$\boxed{\text{Rmk}}$ By reversing ~~time~~ we get ~~the~~ corollary:
$\exists\, r>0$ & C^1 $\psi^u : B(0,r)\cap E^u \to E^s \oplus E^c$ s.t.
$W^u :=$ graph ψ^u is neg inv. & $\phi_t(x)\to 0$ exp as
$t\to -\infty$ $\forall x\in W^u$.

What about E^c?? NO ANALOGOUS THM (existence but no uniqueness or dynamical control)

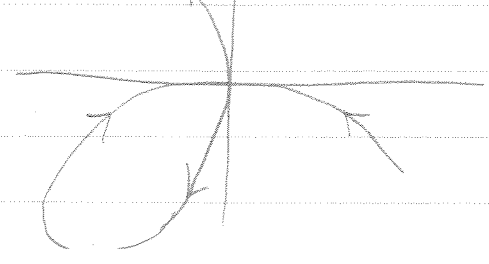
Completion ~~of~~ pf of Hadamard Perron:
$\boxed{C^1}$ $(P_{x_0}x)(t)$ is diff. in x_0 $\forall t$, so $(P_{x_0}^n x)(t)$ is as well. Need to get common modulus of ~~continuity~~ for derivatives
$\boxed{\text{tangent to } E^s}$ can be shown.
$\boxed{\text{positively invariant}}$ use norm in which $|x(t)|$ is decreasing on E^s for linear system.
$\boxed{\text{uniqueness}}$

~~Global stable/unstable~~ mfds:
W^s_{loc} as in thm, $W^s = \bigcup_{t\le 0} \phi_{-t}(W^s_{loc})$
$= \{x \mid \phi_t x \in W^s_{loc}$ for some $t\ge 0\}$
Similarly for W^u.

$\boxed{\text{Example}}$ $\dot{x} = -x - y^2$
$\dot{y} = y + x^2$

Perko p.|||

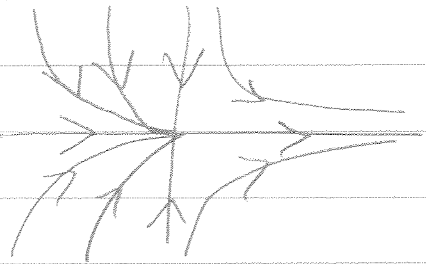
Can use ~~iterative~~ procedure to ~~estimate~~ $\psi^s(x) = -\frac{1}{3}x^2 + O(x^5)$
& $\psi^u(y) = -\frac{1}{3}y^2 + O(y^5)$



This example has a homoclinic orbit. (at least numerically)

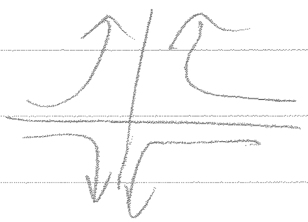
Perko p. 115 $\quad \dot{x} = x^2 \quad \dot{y} = -y$

W^c is not unique nor dynamically characterised.



eg $\quad \dot{x} = -x, \quad \dot{y} = y^3$

nonlinear saddle



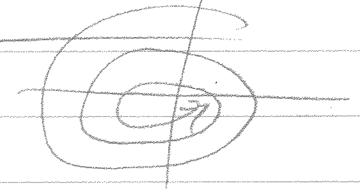
$\dot{y} = -y^3$

$\Rightarrow$ nonlinear sink

eg $\quad \dot{x} = -y - x(x^2+y^2)$

$\dot{y} = x - y(x^2+y^2)$

nonlinear spiral



Re $\lambda = 0$ $\Rightarrow$ linear part not enough

When all eigvals have Re $\lambda \neq 0$, call f.p. hyperbolic.

LEC 21   11/8

Thm (Hartman-Grobman for discrete time)

Let $E \subset \mathbb{R}^n$ be open and $T: E \to \mathbb{R}^n$ a C^1 map. Suppose $0 \in E$, $T(0) = 0$, and 0 is a hyperbolic f.p. Let $L = DT(0)$ be the linear part at 0. Then $\exists$ open sets $U, V \ni 0$ & a homeomorphism $H: U \to V, \subset \mathbb{R}^n$ s.t. $\forall\ x \in U \cap T^{-1}(U)$ we have $H \circ T(x) = L \circ H(x)$

$$\begin{array}{ccc} T(U)\cap U & \xrightarrow{T} & U \\ H \downarrow & & \downarrow H \\ V & \longrightarrow & \mathbb{R}^n \end{array}$$

Pf again, build suitable space, operator, find H as fixed point. Space should be fns $\mathbb{R}^n \circlearrowleft$, but need a complete metric space, so we consider Hölder norm: $\|F\|_\alpha = \|F\|_0 + |F|_\alpha$

$\|F\|_0 = \sup |F(x)|$

$$|F|_\alpha = \sup_{x \neq y} \frac{|F(x) - F(y)|}{|x - y|^\alpha} \qquad \left(\Rightarrow |F(x) - F(y)| \leq |F|_\alpha |x-y|^\alpha \right)$$

Fix $\alpha > 0$ small & let $X = C^\alpha = \{H : \mathbb{R}^n \circlearrowleft \mid \|H\|_\alpha < \infty\}$
(note in particular that $I \in X$).
H does the job $\iff L \circ H = H \circ T \iff H = L^{-1} \circ H \circ T$ / $H = L \circ H \circ T^{-1}$
As in Stable Manifold Theorem, to get contraction we need to consider stable & unstable parts separately, run them in different directions.
Let $H(x) = (\Phi(x), \Psi(x))$, $\Phi : \mathbb{R}^n \to E^s$, $\Psi : \mathbb{R}^n \to E^u$. Need H to have

$$\Psi = C^{-1}\Psi \circ T \qquad C = L|_{E^u}$$
$$\Phi = B\,\Phi \circ T^{-1} \qquad B = L|_{E^s}$$

Let $M = \sup_x \max(\|DT(x)\|, \|DT^{-1}(x)\|)$, $\lambda = \max(\|B\|, \|C^{-1}\|) < 1$
Put $(PH)(x) = (B\Phi \circ T^{-1}, C^{-1}\Psi \circ T)$
Coordinate-wise get $|(P^u\Psi_1 - P^u\Psi_2)(x)| \le \|C^{-1}\| \, |(\Psi_1 - \Psi_2)(Tx)| \le \lambda \|\Psi_1 - \Psi_2\|_0$
& $|(P^u\Psi_1 - P^u\Psi_2)(x) - (\text{---})(x')|$

$$\le \|C^{-1}\| \, |(\Psi_1 - \Psi_2)(Tx) - (\Psi_1 - \Psi_2)(Tx')|$$
$$\le \lambda |\Psi_1 - \Psi_2|_\alpha \, |T(x) - T(x')|^\alpha$$
$$\le \lambda |\Psi_1 - \Psi_2|_\alpha \, M^\alpha |x - x'|^\alpha$$

$\Rightarrow |P^u\Psi_1 - P^u\Psi_2| \le \lambda M^\alpha |\Psi_1 - \Psi_2|_\alpha$
Together & using similar estimates for Φ gives

$$\|PH_1 - PH_2\|_\alpha \le \lambda M^\alpha \|H_1 - H_2\|_\alpha$$

Choose $\alpha > 0$ small enough that $\lambda M^\alpha < 1$.
Then P is a contraction & has a unique fixed point H.

H may not be a global homeomorphism but one can show (we omit the pf) that it is a local homeomorphism. ∎

Thm (Hartman-Grobman for flows)
$E \subset \mathbb{R}^n$ open, $0 \in E$, $f \in C^1(E, \mathbb{R}^n)$, φ_t flow of $\dot{x} = f(x)$, $f(0) = 0$, $A = Df(0)$ hyperbolic. Then $\exists$ open $U, V \ni 0$ & homeom $H : U \to V$ s.t. $\forall x \in U$ $\exists$ open interval $I_0 \subset \mathbb{R}$, $0 \in I_0$ s.t. $\forall t \in I_0$, $H \circ \varphi_t(x) = e^{At} H(x)$

Pf apply previous thm to $\varphi_1 =: T$ & get H_0.
Define H by $H = \int_0^1 e^{-As} H_0 \varphi_s \, ds$

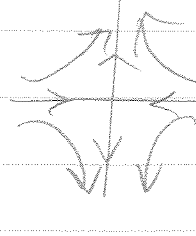
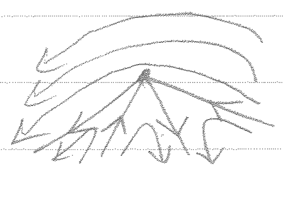
Then
$$e^{At} H = \int_0^1 e^{A(t-s)} H_0 \varphi_{s-t} \, ds \; \varphi_t$$
$$= \int_0^{1-t} e^{-As} H_0 \varphi_s \, ds \; \varphi_t$$
$$+ \left(\int_{-t}^0 e^{-As} H_0 \varphi_s \, ds \right) \varphi_t$$

The last is $\int_{-t}^0 e^{-A(s+1)} H_0 \varphi_{s+1} \, ds$ (since $e^{-A} H_0 \varphi_1 = H_0$)
$$= \int_{1-t}^1 e^{-As} H_0 \varphi_s \, ds$$
$$\therefore e^{At} H = \left(\int_0^1 e^{-As} H_0 \varphi_s \, ds \right) \varphi_t = H \varphi_t \quad \blacksquare$$

4 Dec 22 | 2 Apr 13

H is a topological (C^0) conjugacy — maps orbits into orbits & preserves parametrisation. Does not preserve rates of expansion/contraction:
$$\varphi_t(x) = e^{-t} x \quad \& \quad \psi_t(x) = e^{-2t} x$$
are topologically conjugate via the map $h(x) = \sqrt{x}$:
$$h \circ \psi_t(x) = h(e^{-2t} x) = \sqrt{e^{-2t} x} = e^{-t} \sqrt{x} = \varphi_t \circ h(x)$$

Also does not preserve directions of approach:

 is topologically conjugate to 

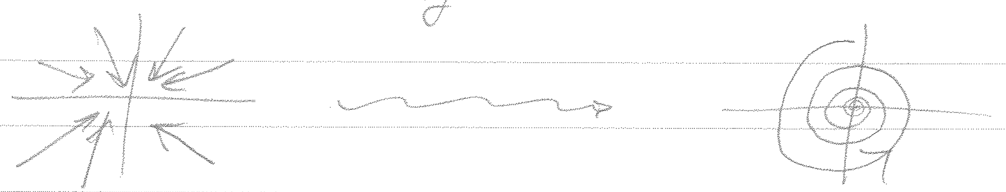
("direction" is a notion in differentiable structure)

Example (§2.10, Eg. 5, p. 141 Perko)

$\dot{x} = -x - y/\log r \qquad \dot{y} = -y + x/\log r \qquad r = \sqrt{x^2+y^2}$

$\Rightarrow \dot{r} = -r \qquad \dot{\theta} = \frac{1}{\log r} \Rightarrow \theta \to \infty$

linear (node)



nonlinear (focus)

C^0-conj is weaker than C^1-conj,
stronger than ~~orbit~~ topological equivalence
(H maps orbits to orbits but may not respect parametrisation)

$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\varphi_t(x) = R(t)x \qquad \psi_t(x) = R(2t)x$

orbit equivalent but not C^0-conj

$\boxed{\text{Thm}}$ (Hartman, 1960)

Under conditions of H-G, if f is C^2 then H is a C^1-diffeo ($H, H^{-1} \in C^1$)

LEC 22 | 11/13

Example (#5, §2.8, p. 127 Perko)

$\dot{x} = 2x, \qquad \dot{y} = 4y + x^2$

If H is C^2 then $\det DH(0) = 0 \Rightarrow H^{-1}$ not diff at 0.

So even if f is analytic, H may only be C^1.
To get $H \in C^r$, $r>1$, need non-resonance condition

$\lambda_j \neq \sum m_i \lambda_i \quad \forall\ m_i \in \mathbb{N} \cup \{0\}, \ \sum m_i > 1.$

Corrections to proof of Hartman-Grobman Thm

PROBLEMS

① $X = \{ H : \mathbb{R}^n \circlearrowleft \mid H \text{ is } \alpha\text{-Hölder} \}$,

$d_\alpha(H, \hat{H}) = |H - \hat{H}|_\alpha$ ← NOT A METRIC

eg. $\hat{H} = H + x_0 \Rightarrow d_\alpha(H, \hat{H}) = 0$, but $H \neq \hat{H}$

② T only defined on E, so PH only defined on $E \cap TE \cap T^{-1}E$, and domain of definition shrinks

③ Did not prove that H is a homeomorphism rather than something silly like $H(x) = 0$.

④ Assumed that $\sup_x \|DT(x)\| < \infty$, but this may fail if T is C^1 & not Lipschitz.

~~①: Let $X = \{ \alpha\text{-Hölder } H : \mathbb{R}^n \circlearrowleft \mid H(x) = x \ \forall |x| \geq r_0 \}$~~

② & ④: Replace T with $\hat{T} : \mathbb{R}^n \circlearrowleft$ s.t.

$$\hat{T}(x) = \begin{cases} Tx & |x| < r_1 \\ Lx & |x| > r_0 \end{cases} \qquad (0 < r_1 < r_0)$$

③ By conjugacy equation, if $Hx = Hy$ then $H\hat{T}^k x = H\hat{T}^k y \quad \forall k \in \mathbb{Z}$.

$\boxed{\text{Lemma}}$ $\forall x \neq 0 \ \exists k \in \mathbb{Z}$ s.t. $|\hat{T}^k x| > r$, where

Pf Uses Stable Mfd Thm: if $\{ \hat{T}^k x \mid k \geq 0 \}$ is bdd then $x \in W^s$ & $|\hat{T}^k x| \to \infty$ as $k \to -\infty$.

① Let $X = \{ \alpha\text{-Hölder } H : \mathbb{R}^n \circlearrowleft \mid$

$$H^s(x) = x^s \quad \forall |x^s| \geq r_0,$$
$$H^u(x) = x^u \quad \forall |x^u| \geq r_0 \}$$

Hartman-Grobman: if $f: U \to \mathbb{R}^n$ is a vector field with a hyperbolic equilibrium point at $\bar{x}$, then any C^1-small perturbation of f is topologically conjugate to f in a nbhd of $\bar{x}$.

Thm: 0 a hyp eq pt $\Rightarrow$ $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\|g - f\|_{C^1} < \delta$ $\Rightarrow g$ has a hyp eq pt in $B(0, \varepsilon)$

"C^1-small": g & Dg small.

$\|g\|_{C^1} = \sup_x |g(x)| + \sup_x \|Dg(x)\|$.

Defn: $f \in C^1(E)$ is structurally stable if $\exists\ \varepsilon > 0$ s.t. $\forall\ g \in C^1(E)$ with $\|g - f\|_{C^1} < \varepsilon$, g & f are topologically equivalent on E. (top equiv: $\exists\ H: E \circlearrowleft$, traj $\circlearrowleft$)

Important question: What vector fields are structurally stable? Hartman-Grobman is a local answer.

LECTURE 23 | 11/14

A point of structural instability is called a bifurcation.

Another way of looking at things:

Let $\mu \in \mathbb{R}$ be a parameter and $f_\mu(x)$ a vector field that depends on μ. f has a bifurcation at μ_0 if $\forall\ \varepsilon > 0$ $\exists\ \mu$, $|\mu - \mu_0| < \varepsilon$ s.t. f_μ & f_{μ_0} are not topologically equivalent.

$\to$ "qualitative behaviour changes at μ_0"

What kinds of bifurcations happen?

* fixed points created or destroyed
* periodic orbits " " "
* stability of f.p. / per orbit changes
* others?

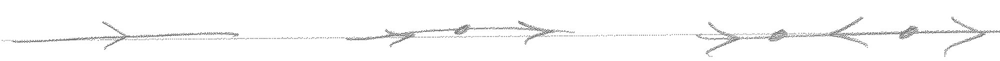
$\left.\right\}$ local bif.

<u>Example</u> $f_\mu(x,y) = (-y + \mu x,\ x + \mu y)$

$\mu = 0$ is a <u>bifurcation point</u> – <u>stability</u> of 0 changes

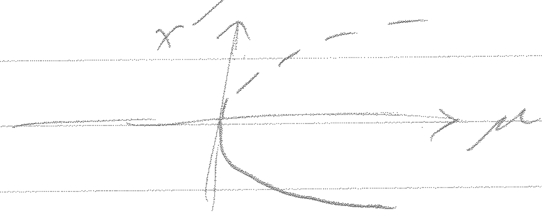
$$Df = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \Longrightarrow \varphi_t(x,y) = e^{\mu t}(\cos t, \sin t)$$

Ex $\dot{x} = \mu + x^2$ <u>saddle-node</u>



$\mu > 0$ $\mu = 0$ $\mu < 0$

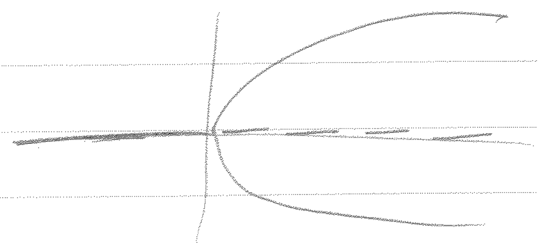
Bifurcation diagram



Ex $\dot{x} = \mu x - x^3$

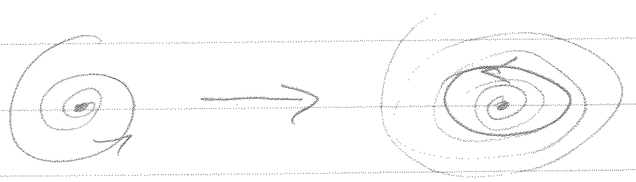
pitchfork

$\mu < 0$ $\mu > 0$



Ex $\dot{x} = -y + x(\mu - x^2 - y^2)$
$\dot{y} = x + y(\mu - x^2 - y^2)$ $\Longrightarrow$ $\dot{r} = \mu r - r^3$
$\dot{\theta} = 1$

$$Df(0) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$$



fixed point <u>changes stability</u> & <u>stable</u> periodic orbit is <u>created</u>

<u>Hopf bifurcation</u>

STABILITY OF PERIODIC ORBITS

also get bifurcations when periodic orbit changes stability: say a periodic orbit γ is

- Lyap stable if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $d(x,\gamma) < \delta$
  $\Rightarrow d(\varphi_t x, \gamma) < \varepsilon \ \forall\ t \geq 0$
- asymp stable if Lyap stable & $d(\varphi_t x, \gamma) \xrightarrow{t\to\infty} 0 \ \forall\, d(x,\gamma) < \delta$.
- unstable if not Lyap stable

Example

$$\dot{x} = -y - x(\mu - (r^2-1)^2)$$
$$\dot{y} = x - y(\mu - (r^2-1)^2)$$

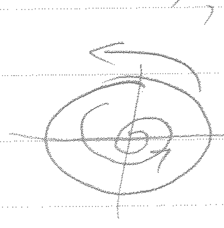
(Perko p. 326-7)

$$\Longrightarrow \dot{r} = -r(\mu - (r^2-1)^2), \quad \dot{\theta} = 1$$

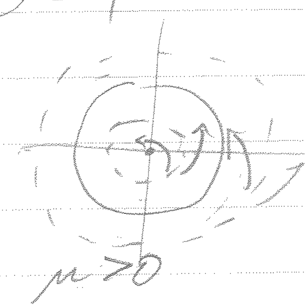
$\mu < 0$	$\mu = 0$ (unstable/ semistable)	$0 < \mu < 1$ outer is unstable, inner is stable	$\mu \geq 1$ unstable

Ex (Perko p. 330) $\dot{r} = r(1-r^2)(\mu - (r^2-1)^2), \quad \dot{\theta} = 1$

- unit circle always per orbit
- changes stability at $\mu = 0$:
- 2 new periodic orbits born at $\mu = 0$.



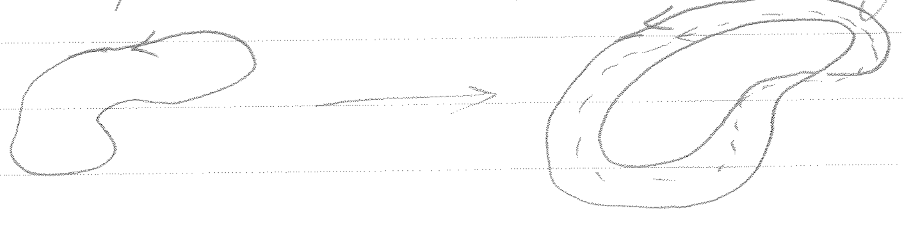
$\mu < 0$



$\mu > 0$

$\boxed{\text{LECTURE 24}}$ 11/19

Other important examples include period-doubling bifurcation:



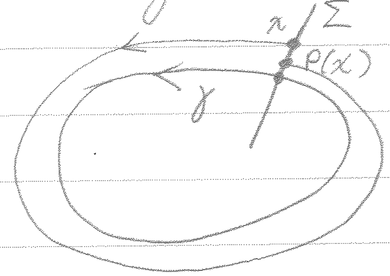
How do we detect stability of a periodic orbit outside of these simple examples?

- Floquet theory
- Poincaré map

$\gamma : \mathbb{R} \to \mathbb{R}^n$ a periodic orbit for $\dot{x} = f(x)$

so $\gamma(T) = \gamma(0)$ $\quad (\gamma(t+T) = \gamma(t) \;\; \forall \; t \in \mathbb{R})$

The following is useful: draw a transversal to γ, follow an orbit from this transversal until it hits it again: Poincaré return map.



$$\tau(x) = \min\{t > 0 \mid \varphi_t(x) \in \Sigma\}$$
$$P(x) = \varphi_{\tau(x)}(x)$$
$$P : \Sigma \to \Sigma$$

$\boxed{\text{Thm}}$ $f \in C^1(U)$, γ a periodic soln, $x_0 \in \gamma$,

$$\Sigma = \{x \in \mathbb{R}^n \mid (x - x_0) \perp f(x_0)\}$$

Then $\exists\ \delta > 0$ and $\tau : B(x_0, \varepsilon) \to (0, \infty)$ s.t.

- τ is C^1, $\tau(x_0) = T$
- $\varphi_{\tau(x)}(x) =: P(x) \in \Sigma \quad \forall \; x \in B(x_0, \varepsilon)$

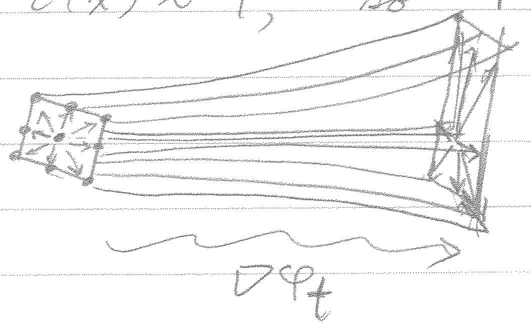
Pf Apply implicit function thm to $F(t,x) = \langle \varphi_t(x) - x_0, f(x_0) \rangle$ using $\partial_t F(t,x) = |f(x_0)|^2 \neq 0$ ■

Using coords on Σ with $x_0 \to 0$, see that P is a map from $\mathbb{R}^{n-1} \circlearrowleft$ with a f.p. at 0. Stability of $\gamma \Longleftrightarrow$ stability of this f.p.

In $\mathbb{R}^2$, ~~this stability~~ is easy to see: is $P(x)$ closer to x_0 than x is? Or further away? (Return map is <u>one-dimensional</u>)

In $\mathbb{R}^n$, return map is $(n-1)$-dimensional so more behaviours are possible. Want to compute $DP(x_0)$ & examine eigenvalues.

$\tau(x) \approx T$, so $P \approx \varphi_T$, so look at $D\varphi_T$



$(D\varphi_t)(v_0)$ is given by $v(t)$ for the solution of $\boxed{\dot{v}(s) = Df(x(s))}$, $v(0) = v_0$.

Let $A(t) = Df(\gamma(t))$, ~~then~~ $A(t+T) = A(t)$ $\forall\, t \in \mathbb{R}$.

The flow φ_t moves vectors transverse to the curve γ according to the non-autonomous ODE

(*) $\dot{v}(t) = A(t)\, v(t)$

$\Phi : \mathbb{R} \to M_{n\times n}$ is a <u>fundamental matrix</u> for (*) if $\dot{\Phi} = A(t)\Phi$: then $v(t) = \Phi(t)\Phi^{-1}(0)\, v(0)$.

Theorem (Floquet) If $A : \mathbb{R} \to M_{n\times n}$ is cts & T-periodic, then any fundamental matrix solution for (*) can be written $\Phi(t) = Q(t)e^{Bt}$ $\forall\, t \in \mathbb{R}$, where $Q(t)$ is inv, diff, T-periodic & B constant.

LECTURE 25 11/26

Informally, solving (*) on $[0,T]$ determines behaviour $\forall$ time. Writing $y = Q^{-1}(t)x$, get $\dot{y} = By$. Eigvals of B are <u>characteristic exponents</u> of $\gamma(t)$ – determine stability of periodic orbit.