λ a char. exp $\rightarrow e^{\lambda T}$ a characteristic multiplier of γ.

$\boxed{\text{Thm}}$ $f \in C^1(U)$, γ periodic, $0 \in \gamma$, $\delta > 0$ small, P Poincaré map
If $\lambda_1, \ldots, \lambda_n$ are CEs of γ then one is 0 (say λ_n) &
$e^{\lambda_1 T}, \ldots, e^{\lambda_{n-1} T}$ are eigvals for $DP(0)$

Pf in Perko, p. 205-206.

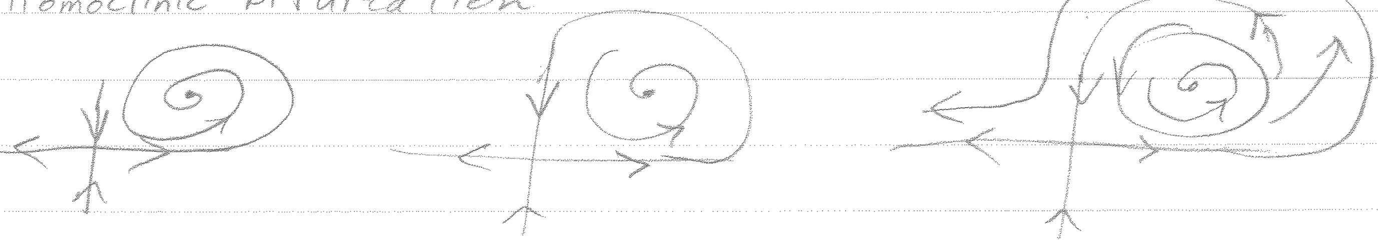
Leads to stable (unstable) Mfd Thm for periodic orbits. Stability properties of orbit change when some CM $e^{\lambda T}$ passes through unit circle $\rightarrow$ bifurcation.

GLOBAL STRUCTURE OF ODE

Previous bifurcations were local. Others are global: change not only in nbhd of f.p. / per orbit, but in global behaviour.

$\boxed{\text{Example}}$

Homoclinic bifurcation



$\boxed{Q}$ What kinds of global behaviour can occur?

Most interested in asymptotic behaviour

$\omega(x) = \{ z \mid \exists\, t_n \nearrow \infty, \text{ s.t. } \varphi_{t_n}(x) \rightarrow z \}$

$\alpha(x) =$ " $\searrow -\infty$ " "

Another example: $\dot{x} = \mu + x^2 - xy$

$\dot{y} = y^2 - x^2 - 1$

f.p. $\Rightarrow$ $x(y-x) = \mu \qquad (x+y)(y-x) = 1$

$$\frac{x}{x+y} = \mu \;\Big/\; x = \mu x + \mu y$$
$$y = \frac{1-\mu}{\mu} x \;\Big/\; y - x = \left(\frac{1-\mu}{\mu} - 1\right)x = \left(\frac{1-2\mu}{\mu}\right)x \;\Big/\; x^2\left(\frac{1-2\mu}{\mu}\right) = \mu$$
$$x^2 = \frac{\mu^2}{1-2\mu} \Rightarrow \boxed{x = \frac{\pm\mu}{\sqrt{1-2\mu}}, \quad y = \frac{\pm(1-\mu)}{\sqrt{1-2\mu}}}$$

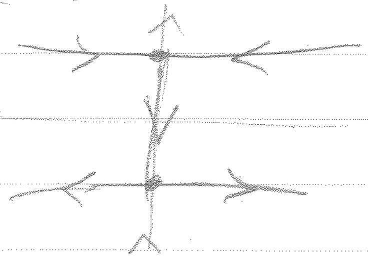
$\boxed{\mu = 0 \Rightarrow x = 0, \quad y = \pm 1}$

perturbation $\Rightarrow$ $(x,y) = (\pm\mu, \pm 1) + O(\mu^2)$

$f_\mu = f_0 + \binom{\mu}{0} = 0 \iff f_0 = \binom{-\mu}{0} \qquad Df_0 = \begin{bmatrix} 2x-y & -x \\ -2x & 2y \end{bmatrix}$

$Df_0(0, \pm 1) = \begin{bmatrix} \mp 1 & 0 \\ 0 & \pm 2 \end{bmatrix}$

$\boxed{\text{For } \mu = 0}$. shows f.p. hyperbolic

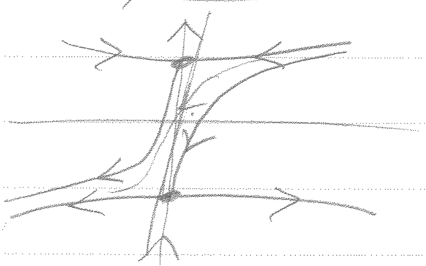


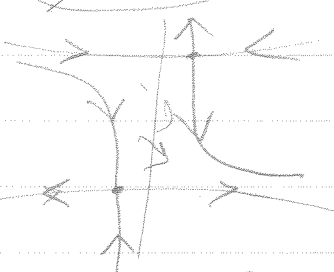
$x = 0 \Rightarrow f_0(x,y) = (0, y^2 - 1)$

So $\{0\} \times (-1, 1)$ is heteroclinic orbit (saddle connection)

For $\mu \neq 0$, $f_\mu(0,y)$ has x-comp $= \mu$, thus

$\mu < 0$ $\qquad\qquad$ $\mu > 0$





local pictures are all the same, but global picture bifurcates at $\mu = 0$.

Thus a complete picture requires study of global behaviour.

$\boxed{\text{LECTURE 26}}$ 11/29

$\boxed{\text{Prop}}$ $\boxed{1}$ $\omega(x)$ & $\alpha(x)$ are closed & invariant

$\boxed{2}$ If trajectory of x is bounded, then $\omega(x)$, $\alpha(x)$ are non-empty, connected, and cpt.

(Recall def of inv, for inv, back inv.)

(Do for ω, then α follows)

Pf INVARIANT: $x \in \omega(x_0) \Rightarrow \exists\, t_n \nearrow \infty$ s.t. $\varphi_{t_n}(x_0) \to x$

$$\Rightarrow \varphi_{t_n+t}(x_0) \to \varphi_t(x)$$

$$\Rightarrow \varphi_t(x) \in \omega(x_0)$$

Uses fact that φ_t is continuous at x.

CLOSED: $x_n \in \omega(\Gamma)$, $x_n \to x$

Then $\forall\, n\ \exists\, t_n$ s.t. $|\gamma(t_n) - x_n| < \frac{1}{n}$

Now $|\gamma(t_n) - x| \le \frac{1}{n} + |x_n - x| \to 0 \quad \therefore\ x \in \omega(\Gamma)$

NON-EMPTY: $\gamma(n)$ a bdd sequence $\Rightarrow \gamma(n_k) \to x \in \omega(\Gamma)$

COMPACT: Heine-Borel

CONNECTED: If disconnected then $\exists$ cpt $A \cap B = \emptyset$ s.t. $\omega(\Gamma) \subset A \cup B$. Let $a(t) = d(\gamma(t), A)$.

Then $\underline{\lim}\ a(t) = 0$, $\overline{\lim}\ a(t) \ge d(A,B) > 0$

a is cts & is $< \delta > \frac{1}{2} d(A,B) =: \delta$ i.o., so $\exists\, t_n \nearrow \infty$ s.t. $d(\gamma(t_n), A) = \delta\ \ \forall\, n$. $\gamma(t_{n_k}) \to x \in \omega(\Gamma)$,

$d(x, A) = d(x, B) = \delta > 0$ $\Rightarrow\Leftarrow$ ∎

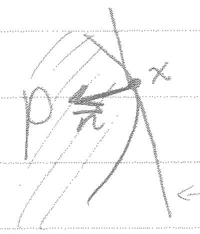
Q Is $\omega(\Gamma)$ path-connected?

Prop If P is cpt & forward-inv, then $\omega(x) \subset P\ \ \forall\, x \in P$. If backward-inv, then $\alpha(x) \subset P\ \ \forall\, x \in P$.

One-sided invariance easier to get than full invariance

Prop Let $U \subset \mathbb{R}^n$ be open, $P \subset U$ the closure of its interior, and sps $f: U \to \mathbb{R}^n$ points into P on ∂P. Then P is forward-invariant.

Pf What does "points into" mean?



$\vec{n}(x)$ unit normal pointing inwards from x.

"points into" means $\langle f(x), \vec{n}(x)\rangle > 0$ $\forall\, x \in \partial P$.

alternately $V: U \to \mathbb{R}$, $P = \{x \mid V(x) \geq a\}$

$\vec{n}(x) = \nabla V(x)$, $\langle f(x), \vec{n}(x)\rangle = \langle f, \nabla V\rangle = \dot{V} > 0$

Now to show forward invariance. Sps $x \in P$ & $t_1 > 0$ is s.t. $\varphi_{t_1}(x) \notin P$. Let

$t_0 = \sup\{\tau \in (0, t_1] \mid \varphi_\tau(x) \in P\}$, $y = \varphi_{t_0}(x)$.

Then $y \in \partial P$ & $\varphi_t(y) \notin P$ $\forall\, t \in (0, t_1 - t_0]$.

$\varphi_t(y) = y + f(y)t + O(t^2)$, so

$\langle f(y), \vec{n}(y)\rangle = \langle \frac{1}{t}(\varphi_t(y) - y) + O(t), \vec{n}(y)\rangle \leq 0$ ↯ ∎

actually get something a little stronger this way: only need $\dot{V} \geq 0$.

Indeed, given $x \in U$, get $\forall\, t \geq 0$,

$V(\varphi_t(x)) \geq V(x)$ $\therefore$ $x \in P \Rightarrow \varphi_t(x) \in P$. ∎

Analogous results for backwards-invariance.

LECTURE 27 12/4

Now we have basic properties of ω, α limit sets & a way to constrain their location. What can they look like?

carefully explain this example

$\mathbb{R}^1$: single pt

$\mathbb{R}^2$: single pt, limit cycle, homoclinic/heteroclinic cycle — anything else?

NO

homoclinic orbit: $\alpha(\Gamma) = \omega(\Gamma)$ is an eq pt

heteroclinic orbit: $\alpha(\Gamma) \neq \omega(\Gamma)$ are 2 eq pts

separatrix cycle: simple closed curve given as union of homo/heteroclinic orbits & eq. pts (compatibly oriented)

compound separatrix cycle / graphic = union of sep. cycles (comp. oriented)

<u>Poincaré - Bendixson Thm</u> cpt

$x \in \mathbb{R}^2$, $\omega(x)$ contains no eq. pt $\Rightarrow \omega(x)$ is a periodic orbit

<u>Generalised PB Thm</u> (if cpt) (& if f has only finitely many eq pts in cpt region containing $\omega(x)$)

$x \in \mathbb{R}^2 \Rightarrow \omega(x)$ is one of

1. eq pt
2. per orbit
3. (compound) separatrix cycle

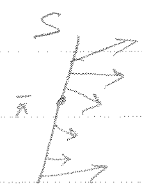
Pf of PB <u>Thm</u>

Recall: $\omega(x)$ closed & inv., also $\subset P$ if $x \in P$, P forward-inv & closed

We will need: ② flow boxes, ① local sections, ③

① $\boxed{\text{Defn}}$ A <u>hyperplane</u> in $\mathbb{R}^n$ is an $n-1$-dim subsp. H.

A <u>local section</u> of a vector field f at $\bar{x}$ is a nbhd of H ~~through~~ $\bar{x}$: $S = (B(0,\delta) \cap H) + \bar{x}$, such ~~that~~ S is <u>transverse</u> to f: $f(x) \notin H \ \forall x \in S$.



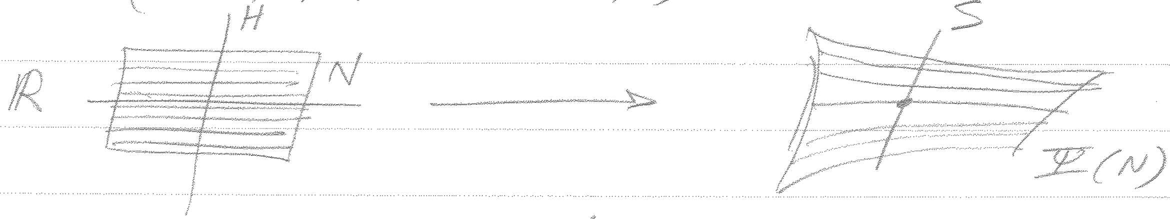
(In particular f does not vanish on S)

So every ~~traj~~ hitting S crosses it.

② A ~~flow box~~ flow box is a diffeo $\underbrace{\mathbb{R}\times H}_{\text{local section at } \bar{x}} \supset N \xrightarrow{\Psi} U$ (onto its image)

where N is a nbhd of $(0,0)$, ~~which transforms~~ $f: U \to \mathbb{R}^n$ to the constant vec field $(1,0)$. That is, (& has $\Psi(0,0)=\bar{x}$)

$$(\nabla\Psi(t,x))(1,0) = f(\Psi(t,x))$$



Then φ_t on $\Psi(N)$ is conj. to $\psi_s(t,y) = \psi(t+s,y)$ on N.

Define Ψ by $\Psi(t,y) = \varphi_t(y)$ – ~~this~~ is C^1 by general theory and $\nabla\Psi(0,0) : (t,v) \mapsto (f(0)+v)$ is an isomorphism ∴ by inv. fn thm Ψ is a diffeo on small nbhd.

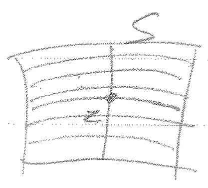
(*) Prop S a local section at $\bar{x}$, $\varphi_{t_0}(z_0) = \bar{x}$. Then $\exists$ open $W \ni z_0$, unique C^1 $\tau : W \to \mathbb{R}$ s.t. $\tau(z_0) = t_0$, $\varphi_{\tau(x)}(x) \in S$ $\forall\, x \in W$.

Pf Implicit fn thm, as for periodic orbits. ∎

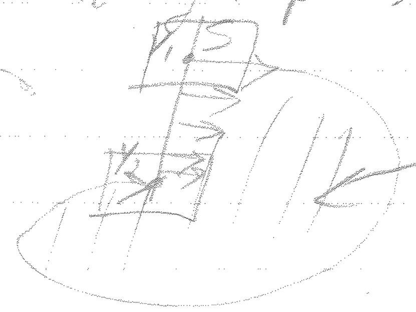
Back to PB thm. Spse $f(y) \neq 0$ $\forall\, y \in \omega(x)$. Fix $y \in \omega(x)$.

Lemma y is periodic.

Pf Fix $z \in \omega(y)$, let S be a local section at z, N a flow box nbhd of z. Now $z \in \omega(y) \Rightarrow \exists\, t_n \nearrow \infty$, $\varphi_{t_n}(y) \to z$. Get $s_n \approx t_n$ s.t. $\varphi_{s_n}(y) \to z$ & $\varphi_{s_n}(y) \in S$.



FACT $\{\varphi_t(y)\}$ crosses S in only 1 point. Indeed, if 2 crossings get



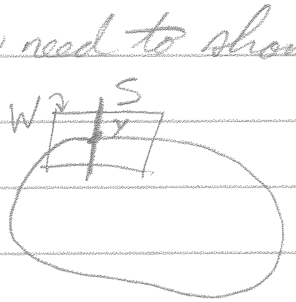
pos. inv.

So after coming near y_2, trajectory of x can never come near y_1.

KEY USE OF 2-DIMENSIONALITY

Thus $\varphi_{s_1}(y) = \varphi_{s_2}(y) = z \Rightarrow y$ periodic ∎

Now need to show $\omega(x) = \omega(y)$.



By Prop ⊛ $\exists\, \tau : W \to \mathbb{R}$ s.t.

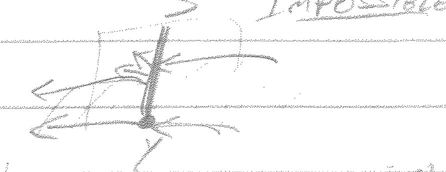
$\varphi_{\tau(z)}(z) \in S \ \ \forall z \in W, \ \ \tau(y) = T$

Let $\hat{\tau} = \sup_{z \in W} \tau(z)$.

$y \in \omega(x) \Rightarrow \varphi_{t_n}(x) \to y$.

For large n, $\varphi_{t_n}(x) \in W \Rightarrow \varphi_{s_n}(x) \in S$.

By the above, $s_{n+1} - s_n \leq \hat{\tau}$.



① $d(\varphi_{s_n}(x), y) \to 0$

② $d(\varphi_t(x), \omega(y)) \to 0$.

(Fix $\varepsilon > 0$, let $\delta > 0$ be s.t.

$|x - u| < \delta$ & $|t| \leq \hat{\tau} \Rightarrow |\varphi_t(x) - \varphi_t(u)| < \varepsilon$)

$\therefore \varphi_{s_n}(x) \to y$ monotonically

▨

Pf of gen PB.

$\boxed{\text{I}}$ $\omega(x)$ no reg $\Rightarrow$ = crit pt

$\boxed{\text{II}}$ only reg $\Rightarrow$ = per orbit

$\boxed{\text{III}}$ If $\omega(x)$ has both reg & crit pts, then = $\bigcup$ lim orbits + crit pts.

$y \in$ a limit orbit.

If $\omega(y)$ contains a reg pt, use flow box & section as before, deduce that $\omega(x) = \omega(y)$ is periodic.

(orbit of y can only cross local section near z once)

Thus $\omega(y)$ = single crit pt, same for $\alpha(y)$.

Now $y \in \omega(x)$ reg, sec $S \ni y$, ~~then~~ $\varphi_{t_n}(x) \in S$ are all on one side of y. Further arguments (omitted here) imply $\omega(x) = \bigcup$ finitely many separatrices & crit pts.

Cod/Lev Thm 3.2

▨

MORAL No chaos in dimension 2.

$\boxed{Q}$ What about ~~statistical~~ behaviour? How much ~~time~~ does $\{\varphi_t(x)\}$ spend near different parts of $\omega(x)$? Given $y \in \omega(x)$, $\varepsilon > 0$, what is $\lim_{T\to\infty} \frac{1}{T} \lambda\{t \in [0,T] \mid \varphi_t(x) \in B(y,\varepsilon)\}$?

More generally, define a measure μ by

$$\mu(E) = \lim_{T\to\infty} \frac{1}{T} \lambda\{t \in [0,T] \mid \varphi_t(x) \in E\}$$

Is μ well-defined? What properties does it have? Does it depend on x?

Dimension 2 – either spend almost all ~~time~~ near a fixed point, or converge to an periodic orbit

Dimension ≥ 3. Life is much more complicated – show slides for Lorenz system.