Can we describe how $x \to 0$ when 0 is a sink?
IDEA — find $V(x) \in \mathbb{R}$ s.t. $\frac{d}{dt} V(x(t)) < 0 \ \forall \ t$.
Note: $\frac{d}{dt} V(x) = \nabla V \cdot \dot{x} = \nabla V \cdot A x$

What about canonical coords? $A = PJP^{-1}$, $y = P^{-1}x \Rightarrow \dot{y} = Jy$,
$\frac{d}{dt}|y|^2 = y^T(J^T + J)y$, no good if Jordan blocks present

Use quadratic forms: $V(x) = x^T Q x$, Q sym & pos def.
Then $\dot{V} = x^T(A^T Q + QA)x$, so we want $A^T Q + QA$
to be neg def — in particular, try $= -I$.

eg $A = \begin{pmatrix} -1 & 3 \\ -6 & -2 \end{pmatrix} \Rightarrow Q = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$, $V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2$
$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow Q = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$, $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_1x_2 + \frac{3}{4}x_2^2$
& in both cases $\dot{V}(x) = -|x|^2$

Thm If 0 is a sink then $\exists$ symmetric positive definite $Q \in M_{n\times n}$ s.t. $A^T Q + QA = -I$ &
hence $V(x) = x^T Q x$ has $\frac{d}{dt} V = -|x|^2$

Pf Let $B(t) = e^{tA^T} e^{tA}$, so $B(0) = I$ & B is
sym & pos def. Now $\dot{B}(t) = A^T B + BA$
$\Rightarrow B(\tau) - I = A^T \int_0^\tau B(t)\,dt + \left(\int_0^\tau B(t)\,dt\right)A$
Since 0 is a sink the entries of B decay
exponentially & $\int_0^\infty B$ exists, and we get
$Q = \int_0^\infty B(t)\,dt \Rightarrow -I = A^T Q + QA$ $\blacksquare$

LECTURE 10   9/27

NONLINEAR THEORY

Consider $\dot{x} = f(x)$, $\quad x \in \mathbb{R}^n$, $\quad f: \mathbb{R}^n \to \mathbb{R}^n$

When $f(x) = Ax$, $A \in M_{n\times n}$, we have seen that

① solution exists: $x(t)$

② it is unique

③ it is defined $\forall\, t \in \mathbb{R}$

④ we can write down an explicit formula $x(t) = e^{At}x_0$ & compute $x(t)$ using Jordan normal form.

For nonlinear, how much extends?

① existence – as long as f is continuous (Peano)

② uniqueness – if f is Lipschitz (Picard–Lindelöf)

③ defined $\forall\, t$ – if f is bounded

④ explicit solution – hardly ever

$\boxed{\text{Thm}}$ (Peano)

Let $U \subset \mathbb{R}^n$ be open and $f: U \times \mathbb{R} \to \mathbb{R}^n$ be continuous.
Consider the IVP (1) $\dot{x}(t) = f(x(t), t)$, $\quad x(t_0) = x_0 \in U$
Then $\exists\ \tau > 0$ s.t. (1) has a solution $x: (t_0 - \tau, t_0 + \tau) \to U$.

$\boxed{\text{Pf}}$ (sketch)

I. Show that (1) $\Leftrightarrow$ (2) $x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\, ds$

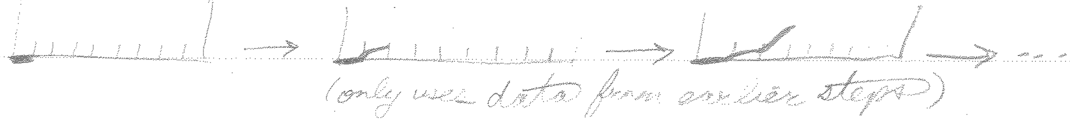
II. Choose τ s.t. no solution of $(*)$ can leave U within time τ.

III. Define $x_m: [t_0, t_0 + \tau)$ by $x_m(t) = x_0 + \int_{t_0}^{t - \tau/m} f(x_m(s), s)\, ds$

IV. Use continuity of f to get $\{x_m\}$ unif. Lipschitz $\therefore$ by Arzela-Ascoli $\exists\ x_{m_k} \to x$

V. Show x solves (2), define on $(t_0 - \tau, t_0]$ via $\dot{y} = -f$ ∎

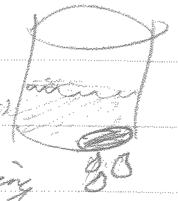
building x_m:



(only uses data from earlier steps)

* *Solutions are not always unique!*

$\boxed{\text{Example}}$ Leaky bucket: $h(t)$ = height of water



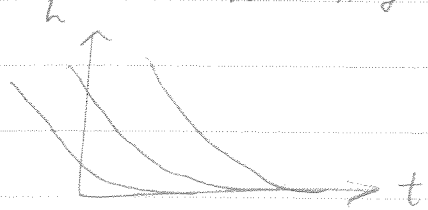
A = area of bucket cross-section

a = hole cross-section $v(t)$ = velocity of water leaving

$Ah' = av$, $mgh = \frac{1}{2}mv^2$ for falling water, (if no energy loss)

(which has $m = (\Delta h)A\rho$), so $v^2 = 2gh$

$\therefore\ h' = -\frac{a}{A}\sqrt{2g}\,\sqrt{h} = -c\sqrt{h}$



Reverse time: $\dot{x} = \sqrt{x}$

multiple solutions!

(If bucket is empty, no way to know when it was full.)

$$x(t) = \begin{cases} 0 & t < t_0 \\ \frac{1}{4}(t-t_0)^2 & t \geq t_0 \end{cases}$$ is a solution $\underline{\forall\, t_0}$.

(Peano's method only gets 0 solution)

More generally $\dot{x} = x^\alpha$, $0 < \alpha < 1$; guess that $x(t) = c(t-t_0)^\beta$ is a solution, get $\dot{x} = c\beta(t-t_0)^{\beta-1} = c^\alpha (t-t_0)^{\alpha\beta}$

$\Rightarrow \beta - 1 = \alpha\beta \Rightarrow \beta = \frac{1}{1-\alpha}$, $c\beta = c^\alpha \Rightarrow c^{1-\alpha} = \frac{1}{\beta} \Rightarrow c = (1-\alpha)^{\frac{1}{1-\alpha}}$

(note: initial condition enters by choice of t_0)

Then $x(0) = 0$ has multiple solutions.

* *Solutions may not exist for all time!*

$\boxed{\text{Example}}$ $\dot{x} = x^{1+\alpha}$, $\alpha > 0$ $\Rightarrow$ as above, $\beta = -\frac{1}{\alpha}$, get

$(x^{-\alpha})' = -\alpha x^{-\alpha-1}\dot{x} = -\alpha \Rightarrow x^{-\alpha} = x_0^{-\alpha} - \alpha t$

$\Rightarrow x(t) = (x_0^{-\alpha} - \alpha t)^{-1/\alpha}$

Now solution is unique (we'll see a pf later), but only defined for $t < \frac{x_0^{-\alpha}}{\alpha}$. Finite time blowup

First we address uniqueness.

$\boxed{\text{Thm}}$ Let $U \subset \mathbb{R}^n$ be open and $f: U \times \mathbb{R} \to \mathbb{R}^n$ be continuous in $t \in \mathbb{R}$, Lipschitz in $x \in U$. Then $\forall\, x_0 \in U$ $\exists\, \tau > 0$ s.t. the IVP (1) has a unique solution $x: (t_0 - \tau, t_0 + \tau) \to U$.

Pf The technique is called Picard iteration – use integral form (2) to produce a sequence of approximate solutions x_m as in Peano's theorem:

(3) $\quad x_{m+1}(t) = x_0 + \int_{t_0}^{t} f(x_m(s), s)\, ds$

$\boxed{\text{EXAMPLE}}$ $\quad \dot{x} = \lambda x,\ t_0 = 0 \Rightarrow x_1(t) = x_0 + \int_0^t \lambda x_0\, ds = x_0(1 + \lambda t)$

$x_2(t) = x_0 + \int_0^t x_1\, ds = x_0 + \int_0^t \lambda x_0 (1 + \lambda s)\, ds = x_0 (1 + \lambda t + \frac{\lambda^2 t^2}{2})$

and so on, building power series for $x(t) = x_0 e^{\lambda t}$

Now choose $B_\varepsilon(x_0) \subset U$, δ & use Lipschitz property to get $K > 0$ s.t. $|f(x,t) - f(y,t)| \le K|x - y| \quad \forall\, x, y \in B_\varepsilon(x_0),\ |t - t_0| < \delta$.

Let $M = \max\{|f(x,t)| : x \in B_\varepsilon(x_0),\ t \in (t_0 - \delta, t_0 + \delta)\}$

Choose $\tau < \frac{\varepsilon}{M}$, $\tau < \frac{1}{K}$, $\tau < \delta$, & define $P: x_m \mapsto x_{m+1}$ by (3). Let $Z = \{x : (t_0 - \tau, t_0 + \tau) \to U \mid x \text{ is cts}\}$

$\boxed{\text{LEMMA}}$ $\quad P: Z \to Z$

Pf $\quad |(Px)(t) - x_0| = \left|\int_{t_0}^t f(x(s), s)\, ds\right| \le \left|\int_{t_0}^t |f(x(s), s)|\, ds\right| \le M|t - t_0|$

$\le M\tau < \varepsilon \Rightarrow (Px)(t) \in B_\varepsilon(x_0) \subset U$.

Continuity follows from definition ∎

$\boxed{\text{LEMMA}}$ $\quad P$ is a contraction in the uniform metric

Pf $\quad \max_t |(Px)(t) - (Py)(t)| \le \max_t \left|\int_{t_0}^t (f(x(s), s) - f(y(s), s))\, ds\right|$

$\le \max_t \left|\int_{t_0}^t K|x(s) - y(s)|\, ds\right| \le K\tau\, d(x, y)$ ∎

Now pf follows from Banach fixed-pt thm. ∎

10/2

LECTURE 11

Banach fixed point theorem

Let (X,d) be a non-empty complete metric space and $T: X \circlearrowleft$ a uniform contraction. Then $\exists!$ fixed point $x^* \in X$. (Moreover, $T^n x \to x^*$ $\forall x \in X$).

Rmk Solution of IVP (1) obtained as limit of x_m. Uniqueness comes because if x, y 2 solutions, then $d(Px, Py) < d(x,y)$ $\lightning$ $(Px = x,\ Py = y)$.

If non-autonomous, let $x_{n+1} = t$ & $F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ $F(x_1, \dots, x_{n+1}) = (f(x_1, \dots, x_n, t), 1)$, then (1) becomes $\dot{x} = F(x)$. Thus we will generally consider autonomous case.

Rmk $f \in C^1 \Rightarrow f \in Lip$: let γ be line from x to y then $|f(x) - f(y)| = |\int_\gamma Df(z)dz| \le \int_\gamma |Df(z)dz| \le \ell(\gamma)\|Df\| = \|Df\||x-y|$

Dependence on initial condition & parameters

How does $x(t)$ vary if we vary x_0 or f? Is it still defined on same interval $(t_0 - \tau, t_0 + \tau)$?

Thm Let $U \subset \mathbb{R}^n$ be open, $x_0 \in U$, & $f \in C^1(U, \mathbb{R}^n)$. Then $\exists\ \tau > 0$ & $\delta > 0$ s.t. $\forall y \in B(x_0, \delta)$ the IVP

$$(*)\quad \dot{x} = f(x), \quad x(0) = y$$

has a unique solution $v(t,y)$ defined on $(-\tau, \tau) \times B(x_0, \delta)$. Moreover, $v(t,y)$ is C^2 in t on $(-\tau, \tau)$ $\forall y \in B(x_0, \delta)$.

Q What is stronger here than in the previous thm?

Pf. We need the following lemma:

$\boxed{\text{Lem}}$ (Gronwall)

If $g : [0,a] \to [0,\infty)$ is cts & $C, k > 0$ are s.t.
$g(t) \le C + k\int_0^t g(s)\,ds \quad \forall\ t \in [0,a]$, ~~then~~
$g(t) \le Ce^{kt} \quad \forall\ t \in [0,a]$.

Pf (easy) Let $h(t) = C + k\int_0^t g(s)\,ds$, ~~then~~ $h(t) \ge g(t)$
& $h > 0$. Also $h' = kg \Rightarrow \frac{h'}{h} = \frac{kg}{h} \le k$
$\Rightarrow \frac{d}{dt}(\log h) \le k \Rightarrow \log h(t) \le kt + \log h(0) = kt + \log C$ ∎

Choose $\varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subset U$.

~~Because $f \in C^1$ $\exists\ \varepsilon > 0$ & $K > 0$ s.t. $|f(x) - f(y)| \le K|x-y|$~~
~~$\forall\ x, y \in B(x_0, \varepsilon)$.~~ Let $N_0 = \bar{B}(x_0, \frac{\varepsilon}{2}) = \{x \mid |x - x_0| \le \frac{\varepsilon}{2}\}$
& $M_0 = \max_{x \in N_0} |f(x)|$, $M_1 = \max_{x \in N_0} \|Df(x)\|$, $\delta = \varepsilon/4$.

Fix $y \in B(x_0, \delta)$ & define $v_0(t,y) = y$,
$v_{k+1}(t,y) = y + \int_0^t f(v_k(s,y))\,ds$ (Picard ~~iteration~~)
for $t \in (-\tau, \tau)$, where $\tau < \min\left(\frac{\varepsilon}{4M_0}, \frac{1}{M_1}\right)$

$\boxed{\text{Lemma}}$ $v_k(t,y) \in N_0 \ \forall\ k \ge 0,\ y \in B(x_0,\delta),\ t \in (-\tau,\tau)$

$\boxed{\text{Pf}}$ $|v_k(t,y) - x| \le |y - x| + \int_0^t |f(v_k(s,y))|\,ds < \frac{\varepsilon}{4} + M_0\tau \le \frac{\varepsilon}{2}$ ∎

- $|v_1(t,y) - v_0(t,y)| \le \tau M_0 < \frac{\varepsilon}{4}$
- $|v_{k+1}(t,y) - v_k(t,y)| \le \left|\int_0^t |f(v_k(s,y)) - f(v_{k-1}(s,y))|\,ds\right| \le \tau M_1 \|v_k - v_{k-1}\| \le (\tau M_1)^k \frac{\varepsilon}{4}$

$\Rightarrow v_k$ is Cauchy in the uniform norm $\Rightarrow v_k \longrightarrow v \in C((-\tau,\tau)\times B(x_0,\delta), N_0)$.

The ~~integrals~~ converge, so $v(t,y) = y + \int_0^t f(v(s,y))\,ds$

By FTC, $\dot{v}(t,y) = f(v(t,y))$. ~~Furthermore~~, f is C^1 $\therefore$
$\ddot{v} = Df(v)\,\dot{v}$, ie., v is C^2 in t.

Uniqueness is as before, using contraction of Picard iteration

What about dependence on y?

Cts in y: If $y_0, y_0+h \in B(x_0,\delta)$, ~~then~~ $\forall\ t\in[0,\tau)$ we have

$$|v(t,y_0+h) - v(t,y_0)| \le |h| + \int_0^t |f(v(s,y_0+h)) - f(v(s,y_0))|\,ds$$

(*) So Gronwall gives $\le |h| e^{M_1 t}$

In fact, we can get differentiability. Consider the matrix ODE

LECTURE 12

10/4

$$\dot{\Phi} = Df(v(t,y_0))\,\Phi$$

RHS is cts in t & linear in Φ $\therefore$ $\exists!$ solution $\Phi(t,y_0)$, the fundamental matrix solution.

Now $f\in C^1 \Rightarrow f(v) - f(v_0) = Df(v_0)(v-v_0) + R(v,v_0)$

where $R(v,v_0) = o(|v-v_0|)$. By FTC,

$$\underbrace{|v(t,y_0) - v(t,y_0+h) + \Phi(t,y_0)h|}_{=:g(t)}$$

$$\le \int_0^t |f(v(s,y_0)) - f(v(s,y_0+h)) + Df(v(s,y_0))\Phi(s,y_0)|\,ds$$

$$= \int_0^t \Big| Df(v(s,y_0))\big(v(s,y_0) - v(s,y_0+h)\big) + R(v(s,y_0), v(s,y_0+h)) + Df(v(s,y_0))\Phi(s,y_0)\Big|\,ds$$

$$\le \int_0^t \|Df(v(s,y_0))\|\,|v(s,y_0) - v(s,y_0+h) + \Phi(s,y_0)|\,ds + \int_0^t R(v(s,y_0), v(s,y_0+h))\,ds$$

$R = o(|v(s,y_0) - v(s,y_0+h)|) \Rightarrow$ writing $\mathcal{E}(h) = \frac{R}{|v(s,y_0) - v(s,y_0+h)|}$ gives $\lim_{h\to 0}\mathcal{E}(h) = 0$. Now using (*) gives

$$g(t) \le M_1\int_0^t g(s)\,ds + \mathcal{E}(h)|h|e^{M_1 t}$$

& another application of Gronwall yields

$$g(t) \le \mathcal{E}(h)|h|e^{2M_1 t} \Longrightarrow g(t) = o(h)$$

Thus $\Phi(t,y_0) = \frac{\partial v}{\partial y}(t,y_0)$. ∎

Rmk $\frac{\partial v}{\partial y} = \Phi \iff \Phi$ is fundamental matrix soln of $\dot{\Phi} = Df(v(t,y))\Phi$ with $\Phi(0,y) = I$.

$\boxed{\text{Rmk}}$ Similar result holds in C^r

Also, lets us deduce regularity of parameter dependence:

$\boxed{\text{Thm}}$ Let $U \subset \mathbb{R}^{n+m}$ be open, $(x_0, \mu_0) \in \mathbb{R}^{n+m}$, $f: U \to \mathbb{R}^n$ C^1.
Then $\exists\ \tau, \delta > 0$ s.t. $\forall\ y \in B(x_0, \delta)$ & $\mu \in B(\mu_0, \delta)$
the IVP $\dot{x} = f(x, \mu)$, $x(0) = y$ has a unique
soln $v(t, y, \mu)$, $v \in C^1((-\tau, \tau) \times B(x_0, \delta) \times B(\mu_0, \delta))$

Pf $\tilde{x}_0 = (x_0, \mu_0)$, $\tilde{x} = (x, \mu)$, $\tilde{f} = f \times 0$, apply prev. thm. ∎

Extending solutions

Prop Sps $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}^n$ C^1, $I \ni t_0$ open interval,
$u, v: I \to U$ both solve $\dot{x} = f(x)$, $x(t_0) = x_0$.
Then $u(t) = v(t)\ \forall\ t \in I$.

$\boxed{\text{NOTE}}$ - not immediate from Picard-Lindelof!

Pf

1. Fake pf of uniqueness for any ODE: If x & y both solve then $\dot{x} = f(x)$, $\dot{y} = f(y)$, $z := x - y \Rightarrow \dot{z} = f(x) - f(y) = 0$ on interval where they agree, so $z \equiv 0$.
2. If Lipschitz, then let $z = |x - y|$, get
$z \leq \int_0^t |f(x(s)) - f(y(s))|\,ds \leq L \int_0^t z(s)\,ds$
$\therefore$ Gronwall gives $z(t) \leq 0 \cdot e^{Lt} = 0$.
3. Another way: let $\hat{t}$ be $\sup\{t \mid x(t) = y(t)\}$, consider IVP with initial condition $x(\hat{t})$, get uniqueness on $(\hat{t} - \tau, \hat{t} + \tau)$, contradicts maximality ∎