8. Infinite series of numbers

The following from Calculus II is almost all review, so we will move quickly. You may need to read it carefully several times. You could also look up several of these topics on wikipedia.

From Calculus II: An 'infinite series' is an expression of the form

$$\sum_{k=m}^{\infty} a_k \;=\; a_m + a_{m+1} + a_{m+2} + \cdots \qquad (*)$$

Let us call this expression (*). The a_k here are real (or complex) numbers.

What does expression (*) mean? In fact we shall see shortly that the expression means two things.

Usually $m = 0$ or 1, that is, (*) usually is

$$a_0 + a_1 + a_2 + \cdots$$

or

$$a_1 + a_2 + a_3 + \cdots .$$

We call the number a_k the *kth term in the series.* Sometimes we will be sloppy and write $\sum_k a_k$ when we mean (*).

The most important question about an infinite series, just as for an infinite sequence, is 1) does the series converge? and 2) if it converges, what is its sum? We will explain these in a minute.

In fact an expression like (*) has two meanings:

Meaning # 1: A 'formal sum'. That is, it is a way to indicate that we are thinking about adding up all these numbers in the expression (*), in the order given. It does not mean that these numbers do add up.

Before we go to Meaning # 2, let me say how you 'add up all the numbers in an infinite series'. To do this, we define the *nth partial sum* s_n to be the sum of the first n terms in the series. In this way we get a *sequence*

$$s_1, s_2, s_3, \cdots$$

called the *sequence of partial sums.* For example, for the series $\sum_{k=0}^{\infty} a_k$, we have $s_n = \sum_{k=0}^{n-1} a_k$. We say the original series *converges* if the *sequence* $\{s_n\}$ converges. If it does not converge then we say it *diverges.* We call $\lim_{n\to\infty} s_n$ the *sum of the series* if this limit exists.

Meaning # 2: $\sum_{k=m}^{\infty} a_k \;=\; \lim_{n\to\infty} s_n$ if this limit exists.

- Cauchy test: $\sum_k a_k$ converges iff given $\epsilon > 0$ there exists an $N \geq 0$ such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$ whenever $m > n \geq N$.

[Proof: Since $s_m - s_n = a_{n+1} + a_{n+2} + \cdots + a_m$, this is just saying that the partial sums $s_n = \sum_{k=1}^{n} a_k$ are a Cauchy sequence. And we know from Theorem 4.8 that a sequence converges iff it is a Cauchy sequence.]

- In a sum like $\sum_{k=1}^{\infty} \frac{1}{(k+1)k}$, the k is a 'dummy index'. That is, it is only used internally inside the sum, and we can feel free to change its name, to $\sum_{j=1}^{\infty} \frac{1}{(j+1)j}$, for example,
- In a series $\sum_{k=m}^{\infty} a_k$ let us call m the 'starting index'. Thus for example, the starting index of $\sum_{k=2}^{\infty} \frac{k-1}{k^2}$ is 2. Any series can be 'renumbered' so that its starting index is 0. That is, any infinite series may be rewritten as $\sum_{k=0}^{\infty} a_k$. For example, $\sum_{k=m}^{\infty} a_k$, which is the same as $a_m + a_{m+1} + a_{m+2} + \cdots$, can be relabelled by letting $j = k - m$, or equivalently $k = j + m$. Then $\sum_{k=m}^{\infty} a_k = \sum_{j=0}^{\infty} a_{j+m}$.

Example: Rewrite $\sum_{k=2}^{\infty} \frac{k-1}{k^2}$ as a series $\sum_{k=0}^{\infty} a_k$.

Solution. Letting $j = k - 2$, so that $k = j + 2$, the sum becomes

$$\sum_{j=0}^{\infty} \frac{j+2-1}{(j+2)^2} = \sum_{j=0}^{\infty} \frac{j+1}{(j+2)^2}$$

Of course j is 'dummy' so we can rewrite this as $\sum_{k=0}^{\infty} \frac{k+1}{(k+2)^2}$.

There is no reason of course why we chose 0 for the starting index. One can make all series begin with the starting index 1 if you wanted to, by a similar trick (let $j = k-m+1$ or equivalently $k = j+m-1$). However it is convenient to fix one starting index, so it may as well be 0. Many of the following results are therefore phrased in terms of series $\sum_{k=0}^{\infty} a_k$.

- **Geometric series:** This is a series of form $c + cx + cx^2 + cx^3 + \cdots$, or $\sum_{k=0}^{\infty} cx^k$, for constants c and x. We call x the 'constant ratio' of the geometric series. Note that if you divide any term in the series by the previous term, you get x. We usually assume $c \neq 0$, otherwise this is the trivial series with sum 0.

  The MAIN FACT about geometric series, is that such a series converges if and only if $|x| < 1$, and in that case its sum is $\frac{c}{1-x}$.

  [Proof: As above assume that $c \neq 0$, else the result is obvious. The sum of the first n terms is

$$s_n = c + cx + cx^2 + \cdots + cx^{n-1} = c(1 + x + x^2 + \cdots + x^{n-1}) \ .$$

There is a well known result in algebra which says that

$$(1+x+x^2+\cdots+x^{n-1})(1-x)=1-x^n$$

(to prove it multiply out the parentheses and cancel). Thus if $x \neq 1$ then $1+x+x^2+\cdots+x^{n-1}=\frac{1-x^n}{1-x}$, so that

$$s_n=c+cx+cx^2+\cdots+cx^{n-1}=c\frac{1-x^n}{1-x}\ .$$

This is the important formula for the sum of n terms of a geometric series. The only thing that depends on n on the right hand side here is the x^n, which converges to 0 if $|x|<1$, and diverges otherwise. Thus

$$\lim_{n\to\infty} s_n=\lim_{n\to\infty} c\frac{1-x^n}{1-x}\ =\ c\frac{1}{1-x}, \qquad |x|<1.$$

If $x=1$ then $s_n=c+c+\cdots+c$ (n times) which equals nc, which diverges as $n\to\infty$. Summarizing: $\lim\limits_{n\to\infty} s_n$ converges to $c\frac{1}{1-x}$ if $|x|<1$, but if $|x|\geq 1$ then $\{s_n\}$ diverges, so that the original series diverges. By Meaning # 2 and the discussion around it (the definition of convergence/divergence of series, we have therefore proved this MAIN FACT.]

- FACT: If $\sum_{k=0}^\infty\ a_k$ and $\sum_{k=0}^\infty\ b_k$ both converge, and if c is a constant, then:
- $\sum_{k=0}^\infty\ (a_k+b_k)$ converges, with sum $\sum_{k=0}^\infty\ a_k+\sum_{k=0}^\infty\ b_k$;
- $\sum_{k=0}^\infty\ (a_k-b_k)$ converges, with sum $\sum_{k=0}^\infty\ a_k-\sum_{k=0}^\infty\ b_k$;
- $\sum_{k=0}^\infty\ (ca_k)$ converges, with sum $c\sum_{k=0}^\infty\ a_k$.

[Proof: We just prove the first and third, the second is quite similar. The nth partial sum of $\sum_{k=0}^\infty\ (a_k+b_k)$ is $\sum_{k=0}^{n-1}\ (a_k+b_k)\ =\ \sum_{k=0}^{n-1} a_k\ +\ \sum_{k=0}^{n-1} b_k$. By Meaning # 2, $(\sum_{k=0}^{n-1} a_k)$ converges, as $n\to\infty$, to $\sum_{k=0}^\infty\ a_k$; and similarly for the b_k's. By a fact about sums of limits of *sequences* from Chapter 4 (Fact 9 (1)), this converges, as $n\to\infty$, to $\sum_{k=0}^\infty\ a_k+\sum_{k=0}^\infty\ b_k$. So the first assertion follows by the definition of convergence and Meaning # 2.

Similarly the nth partial sum of $\sum_{k=0}^\infty\ (ca_k)$ is $\sum_{k=0}^{n-1}\ (ca_k)=c\sum_{k=0}^{n-1} a_k$. By Fact 9 (4) in Chapter 4, this converges, as $n\to\infty$, to $c\sum_{k=0}^\infty\ a_k$. So the third assertion follows by the definition of convergence and Meaning # 2.]

- For any positive integer m we can write $\sum_{k=0}^\infty\ a_k=(a_0+a_1+\cdots+a_{m-1})+\sum_{k=m}^\infty\ a_k$.

Indeed $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=m}^{\infty} a_k$ converges. If these series converge, then their sum also obeys the rule:

$$\sum_{k=0}^{\infty} a_k = (a_0 + a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k \ .$$

[This is because the nth partial sum of the $\sum_{k=0}^{\infty} a_k$ series, and the nth partial sum of the $\sum_{k=m}^{\infty} a_k$ series differ by a fixed constant, namely $a_0 + a_1 + \cdots + a_{m-1}$.]

- From the last fact it follows that the 'first few terms' of a series, do not affect whether the series converges or not. It will affect the sum though.

- **The Divergence Test:** If $\lim\limits_{k \to \infty} a_k \neq 0$ then the series $\sum_k a_k$ diverges.

  A restatement of the Divergence Test (the contrapositive): If $\sum_k a_k$ converges, then $\lim\limits_{k \to \infty} a_k = 0$.

  [Beware: If $\lim\limits_{k \to \infty} a_k = 0$ we cannot conclude that $\sum_k a_k$ converges.

  [Proof: Suppose that $\sum_{k=0}^{\infty} a_k = s$. If s_n is the nth partial sum then $s_n \to s$ as $n \to \infty$ (see Meaning # 2 above). Clearly $s_{n+1} \to s$ too, as $n \to \infty$. Thus $a_n = s_{n+1} - s_n \to s - s = 0$.]

Homework 11.

(1) If $\sum_k a_k$ converges define the *tail* of the series to be the sequence whose nth term is $\sum_{k=n}^{\infty} a_k$. Prove the tail converges to 0 as $n \to \infty$.

(2) If $y \in \mathbb{R}$ write $[y] = \max\{n \in \mathbb{Z} : n \leq y\}$. If $x \in [0, 1)$ write $a_1 = [10x], a_2 = [100(x - \frac{a_1}{10})], a_3 = [1000(x - \frac{a_1}{10} - \frac{a_2}{100})], \cdots$. Prove that $0 \leq a_k \leq 9$ for each k and that $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges, Prove that $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$. Prove that there is no N such that $a_k = 9$ for all $k \geq N$.

(3) Continuing with the last question, if $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$ where $a_n \in \{0, 1, \cdots, 9\}$ then we write $x = 0.a_1a_2a_3\cdots$ and call this a *decimal expansion* of x (or base 10 expansion). Prove that the decimal expansion of $x \in [0, 1)$ is unique provided that we insist that there is no N such that $a_k = 9$ for all $k \geq N$.

(4) Prove that any nonzero real number may be written as $\pm \sum_{n=N}^{\infty} \frac{a_n}{10^n}$ for some integer N, and with $a_n \in \{0, 1, \cdots, 9\}$ for all n, and $a_N \neq 0$; and that this representation is unique provided that we keep the 'recurring 9 convention' in question (2). [Remark: Questions 2–4 can be found in many places on the internet under 'decimal expansion'; or see e.g. the Appendix B to Tao's Analysis I. It can also be done for any 'base', not just base 10, with almost identical proofs.]

9. Nonnegative Series, and tests for series convergence.

- A series $\sum_k a_k$ is called a *nonnegative series* if all the terms a_k are ≥ 0.

- For a nonnegative series, the sequence $\{s_n\}$ of the partial sums is a nondecreasing (or increasing) sequence. Indeed if $s_n = a_0 + a_1 + \cdots + a_{n-1}$ say, then $s_{n+1} = a_0 + a_1 + \cdots + a_{n-1} + a_n$, so that $s_{n+1} - s_n = a_n \geq 0$.

  Therefore, by Theorem 4.4 in Chapter 4 for monotone sequences, we deduce:

  - FACT: A nonnegative series converges if and only if the sequence $\{s_n\}$ of partial sums is bounded above; and in any case the sum of a nonnegative series equals the least upper bound of the sequence $\{s_n\}$.

  Thus the sum of a nonnegative series always 'exists', but will be $+\infty$ if (s_n) is unbounded. In the latter case $\sum_k a_k = \lim_n s_n = +\infty$.

  Repeating, a nonnegative series converges if and only if the $\{s_n\}$ sequence is bounded above. The latter happens if and only if the sum of the series is finite. Thus to indicate that a nonnegative series converges we often simply write $\sum_k a_k < \infty$.

- **Example:** The HARMONIC SERIES is the important series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k} .$$

  This is a nonnegative series, so to see if it converges we need only check if the sequence $\{s_n\}$ is bounded above, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$. A trick to do this is to look at $\int_1^{n+1} \frac{1}{x}\, dx$, interpreted as the shaded area in the graph below [Picture drawn in class]. This shaded area is less than the area of the n rectangles shown. Hence

$$1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + \cdots 1 \cdot \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\, dx.$$

  So $s_n \geq \ln(n+1) - \ln(1) \to \infty$ as $n \to \infty$.
  Thus the harmonic series diverges; it has sum $+\infty$.

- The trick used in the previous example can be used in the same way to prove:

  The Integral Test: If $f(x)$ is a continuous decreasing positive function defined on $[1, \infty)$ [Picture drawn in class], then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges (i.e. is finite).

- **p-series.** An almost identical argument shows that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$. These are called 'p-series'.

- **Basic Comparison Test:** Suppose that $0 \leq a_k \leq b_k$ for all k.
  1) If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.
  2) If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.

  [Proof: We have $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$. So for 1), if $(\sum_{k=1}^{n} b_k)$ is bounded above then $(\sum_{k=1}^{n} a_k)$ is bounded above. That is, by the third 'bullet' in this section, if $\sum_k b_k$ converges, then $\sum_k a_k$ converges.

  Note that 2) is the contrapositive to 1).]

- **Limit Comparison Test:** Suppose that $\sum_k a_k$ and $\sum_k b_k$ are nonnegative series. If $\limsup_{k\to\infty} \frac{a_k}{b_k} < \infty$, and if $\sum_k b_k$ converges then $\sum_k a_k$ converges. If $\liminf_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.

  [Proof: Suppose that $s = \limsup_{k\to\infty} \frac{a_k}{b_k} < \infty$. By (6) in Theorem 4.14 there exists N with $\frac{a_k}{b_k} < s+1$ for $k \geq N$. So $a_k < (s+1)b_k$ for $k \geq N$. By the Basic Comparison Test, if $\sum_k b_k$ converges then $\sum_k a_k$ converges.

  If $s > 0$ then by the liminf variant of (6) in Theorem 4.14 there exists N with $\frac{a_k}{b_k} > s - \frac{s}{2} = \frac{s}{2}$ for $k \geq N$. So $a_k > \frac{s}{2} b_k$ for $k \geq N$. By the Basic Comparison Test, if $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.]

- **Root Test:** Suppose that $\sum_k a_k$ is a nonnegative series with $\limsup_{k\to\infty} (a_k)^{\frac{1}{k}} = r$. If $0 \leq r < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.

  [Proof: Suppose that $r < c < 1$. By (6) in Theorem 4.14 (with $\epsilon = c - r$) there exists N with $a_k^{\frac{1}{k}} < c$ for $k \geq N$. So $a_k < c^k$. Now $\sum_k c^k$ converges (geometric series), so by the Basic Comparison Test, $\sum_k a_k$ converges.

  If $1 < c < r$ then by (6) in Theorem 4.14 there are infinitely many k with $a_k^{\frac{1}{k}} > c$, or equivalently, $a_k > c^k > 1$. So $\sum_k a_k$ diverges by the Divergence Test.]

- **Ratio Test:** Suppose that $\sum_k a_k$ is a nonnegative series with $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = R$ and $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = r$. If $0 \leq R < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.

  [Proof: Suppose that $R < c < 1$. By (6) in Theorem 4.14 (with $\epsilon = c - r$) there exists N with $\frac{a_{k+1}}{a_k} < c$, for $k \geq N$. So $a_{N+1} < ca_N$, $a_{N+2} < ca_{N+1} < c^2 a_N$, etc. Generally $a_{N+k} < c^k a_N$. Now $\sum_k c^k a_N$ converges (geometric series), so by the Basic Comparison Test, $\sum_k a_{N+k}$ converges. So $\sum_k a_k$ converges.

A similar argument does the case $r > c > 1$. By the liminf variant of (6) in Theorem 4.14 there exists N with $\frac{a_{k+1}}{a_k} > c$ for $k \geq N$. In this case $a_{N+k} > c^k a_N$. Now $c^k a_N \to \infty$ as $k \to \infty$, so by the Divergence test, $\sum_k a_{N+k}$ diverges. So $\sum_k a_k$ diverges.]

- From Homework 12 Question 3 (a) below it is easy to see that the root test is more powerful theoretically than the ratio test. That is if the ratio test works to prove convergence or divergence, then the root test would give the same conclusion. However the converse is not true (if the root test works, the ration test might not). See Homework 12 Question 2.

- **Condensation test** Suppose that $a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0$, and that $\lim_k a_k = 0$. Then $\sum_k a_k$ converges iff $\sum_k 2^k a_{2^k}$ converges.

  [Proof: Let $s_n = \sum_{k=0}^n a_k$ and $t_n = \sum_{k=0}^n 2^k a_{2^k}$. If $n < 2^k$ then

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = t_k. \end{aligned}$$

  Thus if (t_k) is bounded then (s_n) is bounded; and so by the fact in the paragraph before the harmonic series a few pages back, $\sum_k a_k$ converges if $\sum_k 2^k a_{2^k}$ converges. If $n > 2^k$ then

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} t_k. \end{aligned}$$

  So $t_k \leq 2 s_n$. Thus if (s_n) is bounded then (t_k) is bounded; and so by the fact in the paragraph before the harmonic series a few pages back, $\sum_k 2^k a_{2^k}$ converges if $\sum_k a_k$ converges.

- As an application of the condensation test note that the harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges because $\sum_k 2^k \frac{1}{2^k} = \sum_k 1 = \infty$.

Homework 12.

(1) Test for convergence (giving reasons): (a) $\sum_{n=1}^\infty \frac{n^3}{2^n}$, (b) $\sum_{n=2}^\infty \frac{1}{(\log n)^2}$, (c) $\sum_{n=1}^\infty n^{-1-\frac{1}{n}}$, (d) $\sum_{n=2}^\infty \frac{1}{(\log n)^{\log n}}$, (e) $\sum_{n=1}^\infty (\sqrt[n]{n} - 1)^n$, (f) $\sum_{n=1}^\infty \frac{\sqrt{1+n^2} - n}{\sqrt{n}}$.

(2) Suppose $\sum_{n=1}^\infty a_n$ is the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \cdots$. Find $\limsup_n \frac{a_{n+1}}{a_n}$, $\liminf_n \frac{a_{n+1}}{a_n}$ and $\limsup_n a_n^{\frac{1}{n}}$. Can we conclude that $\sum_n a_n$ converges using the ratio test? Using the root test?

(3) (a*) Show (or look up) that

$$\liminf_n |s_{n+1}/s_n| \le \liminf_n |s_n|^{1/n} \le \limsup_n |s_n|^{1/n} \le \limsup_n |s_{n+1}/s_n|,$$

for any sequence $\{s_n\}$ in $\mathbb{R}\setminus\{0\}$.

(b) From (a) it is easy to see that the root test is more powerful theoretically than the ratio test. For example, deduce that if $\{|s_{n+1}/s_n|\}$ converges then $\{|s_n|^{1/n}\}$ converges to the same limit.

(c) Calculate $\lim_{n\to\infty} (n!)^{1/n}$.

Try do of as many as possible of the questions in Homework 12 Question 1 using the condensation test.

10. Absolute and conditional convergence

A series $\sum_k a_k$ is called *absolutely convergent* if $\sum_k |a_k|$ converges. Recall that a_k here could be a complex number (you can view complex numbers as elements of $\mathbb{R}^2$ here).

- Any absolutely convergent series is convergent.

  [Proof: By the Cauchy test above (the first and second page of this chapter), since $\sum_k |a_k|$ converges, given $\epsilon > 0$ there exists an $N \ge 0$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \le |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \epsilon, \qquad m > n \ge N.$$

  By the Cauchy test again (but used in the other direction), $\sum_k a_k$ converges.]

- The converse is false, a series may be convergent, but not absolutely convergent. Such a series is called *conditionally convergent.*

- If $\sum_{k=1}^\infty a_k$ converges absolutely then $|\sum_{k=1}^\infty a_k| \le \sum_{k=1}^\infty |a_k|$.

  [Proof. Note $|\sum_{k=1}^n a_k| \le \sum_{k=1}^n |a_k|$. Take the limit as $n \to \infty$ in this inequality, and use Fact 4 about sequences from Chapter 4 to see that $|\sum_{k=1}^\infty a_k| \le \sum_{k=1}^\infty |a_k|$.]

- **The Alternating Series Test (a.k.a. Leibniz Test)/Alternating Series approximation:** Suppose that $a_0 \ge a_1 \ge a_2 \ge \cdots$, and that $\lim_k a_k = 0$. Then $a_0 - a_1 + a_2 - a_3 + \cdots$ (which in sigma notation is $\sum_{k=0}^\infty (-1)^k a_k$) converges, and moreover $|s_n - \sum_{k=0}^\infty (-1)^k a_k| \le a_n$ for all n, where s_n is the nth partial sum $\sum_{k=0}^{n-1} (-1)^k a_k$.

  [Proof: We have noticed before that if $m > n$ then $s_m - s_n = b_n + b_{n+1} + \cdots + b_{m-1}$, where b_k is the kth term. Here $b_k = (-1)^k a_k$, where a_k is as above. Note that $a_{n+k} - a_{n+k+1} \geq 0$, so $a_n - a_{n+1} + a_{n+2} - a_{n+3} + \cdots \geq 0$. Hence

$$|s_m - s_n| = a_n - a_{n+1} + a_{n+2} - \cdots = a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) - \cdots \leq a_n,$$

since $a_{n+k+1} \leq a_{n+k}$. It follows that (s_n) is Cauchy, so convergent. That is, $\sum_{k=0}^{\infty} (-1)^k a_k$ converges. Letting $m \to \infty$ and using Fact 4 about sequences from Chapter 4, we have $|\sum_{k=0}^{\infty} (-1)^k a_k - s_n| \leq a_n$.]

Example. Approximate the sum of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$ with an error of less than 0.001.

Solution. The error in using s_n to approximate the sum is $< |a_{n+1}| = \frac{1}{(n+1)^4}$ (note the starting index of this series is 1, not 0, which means we have to change the formula in the last result slightly). Now $\frac{1}{(n+1)^4} < 0.001$ if $(n+1)^4 > 1000$. Choosing $n = 5$ will work. So an approximation with an error of less than 0.001 is $s_5 = \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k^4} = 0.94754$ (calculator).

- **Exercise.** Test for convergence/absolute or conditional convergence: $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!}$.

Next we prove the 'fundamental fact about power series' (this was already mentioned in the last 'bullet' before Theorem 4.14). Namely consider the power series $\sum_{k=0}^{\infty} c_k x^k$ (here x and the c_k can be complex numbers if you wish). We set $R = \infty$ if $\text{limsup}_n |c_n|^{\frac{1}{n}} = 0$, and $R = 0$ if $\text{limsup}_n |c_n|^{\frac{1}{n}} = \infty$. Otherwise set $R = \frac{1}{\text{limsup}_n |c_n|^{\frac{1}{n}}}$. This is the *radius of convergence* of the power series. In the homework you are asked to show that in place of $\text{limsup}_n |c_n|^{\frac{1}{n}}$ you can use $\text{limsup}_n |\frac{s_{n+1}}{s_n}|$.

Theorem 10.1. *Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series, and define R as above. Then $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely if $|x| < R$, and it diverges whenever $|x| > R$.*

Proof. Set $a_k = c_k x^k$ and we are going to apply the root test to $\sum_k |a_k|$ to check if that converges. Note $|a_n|^{\frac{1}{n}} = |c_n|^{\frac{1}{n}} |x|$. By Theorem 4.14 (4) we have

$$\text{limsup}_n |a_n|^{\frac{1}{n}} = \text{limsup}_n |c_n x^n|^{\frac{1}{n}} = \text{limsup}_n |c_n|^{\frac{1}{n}} |x| = |x| \,\text{limsup}_n |c_n|^{\frac{1}{n}} = \frac{|x|}{R}$$

if $0 < R < \infty$. Note that $\text{limsup}_n |c_n|^{\frac{1}{n}} |x| = 0$ if $R = \infty$, and by the root test $\sum_{k=0}^{\infty} |a_k| = \sum_{k=0}^{\infty} |c_k| \, |x|^k$ converges here, so $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely. Similarly, if $0 < R < \infty$ and $|x| < R$ then by the root test $\sum_{k=0}^{\infty} |c_k| \, |x|^k$ converges and so $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely. If $|x| > R$ then by the root test $\sum_{k=0}^{\infty} |c_k x^k|$ diverges, and indeed if one looks at the proof of the root test one sees that in this case $|c_k x^k| > 1$ for infinitely many k. Thus $\sum_{k=0}^{\infty} c_k x^k$ diverges by the divergence test. $\square$

- If $\sum_{k=1}^\infty a_k$ is a series, and $f : \mathbb{N} \to \mathbb{N}$ is a bijection (that is, is one-to-one and onto), then the series $\sum_{k=1}^\infty a_{f(k)}$ is called a 'rearrangement' of $\sum_{k=1}^\infty a_k$. It is not hard to see that a rearrangement of a convergent series need not converge.

  Theorem Any 'rearrangement' of an absolutely convergent series is convergent and has the same sum.

  [Proof: Suppose that $\sum_{k=1}^\infty a_k$ is absolutely convergent, and that $\epsilon > 0$ is given. Then $\sum_{k=1}^\infty a_k$ is convergent too, with sum s say, so $s_n \to s$ where $s_n = \sum_{k=1}^n a_k$. Choose N with $\sum_{k=N}^\infty |a_k| < \frac{\epsilon}{2}$ (see HW 11 Q 1), and $|s_n - s| < \frac{\epsilon}{2}$ for $n \geq N$. Choose $M \in \mathbb{N}$ such that $\{1, 2, \cdots, N\} \subset \{f(1), f(2), \cdots, f(M)\}$ (this is possible since f is bijective–why?). Let $t_n = \sum_{k=1}^n a_{f(k)}$. If $n > M$ we have that all the terms a_k in $s_n = \sum_{k=1}^n a_k$ with $k \leq N$ cancel with terms $a_{f(j)}$ in $t_n = \sum_{k=1}^n a_{f(k)}$. So (and also using the triangle inequality),
  $$|s_n - t_n| \leq |a_{N+1}| + |a_{N+2}| + \cdots < \frac{\epsilon}{2}.$$
  So
  $$|t_n - s| \leq |t_n - s_n| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad n \geq M.$$
  Hence $t_n \to s$. So the rearrangement is convergent and has the same sum.]

- Dirichlet test: Let $\sum_k a_k$ be a series whose partial sums form a bounded sequence. Suppose (b_n) is a decreasing sequence with limit 0. Then $\sum_k a_k b_k$ converges.
- Abel's test: Suppose that $\sum_k a_k$ converges and b_n is a monotonic convergent sequence. Then $\sum_k a_k b_k$ converges.

We will not prove the last two results (see e.g. Apostol's text for proofs, they are not hard).

Adding parentheses: Let $\sum_{k=1}^\infty a_k$ be a series, and suppose that $1 \leq n_1 < n_2 < \cdots$ are integers. Let $b_1 = \sum_{k=1}^{n_1} a_k, b_2 = \sum_{k=n_1+1}^{n_2} a_k, b_3 = \sum_{k=n_2+1}^{n_3} a_k, \cdots$. We call $\sum_{k=1}^\infty b_k$ a series obtained from $\sum_{k=1}^\infty a_k$ *by adding parentheses.*

Theorem 10.2. *If $\sum_{k=1}^\infty a_k$ converges, and b_k, n_k are as above, then $\sum_{k=1}^\infty b_k$ converges and has sum $\sum_{k=1}^\infty a_k$. Also:*

(a) *If $\sum_{k=1}^\infty a_k$ is a nonnegative series, then $\sum_{k=1}^\infty a_k$ converges iff $\sum_{k=1}^\infty b_k$ converges.*

(b) *If the sequence $(n_{k+1} - n_k)$ is bounded and $\lim_n a_n = 0$, then $\sum_{k=1}^\infty a_k$ converges iff $\sum_{k=1}^\infty b_k$ converges.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. Note that $b_1 = s_{n_1}, b_1 + b_2 = s_{n_2}, \cdots$, generally $b_1 + b_2 + \cdots + b_N = s_{n_N}$. So the partial sums of $\sum_{k=1}^{\infty} b_k$ converge to $\sum_{k=1}^{\infty} a_k$, since they are a subsequence of (s_n), and subsequences of convergent sequences converge with the same limit.

(a) The one direction of the 'iff' is proved above. For the other direction, the key point is the principle (2nd bullet in Section 9) that a nonnegative series converges if and only if its associated sequence of partial sums is bounded above. Since $n_1 < n_2 < \cdots$ are integers, they must be unbounded above. So for any n, there exists a N with $n < n_N$. Then

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n_N} a_k = b_1 + b_2 + \cdots + b_N \leq \sum_{k=1}^{\infty} b_k.$$

So the partial sums of $\sum_{k=1}^{\infty} a_k$ are bounded above by $\sum_{k=1}^{\infty} b_k$. Hence if $\sum_{k=1}^{\infty} b_k$ converges then so does $\sum_{k=1}^{\infty} a_k$ (by the principle mentioned above (second bullet in Section 9)).

(b) The one direction of the 'iff' is proved above (a). For the other direction, suppose that $n_{k+1} - n_k \leq C$ for all k, and suppose that $\sum_{k=1}^{\infty} b_k$ converges with sum s. Let K_1 be such that $|s - \sum_{k=1}^{N} b_k| < \epsilon$ for $N \geq K_1$. Since $a_n \to 0$ there exists a $K_2 \geq n_{K_1}$ such that $|a_n| < \frac{\epsilon}{C}$ whenever $n \geq K_2$. Choose K such that $n_K \geq K_2 \geq n_{K_1}$. Suppose that $n \geq n_K \geq n_{K_1}$. Then there exists a p with $n_K \leq n_p \leq n < n_{p+1}$. So

$$|s - \sum_{k=1}^{n} a_k| \leq |s - \sum_{k=1}^{p} b_p| + |a_{n_p+1}| + |a_{n_p+2}| + \cdots + |a_n| < \epsilon + C\frac{\epsilon}{C} = 2\epsilon, \qquad n \geq n_K,$$

since $n_p \geq n_K \geq K_2$. So $\sum_{k=1}^{\infty} a_k$ converges to s. □

If s_n is the nth partial sum of a series $\sum_k a_k$, define the *Cesáro means* to be the sequence (σ_n) defined by $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. We say that $\sum_k a_k$ is *Cesáro summable* if the sequence (σ_n) converges, and in this case $\lim_n \sigma_n$ is called the *Cesáro sum* of the series $\sum_k a_k$.

Theorem 10.3. *If $\sum_k a_k$ converges with sum s, then it is Cesáro summable and its Cesáro sum is s.*

Proof. Let $t_n = s_n - s, \tau_n = \sigma_n - s$. Then it is easy to see that $\tau_n = \frac{1}{n}(t_1 + t_2 + \cdots + t_n)$. Let $\epsilon > 0$ be given. Since $t_n \to 0$ the sequence is bounded: $|t_n| \leq K$ for all n, say, and there is an N with $|t_n| \leq \epsilon$ for $n \geq N$. Then

$$|\tau_n| \leq \frac{1}{n}(|t_1| + |t_2| + \cdots + |t_N|) + \frac{1}{n}(|t_{N+1}| + \cdots + |t_n|) < \frac{NK}{n} + \epsilon, \qquad n \geq N.$$

From this it is clear that $\tau_n = \sigma_n - s \to 0$. So $\sigma_n \to s$. □

Homework 13.

(1) Suppose that $z_n = a_n + ib_n$, where $a_n,\ b_n \in \mathbb{R}$. Show that (a) $\sum_n z_n$ is convergent if and only if both $\sum_n a_n$ and $\sum_n b_n$ are convergent; (b) $\sum_n z_n$ is absolutely convergent if and only if both $\sum_n a_n$ and $\sum_n b_n$ are absolutely convergent; (c) $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\sqrt{n}} + \frac{i}{n^2} \right)$ is convergent but not absolutely convergent.

(2) If $\sum_n a_n$ converges absolutely, show that $\sum_n a_n^2$ and $\sum_n \frac{a_n}{1+a_n}$ converge absolutely (you may assume if you wish that that no $a_n = -1$). If $\sum_n a_n$ diverges show $\sum_n n\, a_n$ diverges.

(3)* Prove or look up a proof of Riemann's result that any conditionally convergent series has the property that if $a \in \mathbb{R}$ is given, there is a rearrangement of the series which has sum a. Show also that there is a rearrangement of the series which diverges.

(4) Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Does $a_1 + a_2 + \cdots + a_{10} - a_1 + a_{11} + a_{12} + \cdots + a_{20} - a_2 + a_{21} + \cdots + a_{30} - a_3 + a_{31} + \cdots$ converge? If so, prove it, and find the sum; if not find a counterexample.

(5) Show that the radius of convergence of $\sum_{k=0}^{\infty} c_k x^k$ is $\lim_n |\frac{c_n}{c_{n+1}}|$ if this limit exists. You may assume c_k is never 0.

Also: do [Lay, 5th Ed] 8.1 Q 10, 11; 8.2 Q 9, 11; 8.3 Q 7. (These are the same numbers in Sections 32, 33, 34 in the 3rd or 4th Edition of [Lay]. Also, those who did not 'present' in class please read the proof of Theorem 10.2 and turn in an explanation in your own words of the proof.

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