LECTURE 13 10/9

Say $I \subset \mathbb{R}$ is a maximal interval of existence if $\nexists\, J \supsetneq I$ s.t. $x: J \to U$ solving IVP exists

Thm Let $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}^n$ C^1, $x(t)$ solve IVP on maximal interval of existence. Then I is open & if $\sup I < \infty$, then $\forall$ cpt $K \subset U$ $\exists\, t \in I$ s.t. $x(t) \notin K$

Rmk "as $t \to \beta = \sup I$, either $x \to \partial U$ or $|x| \to \infty$"

Pf If closed at either end then can extend using [P-2]. Let $I = (\alpha, \beta)$, fix $\gamma \in (\alpha, \beta)$

Let $M = \max_{x \in K} |f(x)|$ and suppose $x \in K$ $\forall\, t \in I$.

Then $|x(t_0) - x(t_1)| = \left|\int_{t_0}^{t_1} \dot{x}(s)\,ds\right| \le \int_{t_0}^{t_1} |f(x(s))|\,ds \le M(t_1 - t_0)$

$\therefore$ x is Lipschitz $\therefore$ extends to cts map $x: [\gamma, \beta] \to K \subset U$

Furthermore, $\frac{d}{dt} x(t)\big|_{\beta^-} = f(x(\beta))$ & $\dot{y} = f(y)$, $y(\beta) = x(\beta)$ has unique solution on $(\beta - \tau, \beta + \tau)$ $\therefore$ $x = y$ on $(\beta - \tau, \beta]$ $\therefore$ $x = y$ on $(\beta, \beta + \tau)$ extends solution contradicting maximality. ∎

What about continuity wrt. IC on maximal interval of existence? (Not just local interval)

Thm Let $f \in C^1(U, \mathbb{R}^n)$ & $x(t)$ solve (1) $\dot{x} = f(x)$ & $x(t_0) = x_0$ on $[t_0, t_1]$. Then $\exists$ nbhd $x_0 \in V \subset U$ & const K s.t. $\forall\, y_0 \in V$ $\exists!\ y: [t_0, t_1] \to U$. Further, $|x - y| \le K |x_0 - y_0| e^{K(t - t_0)}$ $\forall\, t \in [t_0, t_1]$.

Rmk Gives both existence-uniqueness & continuity.

Lemma If $f: A \to \mathbb{R}^n$ is loc Lip & A is cpt then f is Lip.

Pf If not then $\exists\, |fx_n - fy_n| \ge n|x_n - y_n|$ $\forall\, n$, relabel so $x_n \to x^*$, $y_n \to y^*$, $|f(x)| \le M$ $\forall\, x \in A \Rightarrow |x^* - y^*| \le \frac{1}{n} 2M$ $\forall\, n$ ↯ □

(alternate pf: $L: A \times A \setminus \Delta \to \mathbb{R}$, $(x,y) \longmapsto \frac{|f(x)-f(y)|}{|x-y|}$
extends to upper semicts $L: A \times A \to \mathbb{R}$ $\therefore$ achieves max)

Pf of Thm. $[t_0, t_1]$ cpt $\Rightarrow \exists\, \varepsilon > 0$ s.t. $|z - x(t)| \le \varepsilon$
$\Rightarrow z \in U$ $\forall$ $t \in [t_0, t_1]$. $A := B(x([t_0,t_1]), \varepsilon)$ is cpt
$\therefore f|_A$ is Lip, let k = const.
Choose $\delta > 0$ s.t. $\delta \le \varepsilon$, $\delta e^{k|t_1 - t_0|} \le \varepsilon$.
Fix $|y_0 - x_0| < \delta$. Because $y_0 \in U$, $\exists$ sol'n on max ~~interval~~ $[t_0, \beta)$. If $\beta \le t_1$ ~~then~~ by Gronwall every $t \in [\beta, t_1)$ has $|y(t) - x(t)| \le \delta e^{k|t - t_0|} < \varepsilon$
$\therefore y \in A$ ↯ (~~prev. thm~~). Thus y def on $[t_0, t_1]$
(Rmk: this implies $\beta(x_0)$ is LSC)
Gronwall $\Rightarrow$ $|y - x| \le |y_0 - x_0| e^{k|t - t_0|}$ & ~~this~~
implies uniqueness, continuity ▨

Rmk If $U = \mathbb{R}^n$ (so $f: \mathbb{R}^n \to \mathbb{R}^n$) & $|f(x)| \le M_0 + M_1|x|$,
then all sol'ns defined for all time
Or if y_0 is such that sol'n $y(t)$ is known to stay in cpt set A — then defined $\forall\, t$.

$\boxed{\text{FLOW}}$ of a DE $\qquad f \in C^1(U, \mathbb{R}^n)$

autonomous ODE $\dot{x} = f(x) \Rightarrow$ solutions $\phi(t, x)$
time t, IC x. Let $J(x)$ = max int of existence.
$\Omega \subset \mathbb{R} \times U$, $\Omega = \{(t,x) \mid t \in J(x)\}$, $\phi: \Omega \to U$
ϕ is the flow of the ODE. (vec field).
Often we write $\phi(t, x) := \phi_t(x)$

Thm $\Phi_{s+t}(x) = \Phi_s(\Phi_t(x))$ – if one side is defined, so is the other.

Pf Mostly exercise. Do 1 case: $s, t > 0$ & $\Phi_s(\Phi_t(x))$ def
$\Rightarrow t \in J(x)$ & $s \in J(\Phi_t(x))$, say $J(x) = (\alpha, \beta)$,
show $\beta > s+t$. Define $y : (\alpha, s+t] \to W$ by

$$y(r) = \begin{cases} \Phi(r, x) & \alpha < r \leq t \\ \Phi(r-t, \Phi_t x) & t \leq r \leq t+s \end{cases}$$

Then y solves $\therefore s+t \in J(x)$ etc... ∎

LECTURE 14 | 10/11

Thm Ω is open & Φ is cts.

Pf Given $(t_0, x_0) \in \Omega$ $(t_0 \geq 0)$, get $J(x_0) > t_0$
$\therefore x$ def on $[-\varepsilon, t_0 + \varepsilon]$ $\therefore \exists$ nbhd $V \ni x_0$ s.t.
sol'n defined on $[-\varepsilon, t_0+\varepsilon]$ i.e. $J(y) > t_0 \; \forall y \in V$.
$\therefore (-\varepsilon, t_0 + \varepsilon) \times V \subset \mathcal{U}$.

cts: V as above s.t. $\overline{V} \subset \mathcal{U}$, K Lip const for $f|A$,
$A = \Phi([-\varepsilon, t_0+\varepsilon] \times \overline{V})$, $M = \max\{|f(x)| : x \in A\}$
Fix δ s.t. $\delta < \varepsilon$ & $B(x_0, \delta) \subset \mathcal{U}$. Now $\Delta t, \Delta x_0 < \delta$
$\Rightarrow |\Phi(t_1, x_1) - \Phi(t_0, x_0)| \leq |\Phi(t_1, x_1) - \Phi(t_1, x_0)| + | - |$
$< \delta e^{K\delta} + (\to 0)$ ∎

Rmk In fact if $f \in C^r$ then Φ is C^r ($r=1$ as before).

Given $(t, x_0) \in \Omega$ $\exists$ $V \ni x_0$, $V \subset \mathcal{U}$ s.t. $t \times V \subset \Omega$, & $\Phi_t : V \to \mathcal{U}$

Thm $\Phi_t(V) =: W$ is open, $\Phi_{-t} : W \to V$. $\Phi_{-t}\Phi_t = Id_V$, $\Phi_t \Phi_{-t} = Id_W$

Pf $y = \Phi_t(x) \Rightarrow t \in J(x)$, $-t \in J(y)$
$V^* \supset V$, $= \{z \in \mathcal{U} \mid -t \in J(z)\}$ is open since Ω is
Also Φ_{-t} is cts. So $V = (\Phi_{-t})^{-1}(\mathcal{U})$ is open ∎

STABILITY OF EQUILIBRIA

So far results have been very abstract, don't say much about a particular system.

Natural steps in studying a particular system $\dot{x} = f(x)$: (1)

① find fixed points & determine their stability

② find periodic orbits (if possible) & stability

③ determine what other types of orbits there are.

Def $\bar{x} \in U$ is an *equilibrium* (or *fixed*) *point* of (1) (stationary pt, zero, singular pt) if $f(\bar{x}) = 0$.

Then $x(t) = \bar{x}$ is only soln through $\bar{x}$. (NEEDS Lipschitz)

Def The linear part of f at $\bar{x}$ is $Df(\bar{x}) = \left[\frac{\partial f_i}{\partial x_j}(\bar{x})\right]_{ij} \in M_{n\times n}(\mathbb{R})$

$\bar{x}$ is a *sink* of f if $f(\bar{x}) = 0$ & all eigvals of $Df(\bar{x})$ have negative real parts, and a *source* if $f(\bar{x}) = 0$ & all eigvals of $Df(\bar{x})$ have pos. real parts, *saddle* if all $\mathrm{Re} \neq 0$ & some $+$, $-$

Def

A f.p. $\bar{x}$ is *stable* if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. (Lyapunov stability)

(2) $|x_0 - \bar{x}| < \delta \Rightarrow |\phi_t(x_0) - \bar{x}| < \varepsilon \quad \forall\ t \geq 0$.

It is *asymptotically stable* if (2) holds and also

(3) $|x_0 - \bar{x}| < \delta \Rightarrow \phi_t(x_0) \to \bar{x}$ as $t \to \infty$.

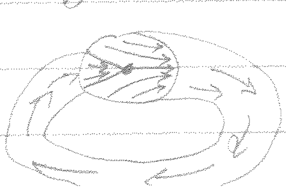
It is *exponentially stable* if $\exists\ C, \alpha, \delta > 0$ s.t.

(4) $|x_0 - \bar{x}| < \delta \Rightarrow |\phi_t(x_0) - \bar{x}| \leq C e^{-\alpha t} \quad \forall\ t \geq 0$.

It is *unstable* if not stable.

Remark * "unstable" $\neq$ "stable in backwards time"

* (4) $\Rightarrow$ (3) but (3) $\not\Rightarrow$ (2).



Exp St $\Rightarrow$ Asy St $\Rightarrow$ Lya Sta

$\not\Leftarrow$ $\qquad$ $\not\Leftarrow$

Examples • A center for a linear system is Lyapunov stable but not asymptotically stable.

• $\dot{x} = -x^3 \Rightarrow -\frac{dx}{x^3} = dt \Rightarrow \frac{1}{2}x^{-2} = t + c$

$\Rightarrow x(t) = 1/\sqrt{2(t+c)} \longrightarrow 0$ as $t \to \infty$

but the convergence is subexponential.

$\boxed{\text{LECTURE 15}}$ $\boxed{10/16}$

Thm Sinks are exponentially stable.

Pf Put $y = x - \bar{x}$, $g(y) = f(x - \bar{x})$, $A = Df(\bar{x}) = Dg(0)$

Choose $0 < a < b < |\mathrm{Re}\,\lambda|$ $\forall$ eigvals λ of A.

Let $Q \in M_{n\times n}$ be symmetric, pos. def, & solve

Lyapunov's eq'n $A^TQ + QA = -I$.

(Take $B(t) = e^{tA^T}e^{tA}$, $Q = \int_0^\infty B(t)\,dt$)

Let $V(y) = y^TQy$. Then

$\dot{V}(y) = \dot{y}^TQy + y^TQ\dot{y} = g(y)^TQy + y^TQg(y)$

Write $g(y) = Ay + R(y)$, where $R(y) = o(|y|)$. ~~then~~

$\dot{V}(y) = y^TA^TQy + R(y)Qy + y^TQAy + y^TQR(y)$

$= -|y|^2 + R(y)Qy + y^TQR(y)$

For small $|y|$ ~~this~~ is $\le -(1-\varepsilon)|y|^2$

BETTER APPROACH: Choose basis in which $A = D + N$ with D diag & $\|N\| < \varepsilon$ (in L^2-norm for that basis). This can be done by #2(c) from HW1. Then if $\varepsilon = \min(|\mathrm{Re}\,\lambda| - b)$, $a = \min|\mathrm{Re}\,\lambda|$

$\langle Ay, y\rangle = \langle Dy, y\rangle + \langle Ny, y\rangle \le -a|y|^2 + \varepsilon|y|^2 < -b|y|^2$

By def of derivative, $|g(y) - Ay| = o(|y|)$ $\therefore$ by Cauchy-~~Schwarz~~

$\frac{\langle g(y) - Ay, y\rangle}{|y|^2} \to 0$, so take δ with

$\langle g(y), y\rangle \le -c|y|^2 \quad \forall\, y \in B(0,\delta)$

Then $\frac{d}{dt}|y| = \frac{1}{|y|}\langle \dot{y}, y\rangle \le -c|y|$ $\therefore$ defined $\forall\, t \ge 0$ & $|y| \le e^{-tc}|y_0|$ ∎

What about initial conditions far from $\bar{x}$? Or eigvals with 0 real part? basin of $\bar{x}$ is $\underbrace{\{x \mid \phi_t x \to \bar{x}\}}_{\mathcal{B}(\bar{x})}$

Did on 10/11

Let $V: U \to \mathbb{R}$ be C^1 & denote $\dot{V}(x) = \langle \nabla V(x), f(x) \rangle$, then $\dot{V}(x) = \frac{d}{dt} V(\phi_t x)|_{t=0}$

$\boxed{\text{Thm } 1}$ (Lyapunov) Let $\bar{x}$ be an eq pt & $V: W \to \mathbb{R}$ a cts fn s.t. V is C^1 on $W \setminus \{\bar{x}\}$ & $(\bar{x} \in W \subset U)$

(a) $V(\bar{x}) = 0$, $V(x) > 0 \;\; \forall\, x \neq \bar{x}$

(b) $\dot{V}(x) \leq 0 \;\; \forall\, x \neq \bar{x}$

Then $\bar{x}$ is stable. If in addition

(c) $\dot{V}(x) < 0 \;\; \forall\, x \neq \bar{x}$

then $\bar{x}$ is asymptotically stable.

Def P is positively invariant if $x \in P \Rightarrow \phi_t x$ defined & $\in P \;\; \forall\, t \geq 0$.

entire orbit is $\{\phi_t x \mid t \in \mathbb{R}\}$

$\boxed{\text{Thm } 2}$ (Lyap) $\bar{x}$ a f.p., V a Lyap fn on W. Let $P \subset W$ be a nbhd of $\bar{x}$ ~~closed in W~~ compact. Sps P is pos inv & $\nexists$ entire orbit in $P \setminus \{\bar{x}\}$ on which V is constant. Then $\bar{x}$ is asymp stable & $P \subset \mathcal{B}(\bar{x})$.

$\boxed{\text{Rmk}}$ Gives results w/o need to solve equations but finding a good Lyap fn can be an art. (Often an energy)

Pf of 1 δ s.t. $B(\bar{x}, \delta) \subset W$, $\alpha = \min\{V(x) \mid x \in \partial B(\bar{x}, \delta)\} > 0$.

$U_1 = \{x \in B(\bar{x}, \delta) \mid V(x) < \alpha\}$, so $x_0 \in U_1 \Rightarrow \phi_t x \in B(\bar{x}, \delta) \;\; \forall\, t \geq 0$.

Now assume (C) & prove ~~asymptotic stability~~. Sps $t_n \to \infty$ is s.t. $x(t_n) \to z_0 \in B(\bar{x}, \delta)$. Then

$V(x(t)) \geq V(z_0)$ $\forall$ $t \geq 0$. If $z_0 \neq \bar{x}$, ~~then~~

we have $V(\phi_t z_0) < V(z_0)$ $\forall$ $t \geq 0$, & by ~~continuity~~ of V, ϕ_t, $V(\phi_t y) < V(z_0)$ $\forall$ $y \sim z_0$, $\not\leq$. ∎

Pf of $\boxed{2}$. Fix $x_0 \in P$ & sps $x(t) \not\to \bar{x}$, then $\exists$ $a \neq \bar{x}$, $a \in P$, $x(t_n) \to a$.

$\alpha := V(a) = \inf \{ V(x(t)) \mid t \geq 0 \}$ ω-limit set

$L = \{ a \in W \mid \exists\, t_n \to \infty,\ x(t_n) \to a \} =: \omega(x_0)$

$L \subset P$ & $a \in L \Rightarrow \phi_t a$ defined & in L $\forall$ $t \geq 0$.

Also defined & in L for every $t \leq 0$, since defined on $[-t_n, 0]$ as $\lim_k \phi_t(x(t_k))$ (valid for $t_k \geq t_n$)

& $x(t_n) \to a \implies x(t_n + s) \to \phi_s(a) \in L$.

(ω-limit sets are pos-& neg-inv. & def. $\forall$ time).

$V(a) = \alpha$ $\forall$ $a \in L \Rightarrow V$ const on entire orbit $\not\leq$ ∎

Def'n $\omega(x)$ & $\alpha(x) \leftarrow (t_n \to -\infty)$

ω-limit set & α-limit set.

Did on 10/11

Example $\dot{x} = -y - x^3$ $\quad$ $Df(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\lambda = \pm i$

$\dot{y} = x - y^3$

Linear part gives no information. Let $V(x,y) = x^2 + y^2$

~~then~~ $\dot{V} = 2x\dot{x} + 2y\dot{y} = 2x(-y - x^3) + 2y(x - y^3)$

$= -2xy - 2x^4 + 2xy - 2y^4 = -2(x^4 + y^4) < 0$

so V is a strict Lyapunov fn on $\mathbb{R}^2$ & $(0,0)$ is asymptotically stable with $\mathcal{B} = \mathbb{R}^2$.

$\boxed{\text{LECTURE 16}}$ 10/18

What about positive eigvals? (H-S p. 187) ← but pf there has errors

$\boxed{\text{Thm}}$ If $\exists\, \lambda$, $\mathrm{Re}\,\lambda > 0$ then not stable.

Pf WLOG $\bar{x} = 0$, $\mathbb{R}^n = E^u \oplus E^{cs}$

Let $a < \min(\mathrm{Re}\,\lambda)\ (>0)$, $0 < b < a$

$A = Df(0)|_{E^u}$, $B = Df(0)|_{E^{cs}}$

Then $\exists$ norm s.t. $\langle Ax, x\rangle \geq a|x|^2 \quad \forall\, x \in E^u$,

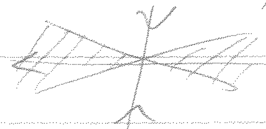
$-c|y|^2 < \langle By, y\rangle \leq b|y|^2 \quad \forall\, y \in E^{cs}$ $\quad c > 0$

$f(x,y) = (Ax + R(x,y),\ By + S(x,y))$

$= (f_1, f_2)$,

$z = (x,y)$, $\quad Q(z) = (R, S)$

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $|z| < \delta \Rightarrow |Q(z)| \leq \varepsilon |z|$

Fix $\gamma > 0$, let $C_\gamma = \{(x,y) \in E^u \oplus E^{cs} \mid |y| \leq \gamma |x|\}$ be the unstable cone  of width γ.

Lemma $\exists\, \delta > 0$ s.t. $\forall\, z \in C_\gamma \cap B(0,\delta)$, ($\gamma > 0$)

① $\gamma^2 \langle x, f_1(x,y)\rangle - \langle y, f_2(x,y)\rangle > 0$ if $x \neq 0$

② $\exists\, \alpha > 0$ with $\langle f(z), z\rangle \geq \alpha |z|^2$

Pf $\langle f(z), z\rangle = \langle Ax, x\rangle + \langle By, y\rangle + \langle Q(z), z\rangle$

$\geq a|x|^2 - c|y|^2 - \varepsilon|z|^2$

$\geq (a - c\gamma)|x|^2 - \varepsilon|z|^2 \qquad |z|^2 \leq |x|^2 + |y|^2$

② : $\geq \left(\frac{a - c\gamma}{1+\gamma^2} - \varepsilon\right)|z|^2 \qquad \leq (1+\gamma^2)|x|^2$

$\geq \alpha |z|^2$ as long as γ, ε small enough

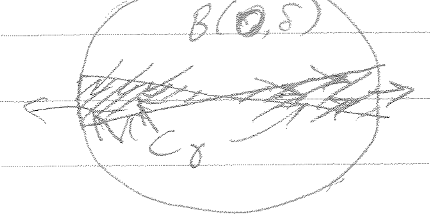
& $0 < \alpha < a$.

LHS of ① $= \gamma^2\langle x, Ax\rangle - \langle y, By\rangle + \gamma^2\langle x, R\rangle - \langle y, S\rangle \rightarrow \leq |\langle 2Kz, Q(z)\rangle|$

$\geq \gamma^2 a|x|^2 - b|y|^2 - 2\varepsilon|z|^2$

$\geq \left(\frac{(a-b)\gamma^2}{1+\gamma^2} - 2\varepsilon\right)|z|^2 > 0$ for small γ, ε.

Given the Lemma, get the Thm as follows:



Let $g(x,y) = \frac{1}{2}(\gamma^2|x|^2 - |y|^2)$

then $C_\gamma = g^{-1}[0,\infty)$ & $g^{-1}(0) = \partial C_\gamma$.

① $\Rightarrow \dot{g} = \langle \nabla g, f \rangle = \langle \gamma^2 x, -y \rangle \cdot \langle f_1, f_2 \rangle = \gamma^2 \langle x, f_1 \rangle - \langle y, f_2 \rangle > 0$ in $B(0,\delta)$

$\therefore$ cannot leave C_γ w/o first leaving $B(0,\delta)$

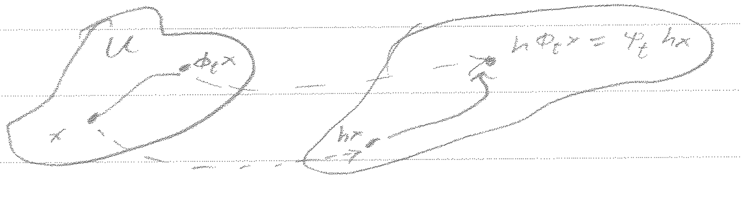
② $\Rightarrow \frac{d}{dt}|z|^2 \geq \alpha |z|^2 \Rightarrow |z|^2$ grows exponentially until leaves ball $\blacksquare$

$\boxed{\text{DO THIS AFTER STABLE MFD}}$

Changes of coordinates ~~often~~ can put things in a nicer form — e.g. JNF, change to basis of gen eigvec. What can we do in the nonlinear setting?

$\boxed{\text{Def}}$ 2 flows $\Phi_t, \Psi_t : U \circlearrowleft$ are ~~topologically conjugate~~ (C^0-conj) if $\exists$ homeomorphism $h : U \to U$ s.t. $h \circ \Phi_t = \Psi_t \circ h \ \forall t$.

(also $\Phi_t : U \circlearrowleft$, $\Psi_t : U' \circlearrowleft$, $h : U \to U'$)



$\boxed{\text{Prop}}$ h takes f.p. to f.p. & periodic orbs to per orbs

$\boxed{\text{Pf}}$ $\Phi_t x = x \Rightarrow (hx) = h\Phi_t x = \Psi_t(hx)$ $\blacksquare$

(& period is preserved)

$\boxed{\text{Def}}$ C^k-conjugacy = same, but require h, h^{-1} ~~to~~ be C^k diff, not just C^0