8. Infinite series of numbers

From Calculus II: An 'infinite series' is an expression of the form

$$\sum_{k=m}^{\infty} a_k \ = \ a_m + a_{m+1} + a_{m+2} + \cdots \qquad (*)$$

Let us call this expression (*). The a_k here are real (or complex) numbers.

What does expression (*) mean? In fact we shall see shortly that the expression means two things.

Usually $m = 0$ or 1, that is, (*) usually is

$$a_0 + a_1 + a_2 + \cdots$$

or

$$a_1 + a_2 + a_3 + \cdots .$$

We call the number a_k the *kth term in the series.* Sometimes we will be sloppy and write $\sum_k a_k$ when we mean (*).

The most important question about an infinite series, just as for an infinite sequence, is 1) does the series converge? and 2) if it converges, what is its sum? We will explain these in a minute.

In fact an expression like (*) has two meanings:

Meaning # 1: A 'formal sum'. That is, it is a way to indicate that we are thinking about adding up all these numbers in the expression (*), in the order given. It does not mean that these numbers do add up.

Before we go to Meaning # 2, let me say how you 'add up all the numbers in an infinite series'. To do this, we define the *nth partial sum* s_n to be the sum of the first n terms in the series. In this way we get a *sequence*

$$s_1, s_2, s_3, \cdots$$

called the *sequence of partial sums.* For example, for the series $\sum_{k=0}^{\infty} a_k$, we have $s_n = \sum_{k=0}^{n-1} a_k$. We say the original series *converges* if the *sequence* $\{s_n\}$ converges. If it does not converge then we say it *diverges.* We call $\lim_{n\to\infty} s_n$ the *sum of the series* if this limit exists.

Meaning # 2: $\sum_k a_k \ = \ \lim_{n\to\infty} s_n$ if this limit exists.

- Cauchy test: $\sum_k a_k$ converges iff given $\epsilon > 0$ there exists an $N \geq 0$ such that $|a_{n+1} + a_{n+2} + \cdots + a_m| < \epsilon$ whenever $m > n \geq N$.

  [Proof: Since $s_m - s_n = a_{n+1} + a_{n+2} + \cdots + a_m$, this is just saying that the partial sums $s_n = \sum_{k=1}^{n} a_k$ are a Cauchy sequence. And we know from Theorem 4.8 that a sequence converges iff it is a Cauchy sequence.]

- In a sum like $\sum_{k=1}^{\infty} \frac{1}{(k+1)k}$, the k is a 'dummy index'. That is, it is only used internally inside the sum, and we can feel free to change its name, to $\sum_{j=1}^{\infty} \frac{1}{(j+1)j}$, for example,

- In a series $\sum_{k=m}^{\infty} a_k$ let us call m the 'starting index'. Thus for example, the starting index of $\sum_{k=2}^{\infty} \frac{k-1}{k^2}$ is 2. Any series can be 'renumbered' so that its starting index is 0. That is, any infinite series may be rewritten as $\sum_{k=0}^{\infty} a_k$.

For example, $\sum_{k=m}^{\infty} a_k$, which is the same as $a_m + a_{m+1} + a_{m+2} + \cdots$, can be relabelled by letting $j = k - m$, or equivalently $k = j + m$. Then $\sum_{k=m}^{\infty} a_k = \sum_{j=0}^{\infty} a_{j+m}$. There is no reason of course why we chose 0 for the starting index. One can make all series begin with the starting index 1 if you wanted to, by a similar trick (let $j = k - m + 1$ or equivalently $k = j + m - 1$).

- **Geometric series:** This is a series of form $c + cx + cx^2 + cx^3 + \cdots$, or $\sum_{k=0}^{\infty} cx^k$, for constants c and x. We call x the 'constant ratio' of the geometric series. Note that if you divide any term in the series by the previous term, you get x. We assume $c \neq 0$, otherwise this is the trivial series with sum 0.

  The MAIN FACT about geometric series, is that such a series converges if and only if $|x| < 1$, and in that case its sum is $\frac{c}{1-x}$.

- FACT: If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ both converge, and if c is a constant, then:
- $\sum_{k=0}^{\infty} (a_k + b_k)$ converges, with sum $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$;
- $\sum_{k=0}^{\infty} (a_k - b_k)$ converges, with sum $\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} b_k$;
- $\sum_{k=0}^{\infty} (ca_k)$ converges, with sum $c\sum_{k=0}^{\infty} a_k$.

  [Proof: We just prove the first and third, the second is quite similar. The nth partial sum of $\sum_{k=0}^{\infty} (a_k + b_k)$ is $\sum_{k=0}^{n-1} (a_k + b_k) = \sum_{k=0}^{n-1} a_k + \sum_{k=0}^{n-1} b_k$. By a fact about sums of limits of *sequences* from Chapter 4 (Fact 9 (1)), this converges, as $n \to \infty$, to $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$.

  Similarly the nth partial sum of $\sum_{k=0}^{\infty} (ca_k)$ is $\sum_{k=0}^{n-1} (ca_k) = c\sum_{k=0}^{n-1} a_k$. By Fact 9 (4) in Chapter 4, this converges, as $n \to \infty$, to $c\sum_{k=0}^{\infty} a_k$.]
- For any positive integer m we can write $\sum_{k=0}^{\infty} a_k = (a_0 + a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k$. Indeed $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=m}^{\infty} a_k$ converges. If these series converge, then their sum also obeys the rule:

$$\sum_{k=0}^{\infty} a_k = (a_0 + a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k \ .$$

- From the last fact it follows that the 'first few terms' of a series, do not affect whether the series converges or not. It will affect the sum though.

- **The Divergence Test:** If $\lim_{k\to\infty} a_k \neq 0$ then the series $\sum_k a_k$ diverges.

  A matching statement (the contrapositive): If $\sum_k a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

  [Beware: If $\lim_{k\to\infty} a_k = 0$ we cannot conclude that $\sum_k a_k$ converges.

  [Proof: Suppose that $\sum_{k=0}^{\infty} a_k = s$. If s_n is the nth partial sum then $s_n \to s$ as $n \to \infty$. Clearly $s_{n+1} \to s$ too, as $n \to \infty$. Thus $a_n = s_{n+1} - s_n \to s - s = 0$.]

9. Nonnegative Series, and tests for series convergence.

- A series $\sum_k a_k$ is called a *nonnegative series* if all the terms a_k are ≥ 0.

- For a nonnegative series, the sequence $\{s_n\}$ of the partial sums is a nondecreasing (or increasing) sequence. Indeed if $s_n = a_0 + a_1 + \cdots + a_{n-1}$ say, then $s_{n+1} = a_0 + a_1 + \cdots + a_{n-1} + a_n$, so that $s_{n+1} - s_n = a_n \geq 0$.

  Therefore, by a fact we saw in Theorem 4.4 in Chapter 4 for monotone sequences, the sum of the series equals the least upper bound of the sequence $\{s_n\}$ of partial sums. Thus the sum of the series always exists, but may be $+\infty$ if (s_n) is unbounded. In the latter case $\sum_k a_k = \lim_n s_n = +\infty$.

  More importantly, a nonnegative series converges if and only if the $\{s_n\}$ sequence is bounded above. The latter happens if and only if the sum of the series is finite. Thus to indicate that a nonnegative series converges we often simply write $\sum_k a_k < \infty$.

  The Integral Test: If $f(x)$ is a continuous decreasing positive function defined on $[1, \infty)$ [Picture drawn in class], then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges (i.e. is finite).

- **p-series.** An almost identical argument shows that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$. These are called 'p-series'.

- **Basic Comparison Test:** Suppose that $0 \leq a_k \leq b_k$ for all k.
  1) If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.
  2) If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.

  [Proof: We have $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$. So for 1), if $(\sum_{k=1}^{n} b_k)$ is bounded above then $(\sum_{k=1}^{n} a_k)$ is bounded above. That is, by the third 'bullet' in this section, if $\sum_k b_k$ converges, then $\sum_k a_k$ converges.

Note that 2) is the contrapositive to 1).]

- **Limit Comparison Test:** Suppose that $\sum_k a_k$ and $\sum_k b_k$ are nonnegative series. If $\limsup_{k\to\infty} \frac{a_k}{b_k} < \infty$, and if $\sum_k b_k$ converges then $\sum_k a_k$ converges. If $\liminf_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.
- **Root Test:** Suppose that $\sum_k a_k$ is a nonnegative series with $\limsup_{k\to\infty} (a_k)^{\frac{1}{k}} = r$. If $0 \leq r < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.
- **Ratio Test:** Suppose that $\sum_k a_k$ is a nonnegative series with $\limsup_{k\to\infty} \frac{a_{k+1}}{a_k} = R$ and $\liminf_{k\to\infty} \frac{a_{k+1}}{a_k} = r$. If $0 \leq R < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.
- From Homework 12 Question 3 (a) it is easy to see that the root test is more powerful theoretically than the ratio test. That is if the ratio test works to prove convergence or divergence, then the root test would give the same conclusion.

10. Absolute and conditional convergence

A series $\sum_k a_k$ is called *absolutely convergent* if $\sum_k |a_k|$ converges. Recall that a_k here could be a complex number (you can view complex numbers as elements of $\mathbb{R}^2$ here).

- Any absolutely convergent series is convergent.

  [Proof: By the Cauchy test above (the first and second page of this chapter), since $\sum_k |a_k|$ converges, given $\epsilon > 0$ there exists an $N \geq 0$ such that

$$|a_{n+1} + a_{n+2} + \cdots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \cdots + |a_m| < \epsilon, \qquad m > n \geq N.$$

  By the Cauchy test again (but used in the other direction), $\sum_k a_k$ converges.]

- The converse is false, a series may be convergent, but not absolutely convergent. Such a series is called *conditionally convergent.*
- If $\sum_{k=1}^{\infty} a_k$ converges absolutely then $|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$.

  [Proof. Note $|\sum_{k=1}^{n} a_k| \leq \sum_{k=1}^{n} |a_k|$. Take the limit as $n \to \infty$ in this inequality, and use Fact 4 about sequences from Chapter 4 to see that $|\sum_{k=1}^{\infty} a_k| \leq \sum_{k=1}^{\infty} |a_k|$.]

- **The Alternating Series Test (a.k.a. Leibniz Test)/Alternating Series approximation:** Suppose that $a_0 \geq a_1 \geq a_2 \geq \cdots$, and that

$\lim_k\ a_k = 0$. Then $a_0 - a_1 + a_2 - a_3 + \cdots$ (which in sigma notation is $\sum_{k=0}^{\infty} (-1)^k a_k$) converges, and moreover $|s_n - \sum_{k=0}^{\infty} (-1)^k a_k| \leq a_n$ for all n, where s_n is the nth partial sum $\sum_{k=0}^{n-1} (-1)^k a_k$.

- The 'fundamental fact about power series' (this was already mentioned in the last 'bullet' before Theorem 4.14). Namely consider the power series $\sum_{k=0}^{\infty} c_k x^k$ (here x and the c_k can be complex numbers if you wish). We set $R = \infty$ if $\text{limsup}_n |c_n|^{\frac{1}{n}} = 0$, and $R = 0$ if $\text{limsup}_n |c_n|^{\frac{1}{n}} = \infty$. Otherwise set $R = \frac{1}{\text{limsup}_n |c_n|^{\frac{1}{n}}}$. This is the *radius of convergence* of the power series. In the homework you are asked to show that in place of $\text{limsup}_n |c_n|^{\frac{1}{n}}$ you can use $\text{limsup}_n |\frac{s_{n+1}}{s_n}|$.

Theorem 10.1. *Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series, and define R as above. Then $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely if $|x| < R$, and it diverges whenever $|x| > R$.*

- If $\sum_{k=1}^{\infty} a_k$ is a series, and $f : \mathbb{N} \to \mathbb{N}$ is a bijection (that is, is one-to-one and onto), then the series $\sum_{k=1}^{\infty} a_{f(k)}$ is called a 'rearrangement' of $\sum_{k=1}^{\infty} a_k$. It is not hard to see that a rearrangement of a convergent series need not converge.

Theorem Any 'rearrangement' of an absolutely convergent series is convergent and has the same sum.

- Dirichlet test: Let $\sum_k a_k$ be a series whose partial sums form a bounded sequence. Suppose (b_n) is a decreasing sequence with limit 0. Then $\sum_k a_k b_k$ converges.
- Abel's test: Suppose that $\sum_k a_k$ converges and b_n is a monotonic convergent sequence. Then $\sum_k a_k b_k$ converges.

Adding parentheses: Let $\sum_{k=1}^{\infty} a_k$ be a series, and suppose that $1 \leq n_1 < n_2 < \cdots$ are integers. Let $b_1 = \sum_{k=1}^{n_1} a_k, b_2 = \sum_{k=n_1+1}^{n_2} a_k, b_3 = \sum_{k=n_2+1}^{n_3} a_k, \cdots$. We call $\sum_{k=1}^{\infty} b_k$ a series obtained from $\sum_{k=1}^{\infty} a_k$ *by adding parentheses.*

Theorem 10.2. *If $\sum_{k=1}^{\infty} a_k$ converges, and b_k, n_k are as above, then $\sum_{k=1}^{\infty} b_k$ converges and has sum $\sum_{k=1}^{\infty} a_k$. Also: (a)] If $\sum_{k=1}^{\infty} a_k$ is a nonnegative series, then $\sum_{k=1}^{\infty} a_k$ converges iff $\sum_{k=1}^{\infty} b_k$ converges. (b) If ...*

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