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad w_3 = Im(v_2) = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \quad w_4 = Re(v_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$Q = \begin{bmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 2 & 1 \\ & & 0 & 1 \end{bmatrix} \Longrightarrow Q^{-1}AQ = \left[\begin{array}{cc|cc} 2 & -1 & & \\ 1 & 2 & & \\ \hline & & 0 & -2 \\ & & 2 & 0 \end{array}\right]$

$$e^{At} = Q \left[\begin{array}{cc|cc} e^{2t}\cos t & -e^{2t}\sin t & & \\ e^{2t}\sin t & e^{2t}\cos t & & \\ \hline & & \cos 2t & -\sin 2t \\ & & \sin 2t & \cos 2t \end{array}\right] Q^{-1}$$

$\boxed{\text{Thm}}$ Let $A \in M_{n\times n}(\mathbb{R})$ have distinct real eigvals λ_j (eigvecs w_j) & distinct complex eigvals $\mu_j = a_j + ib_j$ (eigvecs $x_j = u_j + iv_j$). Then $P = [w_1 \cdots w_k | v_{k+1} u_{k+1} \cdots v_{k+m} u_{k+m}]$ is invertible &

$P^{-1}AP = diag(\lambda_1, \dots, \lambda_k, B_{k+1}, \dots, B_{k+m}), \qquad B_j = \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$

Rmk Works as long as A is semisimple (enough eigvecs)

Not every matrix is semisimple. What if (alg mult) > (geom mult)?

$\boxed{\text{Def'n}}$ $v \in \mathbb{C}^n$ is a generalised eigenvector of $A \in M_{n\times n}$ for ~~the~~ eigval λ if $(A-\lambda I)^k v = 0$ for some $k \in \mathbb{N}$.

$\{$gen eigvec$\}$ = generalised eigenspace. Jordan chain $= \lambda_k \xrightarrow{A-\lambda I} \lambda_{k-1} \mapsto \cdots \mapsto \lambda_1 \mapsto$

Rmk k minimal $\Rightarrow (A-\lambda I)^{k-1} v$ an eigvec $\therefore \lambda$ an eigval

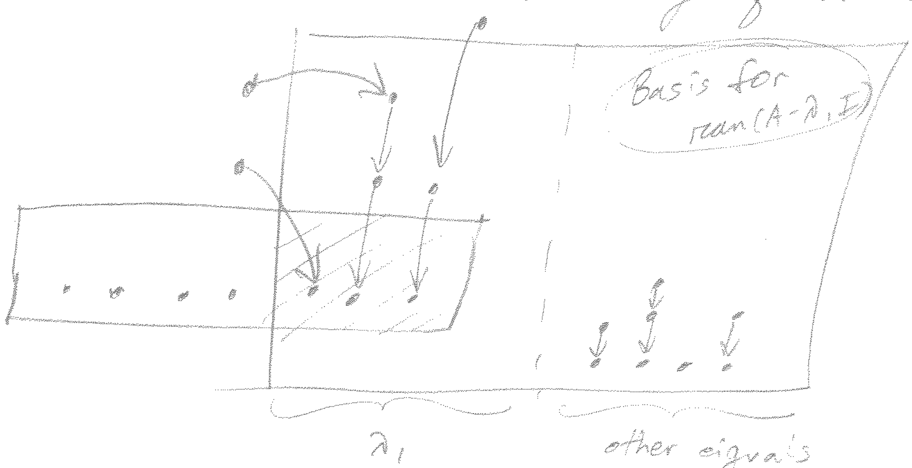
Rmk eigsp $= \ker(A-\lambda I)$, gen eigsp $= \ker(A-\lambda I)^n$

*Why n suff? If k minimal then $\{v, (A-\lambda I)v, \ldots, (A-\lambda I)^{k-1}v\}$ are lin ind. ← EXERCISE, uses fact ~~that~~ $k(v+w) \le \max(k(v), k(w))$

$\boxed{\text{Thm}}$ If $A \in M_{n\times n}(\mathbb{C})$ has eigvals $\lambda_1, \ldots, \lambda_r$, with gen eigspaces $E(\lambda_i)$, ~~then~~ $\mathbb{C}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_r)$

(THERE ARE ENOUGH GENERALISED EIGENVECTORS!)

Pf Use induction. Note ~~that~~ ker & range of $A - \lambda_j I$ are A-invariant



sort into Jordan chains $v \mapsto (A-\lambda_j I)v \mapsto \cdots \mapsto 0$

By inductive hyp, $\operatorname{ran}(A-\lambda_1 I)$ has a basis of GEV. (since A-inv.).

Also $\dim \operatorname{ran} + \dim \ker = n$, so # extra vectors to make basis of $\mathbb{C}^n$ is $\dim(\operatorname{ran} \cap \ker)$. Basis for $\operatorname{ran} \cap \ker \to$ ▭ in picture

$\to$ each chain can be extended outside ran. ■

LECTURE 6

9/13

start with ✱ (p. 16)

Example c.p. $A = (x-\lambda)^n$, but only one eigvec, v_1.

v_2 s.t. $(A-\lambda I)v_2 = v_1$, v_k s.t. $(A-\lambda I)v_k = v_{k-1}$ cf #2(b)

Then $\{v_1, \dots, v_n\}$ a basis (LI by earlier argument)

& with $P = [v_1 \cdots v_n]$ get $P^{-1}AP = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda \end{bmatrix}$ ← Jordan block

$= \lambda I + N$

NB $N^n = 0$, ~~so~~ N nilpotent, & $e^N = \sum_{k=0}^{n-1} \frac{N^k}{k!}$

$$e^{Nt} = \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ & & & 1 & t \\ & & & & 1 \end{bmatrix}$$

This is the last piece we need to compute arbitrary matrix exponentials: $e^{\oplus B_i} = \oplus e^{B_i} = \oplus e^{\lambda_i} e^{N_{n_i}}$

Given $A \in M_{n\times n}(\mathbb{C})$, let $\{v_1, \dots, v_n\}$ be a basis for $\mathbb{C}^n$ of gen eigvecs for A, w/ eigvals $\underbrace{\lambda_1, \dots, \lambda_n}$. Let $P = [v_1 \cdots v_n]$, ~~then~~ ↖ repetition allowed

$$P^{-1}AP = \operatorname{diag}\{B_1, \dots, B_r\} \qquad B_i = \lambda_i I_{n_i} + N_{n_i}$$

↖ # gen eigvec in chain

$$= \underbrace{\operatorname{diag}(\lambda_1 I_{n_1}, \dots, \lambda_r I_{n_r})}_{S'} + N'$$

$N' = \sum_{i=1}^{r} N_{n_i}$, all nonzero entries immediately above main diag $\therefore$ nilpotent

$$A = \underbrace{PS'P^{-1}}_{=S} + \underbrace{PN'P^{-1}}_{=N}$$

↖ semisimple ↖ nilpotent

EXERCISE $SN = NS$

Now $e^A = e^S e^N$, & both e^S, e^N are easy to compute.

✱ If $A \in M_{n\times n}(\mathbb{R})$, can find $A = S + N$, $S, N \in M_{n\times n}(\mathbb{R})$ by using real, imaginary parts of complex gen eigvecs & working with 2×2 blocks.

<u>real</u> JNF if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is

$$\begin{bmatrix} R_\lambda & I & & \\ & R_\lambda & I & \\ & & R_\lambda & I \\ & & & R_\lambda \end{bmatrix}$$

2×2 blocks

<u>Example</u>

$$A = \begin{pmatrix} 3 & 0 & 0 & -2 \\ 1 & 3 & -2 & -2 \\ -2 & 2 & 3 & 1 \\ 2 & 0 & 0 & 3 \end{pmatrix} \qquad \det(A - \lambda I) = \left((3-\lambda)^2 + 4\right)^2$$

$\Rightarrow$ eigvals are $3 \pm 2i$

One eigvec is $\begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}$, no others, but $(A - \lambda I)\begin{pmatrix} i \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}$

so $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ has $P^{-1}AP = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$

Can compute S, N from here, get e^{At}

$$e^{At} = P e^{3t} \begin{bmatrix} \cos 2t & -\sin 2t & & \\ \sin 2t & \cos 2t & & \\ & & \cos 2t & -\sin 2t \\ & & \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}$$

① ②

& e.g. from p. 13

<u>Do before p. 15</u> | $T \in \mathcal{L}(\mathbb{C}^n)$

$$v_k \xrightarrow{T-\lambda I} v_{k-1} \xrightarrow{T-\lambda I} \cdots \xrightarrow{T-\lambda I} v_1 \xrightarrow{T-\lambda I} 0 \quad \text{a Jordan chain,}$$

cyclic subspace

$E = \operatorname{span}\{v_1, \ldots, v_k\} \subset G_\lambda$ = gen eigsp = $\ker(A - \lambda I)^n$

In the basis $\{v_j\}$, find $[T|_E]$:

$$T v_1 = \lambda v_1, \quad T v_j = \lambda v_j + v_{j-1} \Rightarrow [T|_E] = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} =: B_{\lambda,k}$$

(reversing order gives lower Jordan block)

① Let $\lambda_1, \ldots, \lambda_r$ be all the ~~distinct~~ eigvals of T

② Let $l_i = \dim \ker(T - \lambda_i I)$ = # of lin ind eigvecs for λ_i

③ Denote Jordan chains for λ_i by $(1 \le j \le l_i)$

$$v^{i,j}_{k_j} \longrightarrow v^{i,j}_{k_j - 1} \longrightarrow \cdots \longrightarrow v^{i,j}_1 \longrightarrow 0$$

NB May not work $\forall$ choices of $v_1^{i,j}$! eg. $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

④ $E_i^j = \operatorname{span}\{v_1^{i,j}, \ldots, v_{k_j}^{i,j}\}$, $\quad G_{\lambda_i} = \bigoplus_j E_i^j$

By <u>Thm</u> from last time, $\mathbb{C}^n = \bigoplus_{i,j} E_i^j = \bigoplus_i G_{\lambda_i}$

$\dim G_{\lambda_i}$ = <u>alg</u> mult of λ_i. $\quad T = \bigoplus_{i,j} B_{\lambda_i, k_j}$ ← JNF

* Back to p. 15

Cor Cayley-Hamilton Thm: $p(A)=0$

Pf $B_{\lambda,k}$ a Jordan block $\Rightarrow (B_{\lambda,k}-\lambda I)^k = 0$. ■

Rmk numerically unstable since generic matrices have distinct eigvals. Most important as theoretical tool - although sometimes symmetries force a non-generic situation.

Cor Trace $= \Sigma$ eigvals Det $= \Pi$ eigvals

Cor $\det e^A = e^{\operatorname{Tr} A}$

Cor A & A^t are conjugate

LECTURE 7

9/18

Example $\dot{x} = Ax$ $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

c.p. $= (\lambda-1)(\lambda-2)^2$

$\lambda = 1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$\lambda = 2 \rightarrow \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, but no v_3, so

$\begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$e^{At} = P \begin{bmatrix} e^t & & \\ & e^{2t} & te^{2t} \\ & & e^{2t} \end{bmatrix} P^{-1}$

<u>Alternately</u>, use $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, $S' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

to get $A = S + N = P(S'+N')P^{-1} = PS'P^{-1} + N$

$\Rightarrow N = A - S = A - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow e^{Nt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & t & 1 \end{bmatrix}$

$\Rightarrow x(t) = P \begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{2t} \end{bmatrix} P^{-1} \begin{bmatrix} 1 & & \\ & 1 & \\ -t & t & 1 \end{bmatrix} x_0$

$= \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix} x_0$

General higher-order equations — $x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0$

FACT Only 1 Jordan block for each eigval

Pf $\begin{bmatrix} -\lambda & 1 & & \\ & -\lambda & 1 & \\ & & -\lambda & \ddots & 1 \\ -a_n & \cdots & & \lambda - a_1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix} = 0 \Rightarrow \boxed{\begin{matrix} \lambda v_1 = v_2 \\ \lambda v_2 = v_3 \\ \cdots \\ \lambda v_{n-1} = v_n \end{matrix}} \Rightarrow$ only 1 eigvec $\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$

$\begin{bmatrix} -\lambda & 1 & & \\ & -\lambda & 1 & \\ & & \ddots & 1 \\ -a_n & \cdots & & \lambda - a_1 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} \Rightarrow$
$v_2 = \lambda v_1 + 1$
$v_3 = \lambda v_2 + \lambda = \lambda(\lambda v_1 + 1) + \lambda = \lambda^2 v_1 + 2\lambda$
$\cdots$
$-a_n v_1 - a_{n-1} v_2 - \cdots - a_1 v_n - \lambda v_n = \lambda^{n-1}$

$v_4 = \lambda v_3 + \lambda^2 = \lambda^3 v_1 + 3\lambda^2$

$v_k = \lambda^{k-1} v_1 + (k-1)\lambda^{k-2}$

$\lambda^{n-1} + \lambda v_n + a_1 v_n + a_2 v_{n-1} + \cdots + a_{n-1} v_2 + a_n v_1 = 0$

$\lambda^{n-1} + \lambda(\lambda^{n-1} v_1 + (n-1)\lambda^{n-2}) + a_1(\lambda^{n-1} v_1 + (n-1)\lambda^{n-2})$
$\quad + a_2(\lambda^{n-2} v_1 + (n-2)\lambda^{n-3}) + \cdots + a_n v_1$
$= n\lambda^{n-1} + a_1(n-1)\lambda^{n-2} + \cdots + a_{n-2}(2\lambda) + a_{n-1} = 0$

$v_1 = 0 \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 2\lambda \\ 3\lambda^2 \\ \cdots \\ (n-1)\lambda^{n-2} \end{bmatrix}$ etc. $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 6\lambda \\ \vdots \\ \vdots \end{bmatrix} \cdots$ ∎

$\boxed{\text{Thm}}$

If $\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$ has roots $\lambda_1, \ldots, \lambda_r$ & λ_j has multiplicity m_j, then eigvecs of $A = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ -a_n & & \cdots & -a_1 \end{bmatrix}$ are $\{v_{j,k} \mid 1 \le j \le r,\ 0 \le k < m_j\}$, where $(v_{j,k})_i = \left(\frac{d^k}{d\lambda^k}\lambda^{i-1}\right)\Big|_{\lambda=\lambda_j}$ & every soln of $x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0$ is a linear combination of the following functions

$$\{t^k e^{\alpha_j t}\cos\beta_j t,\ t^k e^{\alpha_j t}\sin\beta_j t \mid 1 \le j \le r,\ \lambda_j = \alpha_j + \beta_j i,\ 0 \le k < m_j\}$$

Pf JNF. ∎

Operators on Function Spaces

For an alternate approach to higher-order systems, let $C^\infty = \{f : \mathbb{R} \circlearrowleft \mid f \text{ is infinitely differentiable}\}$, and let $D : C^\infty \circlearrowleft$ be differentiation $Df = f'$.

Then $D \in \mathcal{L}(C^\infty)$ (it is a linear operator) and moreover D is onto. The map $f \mapsto tf$ is also linear: denote it by E.

Given $p \in \mathbb{R}[s]$, $p(s) = \sum_{k=0}^n c_k s^k$, let $p(D) = \sum_{k=0}^n c_k D^k$, where $(D^k f)(t) = f^{(k)}(t)$. (iteration)

Thus solving $f^{(n)} + a_1 f^{(n-1)} + \dots + a_{n-1} f' + a_n f = 0$ is equivalent to finding $\ker p(D)$, $p(s) = s^n + a_1 s^{n-1} + \dots + a_n$

- If $p(x) = q(x) r(x)$ then $\ker q(D) \subset \ker p(D)$ (also for r) – note that all polynomials of D commute.
- If $f \in \ker q(D)$ & $g \in \ker r(D)$ then $f + g \in \ker p(D)$

$\boxed{\text{Prop}}$ $t^k e^{t\lambda} \in \ker p(D) \Longleftarrow (x-\lambda)^{k+1} \mid p(x)$

Pf ($\Leftarrow$) Suffices to show $(D-\lambda)^{k+1}(t^k e^{t\lambda}) = 0$.

$$((DE^k)f)(x) = \frac{d}{dx}(x^k f(x)) = x^{k-1} f(x) + x^k f'(x)$$

$$\Rightarrow DE^k = E^{k-1} + E^k D \Rightarrow [D, E^k] = DE^k - E^k D = E^{k-1}$$

$$\Rightarrow [(D-\lambda), E^k] = E^{k-1}$$

So $(D-\lambda)^{k+1}(t^k e^{t\lambda}) = (D-\lambda)^{k+1} E^k (e^{t\lambda})$

$$= (D-\lambda)^k [(D-\lambda), E^k](e^{t\lambda}) = (D-\lambda)^k E^{k-1}(e^{t\lambda})$$

$$= \dots = 0. \qquad \blacksquare$$

Holds for complex λ too, and then for $\lambda = a + ib$

$t^k e^{t\lambda}, t^k e^{t\bar{\lambda}} \in \ker p(D) \Rightarrow t^k e^{ta} \cos bt,\ t^k e^{at} \sin bt \in \ker p(D)$

- Can also be shown that these are all solutions

What about non-homogeneous equations?

eg. $\ddot{x} + \omega^2 x = \varphi(t)$ ← forced harmonic oscillator

Consider $x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = \varphi(t)$,

turn it into $\boxed{\dot{x} = Ax + B(t)}$, $x \in \mathbb{R}^n$, $A \in M_{n\times n}$, $B: \mathbb{R} \to \mathbb{R}^n$ (*)

Variation of constants: Without $B(t)$ term, solution would be $x(t) = e^{At}x_0$, so in particular $y(t) = e^{-At}x(t)$ is constant. What if $B(t) \neq 0$? Then

$$\begin{aligned}\dot{y} &= -Ae^{-At}x(t) + e^{-At}\dot{x}(t)\\ &= -Ae^{-At}x + e^{-At}Ax + e^{-At}B(t)\\ &= e^{-At}B(t) \qquad \text{(NOTE ORDER!)}\end{aligned}$$

Integrating gives

$$y(t) = \int_0^t e^{-As}B(s)\,ds + \underbrace{y_0}_{=x_0}$$

$$\Rightarrow \boxed{x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}B(s)\,ds} \quad (**)$$

$\boxed{\text{Thm}}$ (**) is the unique solution of (*).

LECTURE 8 9/20

$$(**) = \underbrace{\underbrace{(\text{general solution of } \dot{x} = Ax)}_{\text{linear subspace}} + (\text{particular solution of } (*))}_{\text{affine subspace}}$$

For forced oscillator $y = \binom{x}{\dot{x}/\omega}$, $\dot{y} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} y + \begin{bmatrix} \varphi(t) \\ 0 \end{bmatrix}$

$e^{At} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} \Rightarrow$

$$x(t) = x(0)\cos\omega t + \dot{x}(0)\sin\omega t + \int_0^t \cos(\omega(t-s))\varphi(s)\,ds$$

- $\dot{x} = A(t)x + B(t) \Rightarrow$ solve $\dot{x} = A(t)x$ for $x(t) = e^{\Psi(t)}x_0$, proceed as before. $\boxed{\text{NB}}$ $\frac{d}{dt}(e^{\Psi}) \neq \dot{\Psi}e^{\Psi}$!!

Qualitative behaviour of $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $n > 2$

More possibilities in higher dimensions, but everything is determined by Jordan blocks:

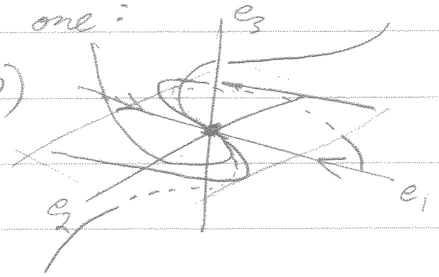
$$A = P(J_1 \oplus \cdots \oplus J_r)P^{-1} \qquad E_j \text{ corresponding subspaces,}$$

$x = (x_1, \dots, x_r)$, then $x_j = e^{J_j t} x_j(0)$

In $\mathbb{R}^3$, could have one Jordan block $\begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$ or multiple

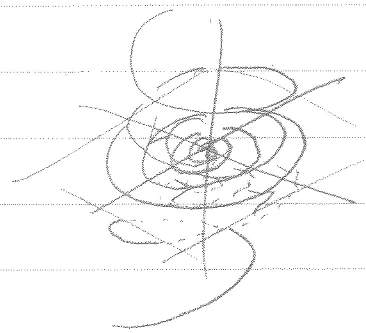
If one:

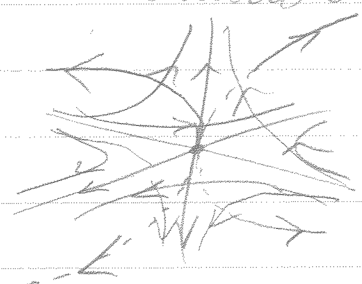
($\lambda < 0$)



$$x = e^{\lambda t}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ & 1 & t \\ & & 1 \end{bmatrix} x_0$$

If multiple, add one independent direction to one of $\mathbb{R}^2$ pictures:





etc.

In $\mathbb{R}^4$ have possibility for Jordan block involving complex eigenvalues (& hence rotations).

General principle: $\mathrm{Re}\,\lambda < 0 \Rightarrow$ sink, $> 0 \Rightarrow$ source

STABILITY THEORY

Given $\dot{x} = Ax$, $A \in M_{n \times n}$, let $\lambda_j = a_j + ib_j$ be the eigvals of A (b_j can be 0) and $E(\lambda_j)$ the corresponding generalised eigenspaces, so

$$\mathbb{R}^n = E(\lambda_1) \oplus \cdots \oplus E(\lambda_r)$$

Define E^s (stable subspace) $= \bigoplus_{a_j<0} E(\lambda_j)$

E^u (unstable subspace) $= \bigoplus_{a_j>0} E(\lambda_j)$

E^c (centre subspace) $= \bigoplus_{a_j=0} E(\lambda_j)$

Then $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$ and $E^{s,c,u}$ are flow invariant.

Def If $E^c = \emptyset$ then $\dot{x} = Ax$ is called hyperbolic. If $E^s = \mathbb{R}^n$ the origin is a sink, if $E^u = \mathbb{R}^n$ it is a source.

eg $\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow E^s = \langle e_1, e_2 \rangle$, $E^u = \langle e_3 \rangle$, $E^c = \emptyset$

LECTURE 9 $\boxed{9/25}$

$\boxed{\text{Thm}}$ $\exists\ m, M, a, c > 0$ s.t.

① $x_0 \in E^s \Rightarrow m e^{-ct}|x_0| \le |e^{At}x_0| \le M e^{-at}|x_0|$

② $x_0 \in E^u \Rightarrow m e^{at}|x_0| \le |e^{At}x_0| \le M e^{ct}|x_0|$

Pf Suffices to prove ①.

Take $\lambda_1 < \lambda_2 < \cdots < \lambda_s < 0$ & let $c > |\lambda_1|$, $a < |\lambda_s|$.

Then $x_0 \in E^s \Rightarrow x_0 = \sum_{j=1}^s y_j$, $y_j \in E(\lambda_j)$

$\Rightarrow e^{At} y_j = e^{\lambda t} e^{N_j t} y_j$

But the entries of $e^{N_j t}$ are of polynomial order

so $\|e^{N_j t}\| \le C_1 e^{\delta t}$ & $\|e^{-N_j t}\| \le C_2 e^{\delta t}$

$\Rightarrow |e^{At} y_j| \le C_1 e^{(\lambda_j + \delta)t} |y_j|$

& $|e^{At} y_j| \ge C_2^{-1} e^{(\lambda_j - \delta)t} |y_j|$

Take δ s.t. $|\lambda_1| + \delta \le c$ & $|\lambda_s| - \delta \ge a$ ■

$\boxed{\text{Rmk}}$ $|x(t)|$ may not be ~~strictly~~ decreasing: could be