Homework 7 solutions (Chapter 6)

5. a. Exponential(100) (time measured in hours) or Exponential$\left(\frac{5}{3}\right)$ (time measured in minutes).

c. Lognormal($\mu$, $\sigma$). There is insufficient information in the statement of the question to specify $\mu$ and $\sigma$. Note that this is an approximate model. The precise value of the security after one year is $X_1 X_2 \cdots X_n$ where $X_j$ is the accumulation of a \$1 investment in the $j$-th day and $n$ is the number of trading days. By assumption, the $X_j$ are independent, identically distributed, and positive random variables. Since $n$ is reasonably large, the multiplicative form of the central limit theorem applies. Hence $X_1 X_2 \cdots X_n \approx$ Lognormal($\mu$, $\sigma$) where $\mu, \sigma$ are the mean and standard deviation of $\log[X_1 X_2 \cdots X_n]$.

d. Exponential(3), time measured in months. Since the failure rate is constant, there is no aging. Consequently, the distribution is exponential.

e. Uniformly distributed on (0, 50), i.e., DeMoivre(50), in miles.

f. Gamma(1000, 5) in minutes.

g. Normal$\left(67, \sqrt{2}\right)$ inches. Sample data on populations suggest that heights have a distribution that is approximately normal.

i. Exponential(3/2) in minutes. Suppose that $T_1$ is the time in minutes for the experienced representative to finish serving the current customer and $T_2$ is the time in minutes for the trainee to finish serving the current customer. Then $X = \min(T_1, T_2)$. A reasonable model for individual service times is the exponential distribution. Under this assumption, $T_1 \sim$ Exponential(1) and $T_2 \sim$ Exponential$\left(\frac{1}{2}\right)$. (Note that $T_1$ and $T_2$ represent the *remaining* service times. However, since exponential distributions have the memoryless property, the distribution of the remaining service time for an individual is the same as the distribution of the total service time for that individual.) If $T_1$ and $T_2$ are independent (a reasonable assumption for service times), then their minimum must also have an exponential distribution with parameter equal to the sum of the parameters of the $T_j$ (see section 6.1.1). Hence $X$ is exponentially distributed with parameter $1 + \frac{1}{2} = \frac{3}{2}$, as claimed.

6. This exercise is similar to exercises 7 amd 8 of section 5.5 where students were asked to fit a discrete distribution to a given set of data. In this exercise, students are asked to fit a continuous distribution to a given set of data that is associated with a particular monetary quantity.

a. Strictly speaking, monetary quantities are discrete since every monetary amount is a discrete multiple of the smallest monetary unit available. For example, every insurance claim amount can be expressed as a whole number of pennies. With this approach, the number of possible monetary values is large and the probability of the given quantity (in this case the claim size) assuming any particular value is small. From section 4.1.3, we know that a discrete quantity with these characteristics can be approximated quite well using a continuous distribution. By taking this approach, we can obtain a model for the claim size that is considerably simpler, but not significantly less accurate, than a discrete model.

b. Let $X$ be the claim amount in hundreds of dollars and let $\hat{f}_X$ denote the empirical relative frequency density function determined in the way outlined in the question. Then $\hat{f}_X$ is given by

$\hat{f}_X[0] = 0.056$, $\hat{f}_X[0.5] = 0.304$, $\hat{f}_X[1.0] = 0.368$, $\hat{f}_X[1.5] = 0.334$, $\hat{f}_X[2.0] = 0.270$, $\hat{f}_X[2.5] = 0.206$, $\hat{f}_X[3.0] = 0.150$, $\hat{f}_X[3.5] = 0.118$, $\hat{f}_X[4.0] = 0.074$, $\hat{f}_X[4.5] = 0.050$, $\hat{f}_X[5.0] = 0.034$, $\hat{f}_X[5.5] = 0.022$, $\hat{f}_X[6.0] = 0.014$.

Note that since the values of $\hat{f}_X$ represent relative frequency *densities*, not relative frequencies, we should not expect these values to sum to 1. In determining the values of $\hat{f}_X[0]$ and $\hat{f}_X[6.0]$, it was assumed that the probability on the intervals $x < .25$ and $x > 5.75$ respectively is distributed over intervals of length $\frac{1}{2}$. This assumption is a bit arbitrary, but it is reasonable to make since the lengths of the intervals used in the estimation of $\hat{f}_X[x]$ for $x = 0.5,\ 1.0,\ \ldots,\ 5.5$ are all equal to $\frac{1}{2}$.

c. Using the formula

$$E\left[X^k\right] \approx \sum x^k\, \hat{f}_X[x]\ (\Delta\, x),$$

the implied first and second moments are respectively

$$E[X] \approx (0)\, \hat{f}_X[0] \cdot \left(\frac{1}{2}\right) + (0.5)\, \hat{f}_X[0.5] \cdot \left(\frac{1}{2}\right) + \cdot\cdot\cdot + (6.0)\, \hat{f}_X[6.0] \cdot \left(\frac{1}{2}\right) =$$
$$(0)\,(0.056)\,(0.5) + (0.5)\,(0.304)\,(0.5) + \cdot\cdot\cdot + (6.0)\,(0.014)\,(0.5) = 1.9175$$

and

$$E\left[X^2\right] \approx (0)^2\, \hat{f}_X[0] \cdot \left(\frac{1}{2}\right) + (0.5)^2\, \hat{f}_X[0.5] \cdot \left(\frac{1}{2}\right) + \cdot\cdot\cdot + (6.0)^2\, \hat{f}_X[6.0] \cdot \left(\frac{1}{2}\right) = 5.24963.$$

Hence the implied mean and variance are

$$E[X] = 1.9175,$$

$$\text{Var}\,(X) = 5.24963 - (1.9175)^2 = 1.57282375.$$

d. For the gamma distribution with parameters $r$ and $\lambda$ we have

$$E[X] = \frac{r}{\lambda},$$

$$\text{Var}\,(X) = \frac{r}{\lambda^2}.$$

Equating these expressions to the implied mean and variance determined in part c, we have

$$\frac{r}{\lambda} = 1.9175,$$

$$\frac{r}{\lambda^2} = 1.57282375.$$

Dividing the first of these equations by the second, we get

$$\lambda = \frac{1.9175}{1.57282375} = 1.21914487.$$

Substituting this into either equation, we then obtain
$r = 2.33771028$.

Hence to the nearest whole integer, we have
$r = 2$ and $\lambda = 1$.

For these values of the parameters, the gamma distribution has density function
$f[x] = x\, e^{-x}$ for $x \geq 0$

and survival function
$S[x] = (1 + x)\, e^{-x}$ for $x \geq 0$.

The table that follows compares this gamma density to the empirical relative frequency density function $\hat{f}_X$:

| $x$ | $\hat{f}_X[x]$ | $x\, e^{-x}$ | $\hat{f}_X[x] \big/ (x\, e^{-x})$ |
|---|---|---|---|
| 0 | 0.056 | 0 | undefined |
| 0.5 | 0.304 | 0.303265 | 1.00242 |
| 1.0 | 0.368 | 0.367879 | 1.00033 |
| 1.5 | 0.334 | 0.334695 | 0.997923 |
| 2.0 | 0.270 | 0.270671 | 0.997523 |
| 2.5 | 0.206 | 0.205212 | 1.00384 |
| 3.0 | 0.150 | 0.149361 | 1.00428 |
| 3.5 | 0.118 | 0.105691 | 1.11646 |
| 4.0 | 0.074 | 0.0732626 | 1.01007 |
| 4.5 | 0.050 | 0.0499905 | 1.00019 |
| 5.0 | 0.034 | 0.0336897 | 1.00921 |

| | | | |
|---|---|---|---|
| 5.5 | 0.022 | 0.0224772 | 0.978768 |
| 6.0 | 0.014 | 0.0148725 | 0.941334 |

From this table, the fit of the gamma distribution with $r = 2$ and $\lambda = 1$ appears to be quite good. To confirm this belief, we would need to perform a statistical test that measures in a quantitative way the fit of this distribution. Such tests are covered in books on statistical estimation and lie beyond the scope of this book.

e. Let $\hat{F}_X$ be the cumulative relative frequency function for the given data set. In this part, we analyze the fit of the gamma distribution with parameters $r = 2$ and $\lambda = 1$ by comparing $\hat{F}_X$ to the function $1 - (1 + x)\,e^{-x}$, which is the distribution function for Gamma(2, 1), at the points $x = 0.25,\ 0.75,\ 1.25,\ \ldots,\ 5.75$. Note that the comparison points $x$ used here are different from the ones used in part d. The reason for this is that the values of the empirical function $\hat{F}_X$ are only known at these points.

The table that follows compares $\hat{F}_X$ to the function $1 - (1 + x)\,e^{-x}$:

| $x$ | $\hat{F}_X[x]$ | $1 - (1 + x)\,e^{-x}$ | $\hat{F}_X[x] - \{1 - (1 + x)\,e^{-x}\}$ |
|---|---|---|---|
| 0.25 | 0.028 | 0.026499 | 0.00150098 |
| 0.75 | 0.180 | 0.173359 | 0.00664147 |
| 1.25 | 0.364 | 0.355364 | 0.00863579 |
| 1.75 | 0.531 | 0.522122 | 0.00887834 |
| 2.25 | 0.666 | 0.657453 | 0.00854748 |
| 2.75 | 0.769 | 0.760271 | 0.00872948 |
| 3.25 | 0.844 | 0.83521 | 0.00879038 |
| 3.75 | 0.903 | 0.888291 | 0.0147093 |
| 4.25 | 0.940 | 0.925113 | 0.0148872 |
| 4.75 | 0.965 | 0.950253 | 0.0147472 |

| | | | |
|---|---|---|---|
| 5.25 | 0.982 | 0.967203 | 0.014797 |
| 5.75 | 0.993 | 0.978516 | 0.0144838 |

This table, like the one in part d, suggests that the fit of the gamma distribution with $r = 2$ and $\lambda = 1$ is good.

f. Using the approximation $X \approx \text{Gamma}(2, 1)$ we have

$$\Pr[4 \le X \le 4.25] = \Pr[X \le 4.25] - \Pr[X \le 4] + \Pr[X = 4] = \\ \left\{1 - (1 + 4.25)\, e^{-4.25}\right\} - \left\{1 - (1 + 4)\, e^{-4}\right\} + 0 = 5\, e^{-4} - 5.25\, e^{-4.25} \approx .01669097,$$

$$\Pr[X \ge 6] = \Pr[X > 6] + \Pr[X = 6] = (1 + 6)\, e^{-6} + 0 = 7\, e^{-6} \approx .01735127,$$

and

$$\Pr[X \le 0.25] = 1 - (1 + 0.25)\, e^{-0.25} = 1 - 1.25\, e^{-0.25} \approx .02649902.$$

Note that there is no correction for continuity in this calculation because $X$ can assume fractional as well as whole number values. Continuity corrections are generally only applied when the discrete variable in question has integer values.

7. The purpose of this question is to give the reader practice recognizing the moment generating functions of special continuous distributions, and to remind the reader of the uniqueness property for moment generating functions (section 4.3.1).

   a. Exponential(1) or equivalently, Gamma(1,1)

   b. $\text{Gamma}\left(\frac{1}{2}, \frac{1}{3}\right)$

   c. Normal(0, 1)

   d. $\text{Normal}(1, \sqrt{2}\,)$

8. In each part of this question, one must first recognize the given moment generating function as the moment generating function of a particular special distribution. Then using the uniqueness property of the moment generating function and properties of the identified special distribution, it is straightforward to determine $E[X]$, $\text{Var}(X)$, $\Pr[X > 1]$, and $\Pr[-1 < X < 1]$.

   a. A random variable $X$ with moment generating function $M_X[t] = \exp\left[t + \frac{t^2}{2}\right]$ must have a normal distribution with mean 1 and standard deviation 1, i.e., $X \sim \text{Normal}(1, 1)$.

Hence

$E[X] = 1,$

$\text{Var}(X) = 1,$

$$\Pr[X > 1] = \Pr\left[\frac{X-1}{1} > 0\right] = \Pr[Z > 0] = \frac{1}{2},$$

and

$$\Pr[-1 < X < 1] = \Pr\left[-2 < \frac{X-1}{1} < 0\right] =$$
$$\Pr[-2 < Z < 0] = \Pr[0 < Z < 2] = \Phi[2] - \Phi[0] \approx .9772 - .5 = .4772,$$

where $Z \sim \text{Normal}(0,\ 1)$ and $\Phi$ is the distribution function for $Z$. The numerical values of $\Phi$ are obtained from Appendix E of the textbook.

b. A random variable $X$ with moment generating function $M_X[t] = (1/(1-t))^3$ must have a gamma distribution with parameters $r = 3$ and $\lambda = 1$, i.e., $X \sim \text{Gamma}(3,\ 1)$. Hence

$E[X] = 3,$

$\text{Var}(X) = 3,$

$$\Pr[X > 1] = S_X[1] = \sum_{n=0}^{2} \frac{1^n\, e^{-1}}{n!} = e^{-1}\left\{\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\right\} = \frac{5}{2}\, e^{-1},$$

and

$$\Pr[-1 < X < 1] =$$
$$\Pr[0 \le X < 1] = 1 - \Pr[X \ge 1] = 1 - \Pr[X > 1] - \Pr[X = 1] = 1 - \frac{5}{2}\, e^{-1} - 0 = 1 - \frac{5}{2}\, e^{-1}.$$

Note the use of the formula

$$S_X[x] = \sum_{n=0}^{r-1} \frac{(\lambda\, x)^n\, e^{-\lambda\, x}}{n!}$$

for $r$ a positive integer, which is derived in section 6.1.2 of the textbook. Note further that $\Pr[-1 < X < 1] = \Pr[0 \le X < 1]$ since the gamma distribution assigns no probability to negative values and $\Pr[X = 1] = 0$ since the gamma distribution is continuous.

c. A random variable $X$ with moment generating function $M_X[t] = 3/(3-t)$ must have an exponential distribution with parameter $\lambda = 3$. Hence

$$E[X] = \frac{1}{3},$$

$$\mathrm{Var}(X) = \frac{1}{3^2} = \frac{1}{9},$$

$$\Pr[X > 1] = S_X[1] = e^{-(3)(1)} = e^{-3},$$

and

$$\Pr[-1 < X < 1] = \Pr[0 \le X < 1] = 1 - \Pr[X \ge 1] = 1 - \Pr[X > 1] - \Pr[X = 1] = 1 - e^{-3} - 0 = 1 - e^{-3}.$$

9. Let $T_l$, $T_r$ be the total service times for the left and right machines respectively and let $T_l^*$, $T_r^*$ be the corresponding remaining service times. Let $T$ be the waiting time until the first machine becomes available when both machines are in use. Suppose that $T_l$, $T_r$, $T_l^*$, $T_r^*$, and $T$ are all measured in seconds. Then $T = \min(T_l^*,\ T_r^*)$.

We are not explicitly told what models to use for $T_l$ and $T_r$. In the interest of simplicity, let's assume that both $T_l$ and $T_r$ have exponential distributions. Since the exponential distribution has the memoryless property it follows from this assumption that $T_l^*$ and $T_r^*$ are exponentially distributed with $T_l^* \sim T_l$ and $T_r^* \sim T_r$. Note that in this context the memoryless property means that knowledge of the time that a machine has already spent servicing a customer has no effect on the distribution of the remaining service time. This is not an unreasonable assumption to make in this context as anyone who has stood behind a customer performing multiple transactions can attest! Since the average service times are 30 seconds and 20 seconds for the left and right machines respectively, it follows that $T_l \sim \text{Exponential}\left(\frac{1}{30}\right)$, $T_r \sim \text{Exponential}\left(\frac{1}{20}\right)$ and also $T_l^* \sim \text{Exponential}\left(\frac{1}{30}\right)$, $T_r^* \sim \text{Exponential}\left(\frac{1}{20}\right)$.

In section 6.1.1, it was shown that if $T_1 \sim \text{Exponential}(\lambda_1)$, $T_2 \sim \text{Exponential}(\lambda_2)$ and $T_1$, $T_2$ are independent then $\min(T_1,\ T_2) \sim \text{Exponential}(\lambda_1 + \lambda_2)$. Since $T = \min(T_l^*,\ T_r^*)$, it follows that $T$ has an exponential distribution with parameter $\lambda = \frac{1}{30} + \frac{1}{20} = \frac{1}{12}$, i.e., $T \sim \text{Exponential}(\frac{1}{12})$. This fact will be used to answer parts a through e.

a. Since $T \sim \text{Exponential}(\frac{1}{12})$, we have $E[T] = 12$. Hence the person at the front of the line should expect to wait 12 seconds.

b. The desired probability is

$\Pr[T > 15] = e^{-15/12} = e^{-5/4} \approx .2865$.

c. From part a, the expected waiting time for a person at the front of the line is 12 seconds. Hence we should expect the line to move every 12 seconds. It follows that the person who is currently third in line should expect to wait 36 seconds. This result can also be derived more formally using the approach outlined in part d.

e. To answer the question of this part, we need only consider the machine on the left. The desired probability is

$\Pr[T_l > 60] = e^{-60/30} = e^{-2} \approx .1353$.

10. On a previous key.

11. Let $T_1$ be the time required to make a withdrawal, let $T_2$ be the time required to pay a bill, and let $T_3$ be the time required to make a deposit. Let $J$ be a random variable that indicates the type of transaction defined as follows:

$J = 1$ if transaction is withdrawal,
$J = 2$ if transaction is bill payment,
$J = 3$ if transaction is deposit.

Let $T$ be the service time for the next customer. Suppose that $T_1$, $T_2$, $T_3$, and $T$ are measured in seconds. Then from the given information we have

$$T_1 \sim \text{Exponential}\left(\frac{1}{15}\right), \quad T_2 \sim \text{Exponential}\left(\frac{1}{25}\right), \quad T_3 \sim \text{Exponential}\left(\frac{1}{50}\right),$$

and

$p_J[1] = .70, \quad p_J[2] = .20, \quad p_J[3] = .10.$

Hence by the law of total probability, the desired probability is

$$\begin{aligned}
\Pr[T > 20] &= \Pr[T > 20 \mid J = 1]\Pr[J = 1] + \\
&\Pr[T > 20 \mid J = 2]\Pr[J = 2] + \Pr[T > 20 \mid J = 3]\Pr[J = 3] = \\
&\Pr[T_1 > 20]\Pr[J = 1] + \Pr[T_2 > 20]\Pr[J = 2] + \Pr[T_3 > 20]\Pr[J = 3] = \\
&e^{-20/15}(.70) + e^{-20/25}(.20) + e^{-20/50}(.10) = \\
&.70\,e^{-4/3} + .20\,e^{-4/5} + .10\,e^{-2/5} \approx .34141579 \approx 34\,\%.
\end{aligned}$$

12. a. Let $T$ be the time in minutes until the next printing job arrives. Since print jobs arrive randomly and independently at a constant average rate of 20 per hour, a reasonable model for $T$ is Exponential($\lambda$) where $\lambda = 20/60 = 1/3$. Hence the desired probability is

$\Pr[T > 5] = e^{-5/3} \approx 18.89\,\%$.

b. Let $T^*$ be the time in minutes to process a given print job. By assumption $T^*$ is exponentially distributed. Since the average job is 10 pages long and the printer is capable of printing 8 pages per minute, it follows that the average job takes $\frac{10}{8}$ minutes to print. Hence $T^* \sim$ Exponential(0.8) and the desired probability is

$\Pr[T^* > 2] = e^{-(0.8)(2)} = e^{-1.6} \approx 20.19\,\%$.

c. Let $T^*$ be the total time in minutes to process the current job. From part b, it follows that $T^* \sim$ Exponential(0.8). Hence the desired probability is

$$\Pr[T^* > 6 \mid T^* > 5] = \frac{\Pr[T^* > 6]}{\Pr[T^* > 5]} = \frac{e^{-(0.8)(6)}}{e^{-(0.8)(5)}} = e^{-0.8} \approx 44.93\,\%.$$

We could also have derived this result by considering the time remaining, which by the memoryless property of the exponential distribution is Exponential(0.8).

d. Let $T_1^*$ be the time remaining to process the active job and let $T_2^*, T_3^*, T_4^*, T_5^*$ be the process times for the jobs waiting in the queue ahead of the job just submitted. Let $T$ be the time until processing of the job just submitted begins. Then $T = T_1^* + T_2^* + T_3^* + T_4^* + T_5^*$. Suppose that $T, T_1^*, T_2^*, T_3^*, T_4^*, T_5^*$ are measured in minutes. Then arguing as in part b we have $T_j^* \sim$ Exponential(0.8) for $j = 2, 3, 4, 5$. We also have $T_1^* \sim$ Exponential(0.8) using the memoryless property of the exponential distribution (as discussed in part c). Since the $T_j^*$ are independent, it follows from section 6.1.2 that $T_1^* + \cdots + T_5^* \sim$ Gamma(5, 0.8). Hence for $t > 0$,

$$\Pr[T > t] = \sum_{n=0}^{4} \frac{(0.8\,t)^n\, e^{-0.8\,t}}{n!}.$$

Therefore the desired probabilities are

$$\Pr[T > 5] = \sum_{n=0}^{4} \frac{4^n\, e^{-4}}{n!} = e^{-4}\left\{1 + 4 + 8 + \frac{32}{3} + \frac{32}{3}\right\} = \frac{103}{3}\, e^{-4} \approx .62883694$$

and

$$\Pr[T > 10] = \sum_{n=0}^{4} \frac{8^n\, e^{-8}}{n!} = e^{-8}\left\{1 + 8 + 32 + \frac{256}{3} + \frac{512}{3}\right\} = 297\, e^{-8} \approx .09963240.$$

13. Let $X_j$ be the accumulation factor for the $j$-th trading day. By assumption, the $X_j$ are independent and identically distributed with probability distribution given by

$$X_j = \begin{cases} 1.02 & \text{with probability } .50, \\ 0.99 & \text{with probability } .50. \end{cases}$$

Since the current price of the stock is \$100, its price $n$ trading days hence is $S_n = 100\, X_1\, X_2 \cdots X_n$. We are interested in determining $\Pr[S_{50} > 200]$.

Note that the possible values of $S_n$ are

$$100\,(0.99)^n,\;\; 100\,(0.99)^{n-1}\,(1.02),\;\; 100\,(0.99)^{n-2}\,(1.02)^2,$$
$$\ldots,\;\; 100\,(0.99)\,(1.02)^{n-1},\;\; 100\,(1.02)^n$$

with respective probabilities

$$(.50)^n,\ \binom{n}{1}(.50)^n,\ \binom{n}{2}(.50)^n,\ \ldots,\ \binom{n}{n-1}(.50)^n,\ \binom{n}{n}(.50)^n.$$

Since

$$\begin{aligned}&100\,(0.99)^{n-k}\,(1.02)^k > 200 \iff (0.99)^{n-k}\,(1.02)^k > 2 \iff\\ &\quad (n-k)\log[0.99] + k\log[1.02] > \log[2] \iff\\ &\quad k\,\{\log[1.02] - \log[0.99]\} > \log[2] - n\log[0.99] \iff\\ &\quad k\log\left[\frac{1.02}{0.99}\right] > \log[2] - n\log[0.99] \iff k > \frac{\log[2]\ -\ n\log[0.99]}{\log\left[\frac{1.02}{0.99}\right]}\end{aligned}$$

it follows that

$$\Pr[S_n > 200] = \sum_{k=k^*}^{n}\binom{n}{k}(.50)^n$$

where

$$k^* = \mathrm{Int}\left[\frac{\log[2] - n\log[0.99]}{\log[1.02/0.99]}\right] + 1.$$

Here $\mathrm{Int}[x]$ denotes the integer part of $x$, i.e., the greatest integer less than or equal to $x$. For $n = 50$ we have $k^* = 41$. Hence

$$\Pr[S_{50} > 200] = \sum_{k=41}^{50}\binom{50}{k}(.50)^{50}.$$

14. Let $X_j$ be the dollar increase on the $j$-th trading day. By assumption the $X_j$ are independent and identically distributed with probability distribution given by

$$X_j = \begin{cases} 2 & \text{with probability } .50,\\ -1 & \text{with probability } .50.\end{cases}$$

Since the current price of the stock is \$100, its price $n$ trading days hence is

$S_n = 100 + X_1 + X_2 + \cdots + X_n.$

We are interested in determining $\Pr[S_{50} > 145]$.

Let $I_j$ be an indicator of a price increase on the $j$-th trading day. Then
$I_j \sim \text{Binomial}[1, .50]$ and
$X_j = 3\, I_j - 1.$

Hence
$S_n = 100 + 3\,(I_1 + \cdots + I_n) - n = 100 - n + 3\,Y$

where $Y = I_1 + \cdots + I_n \sim \text{Binomial}[n, .50]$. Consequently,

$$\Pr[S_n > 145] = \Pr[100 - n + 3\,Y > 145] = \Pr\left[Y > 15 + \frac{n}{3}\right] = \sum_{k=k^*}^{n} \binom{n}{k} (.50)^n$$

where $k^* = 15 + \left[\frac{n}{3}\right] + 1 = 16 + \left[\frac{n}{3}\right]$. Here $[x]$ denotes the integer part of $x$, i.e., the greatest integer less than or equal to $x$. For $n = 50$ we have $k^* = 32$. Hence

$$\Pr[S_{50} > 145] = \sum_{k=32}^{50} \binom{50}{k} (.50)^{50}.$$

Hence

$\Pr[S_{50} > 145] \approx .0324543.$

An alternative approach to determining $\Pr[S_{50} > 145]$ is to use a normal approximation for $S_n$. From the definition of $X_j$ we have

$$E\left[X_j\right] = (2)\,(.50) + (-1)\,(.50) = 0.50,$$

$$\operatorname{Var}\left(X_j\right) = E\left[X_j^2\right] - E\left[X_j\right]^2 = \left\{(2)^2\,(.50) + (-1)^2\,(.50)\right\} - (0.50)^2 = 2.25.$$

Hence

$$E[S_n] = 100 + \sum_{j=1}^{n} E\left[X_j\right] = 100 + \frac{n}{2},$$

$$\operatorname{Var}\,(S_n) = \sum_{j=1}^{n} \operatorname{Var}\left(X_j\right) = 2.25\,n,$$

where the formula for the variance follows from the independence of the $X_j$. It follows that for $n$ sufficiently large,

$$S_n \approx \text{Normal}\left(100 + \frac{n}{2},\, 1.5\,\sqrt{n}\right).$$

Using this approximation and correcting for continuity we have

$\Pr[S_{50} > 145] =$

$$\Pr[S_{50} \geq 145.5] = \Pr\left[\frac{S_{50} - 125}{1.5\,\sqrt{50}} \geq \frac{145.5 - 125}{1.5\,\sqrt{50}}\right] \approx \Pr[Z \geq 1.9328] = 1 - \Phi[1.9328]$$

where $Z \sim \text{Normal}(0,\, 1)$ and $\Phi$ is the distribution function of $Z$. From the tables in Appendix E and using linear interpolation we have

$$\Phi[1.9328] \approx (.72)\,\Phi[1.93] + (.28)\,\Phi[1.94] = (.72)\,(.9732) + (.28)\,(.9738) = .973368.$$

Consequently,

$$\Pr[S_{50} > 145] \approx 1 - \Phi[.9328] \approx 1 - .973368 \approx .02663.$$

15. a. Binomial(500, .05)

b. Poisson(125)

c. Exponential($\lambda$). There is insufficient information to specify $\lambda$.

d. Exponential($\lambda$). There is insufficient information to specify $\lambda$. Note the use of the memoryless property.

e. Poisson(250)

f. Geometric(.10)

g. Exponential(145) years. Suppose that $T_1$ is the time in years until the next variable rate mortgage is prepaid, $T_2$ is the time in years until the next 15-year fixed rate mortgage is prepaid, and $T_3$ is the time in years until the next 30-year fixed rate mortgage is prepaid. Then $X = \min(T_1,\ T_2,\ T_3)$ and $T_1 \sim$ Exponential(100), $T_2 \sim$ Exponential(35), $T_3 \sim$ Exponential(10). Since the $T_j$ are independent and exponentially distributed, their minimum is also exponentially distributed with parameter equal to the sum of the parameters of the $T_j$ (see section 6.1.1, "Effect of Arithmetic Operations"). Hence $X$ is exponentially distributed with parameter 100 + 35 + 10 = 145, as claimed.

i. Poisson(110). Let $N_1$ be the number of health-related claims in a given day, let $N_2$ be the number of auto-related claims in a given day, and let $N_3$ be the number of life-related claims in a given day. Since claims arrive randomly and independently at a constant rate, it is reasonable to model $N_1, N_2$, and $N_3$ using mutually independent Poisson distributions. Let $\lambda_1, \lambda_2, \lambda_3$ be the parameters for these three respective Poisson distributions. Then the total number claims in a given day, $N_1 + N_2 + N_3$, has a Poisson distribution with parameter $\lambda_1 + \lambda_2 + \lambda_3$. Since the average number of claims received in total on a given day is 100, the value of the Poisson parameter for $N_1 + N_2 + N_3$ is 100, i.e., $\lambda_1 + \lambda_2 + \lambda_3 = 100$. We are given that 55% of all claims are health-related. Hence the average number of health-related claims received on a given day is 55 (55% of 100). It follows from this that $N_1 \sim \text{Poisson}(\lambda_1)$ with $\lambda_1 = 55$. Now suppose that $N_1^{(1)}$ is the number of health-related claims received today and $N_1^{(2)}$ is the number of health-related claims received tomorrow. Then by the foregoing argument $N_1^{(j)} \sim \text{Poisson}(55)$ for $j = 1, 2$ and the $N_1^{(j)}$ are independent. Hence $N_1^{(1)} + N_1^{(2)}$, which is the number of health-related claims in the next two days, has a Poisson distribution with parameter $\lambda = 55 + 55 = 110$. Consequently, an appropriate model for the number of health-related claims in the next two days is Poisson(110), as indicated.

*Comment:* Part i illustrates the *disaggregation property* of the Poisson distribution. This is a technique that under certain conditions allows one to decompose a Poisson distribution into a sum of independent Poisson distributions. In part i, the total number of claims in a given day has a Poisson distribution and can be determined by adding together the number of claims of three identifiable types --- health-related claims, auto-related claims, and life-related claims. The disaggregation property of the Poisson distribution asserts that for each claim type, the number of claims of that type in a given day has a Poisson distribution with parameter $p\,\lambda$ where $p$ is the fraction of claims of the given type and $\lambda$ is the Poisson parameter for the distribution of the total number of claims in a given day. This property holds in this case because the claim number random variables for each claim type are mutually independent.

j. NegativeBinomial(2, .05)

k. Gamma(5, 25) hours

l. Geometric(.01)

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b. When $r$ is a positive integer, Gamma($r$, $\lambda$) can be interpreted as the waiting time until the $r$-th arrival where $\lambda$ arrivals are expected per unit time (see section 6.1.2). Hence for $X \sim \text{Gamma}(2, 3)$, the statement $X \geq 1$ has the following interpretation: Either there are no arrivals in the first unit of time or there is exactly one arrival. Let $N$ be the number of arrivals in the time interval (0, 1). Then $N \sim \text{Poisson}(\lambda)$ where $\lambda = 3$ and so

$$\Pr[X \geq 1] = \Pr[N = 0] + \Pr[N = 1] = \sum_{n=0}^{1} \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} + \lambda\, e^{-\lambda} = (\lambda + 1)\, e^{-\lambda} = 4\, e^{-3}.$$

17. Let $R$ be the continuously compounded rate of return over the coming year. Then the price of the stock one year from now is $50\,e^R$ and the desired probability is $\Pr[55 \le 50\,e^R \le 60]$. By assumption, $R \sim \text{Normal}(.10,\ .20)$. Hence the desired probability is

$$\Pr[55 \le 50\,e^R \le 60] = \Pr\left[\frac{55}{50} \le e^R \le \frac{60}{50}\right] = \Pr\left[\log\left[\frac{11}{10}\right] \le R \le \log\left[\frac{6}{5}\right]\right] =$$
$$\Pr\left[\frac{\log[11/10] - .10}{.20} \le \frac{R - .10}{.20} \le \frac{\log[6/5] - .10}{.20}\right] \approx \Pr[-.0234 \le Z \le .4116] =$$
$$\Phi[.4116] - \Phi[-.0234] = \Phi[.4116] - (1 - \Phi[.0234]) = \Phi[.4116] + \Phi[.0234] - 1$$

where $Z \sim \text{Normal}(0,\ 1)$ and $\Phi$ is the standard normal distribution function. Using the tables in Appendix E of the textbook and linear interpolation as appropriate, we have

$$\Phi[.4116] \approx (.84)\,\Phi[.41] + (.16)\,\Phi[.42] = (.84)\,(.6591) + (.16)\,(.6628) = .659692$$

and

$$\Phi[.0234] \approx (.66)\,\Phi[.02] + (.34)\,\Phi[.03] = (.66)\,(.5080) + (.34)\,(.5120) = .50936.$$

Consequently, the desired probability is

$$\Pr[55 \le 50\,e^R \le 60] \approx .659692 + .50936 - 1 = .169052 \approx 17\,\%.$$

Note that the median of $50\,e^R$ is $50\,e^{.10} \approx 55.26$ and the expected value of $50\,e^R$ is

$$E[50\,e^R] = 50\,e^{.10+(0.5)\,(.20)^2} = 50\,e^{.12} \approx 56.37.$$

This helps to explain why the probability of the future stock price lying between 55 and 60 is not particularly large.

21. Let $X$ be the number of boys in a nursery with $n$ newborns. An exact model for $X$ is Binomial($n$, $p$) where $p$ is the probability of getting a boy. We are given that $p = .51$. We are interested in analyzing the behavior of $\Pr[X \le \frac{n}{2}]$ as $n$ increases.

a. When $n$ is even, $\Pr[X \le \frac{n}{2}]$ can be determined using a normal approximation with continuity correction in the following way:

$$\Pr\left[X \le \frac{n}{2}\right] = \Pr\left[X \le \frac{n}{2} + \frac{1}{2}\right] =$$
$$\Pr\left[\frac{X - n\,p}{\sqrt{n\,p\,(1-p)}} \le \frac{\frac{n}{2} + \frac{1}{2} - n\,p}{\sqrt{n\,p\,(1-p)}}\right] \approx \Pr\left[Z \le \frac{\frac{n}{2} + \frac{1}{2} - .51\,n}{\sqrt{n\,(.51)\,(.49)}}\right] = \Pr\left[Z \le \frac{.50 - .01\,n}{\sqrt{.2499\,n}}\right]$$

where $Z \sim \text{Normal}(0,\ 1)$. Using the tables in Appendix E and linear interpolation as appropriate we obtain the following:

For $n = 100$,

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Phi[-.10002001] \approx 1 - \Phi[.1000] \approx 1 - .5398 = .4602.$$

For $n = 500$,

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Phi[-.40257276] \approx 1 - \Phi[.4026] \approx$$
$$1 - \{(.74)\,\Phi[.40] + (.26)\,\Phi[.41]\} \approx 1 - \{(.74)\,(.6554) + (.26)\,(.6591)\} = .343638.$$

For $n = 1000$,

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Phi[-.60095296] \approx 1 - \Phi[.6010] \approx$$
$$1 - \{(.90)\,\Phi[.60] + (.10)\,\Phi[.61]\} \approx 1 - \{(.90)\,(.7257) + (.10)\,(.7291)\} = .27396.$$

From the general formula

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Pr\left[Z \le \frac{.50 - .01\,n}{\sqrt{.2499\,n}}\right]$$

it follows that

$$\Pr\left[X \le \frac{n}{2}\right] \to 0 \quad \text{as } n \to \infty.$$

This means that as the number of babies in the nursery increases, it becomes more and more likely that more than half of them are boys. This phenomenon is studied in detail in section 8.4 where the law of large numbers is discussed.

b. When the calculations are done without correcting for continuity, we have

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Pr\left[Z \le \frac{-.01\,n}{\sqrt{.2499\,n}}\right].$$

In particular, for $n = 100$

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Phi[-.20004001] \approx 1 - \Phi[.2000] \approx 1 - .5793 = .4207,$$

for $n = 500$

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Phi[-.44730306] \approx 1 - \Phi[.4473] \approx$$
$$1 - \{(.27)\,\Phi[.44] + (.73)\,\Phi[.45]\} \approx 1 - \{(.27)\,(.6700) + (.73)\,(.6736)\} = .327372,$$

and for $n = 1000$

$$\Pr\left[X \le \frac{n}{2}\right] \approx \Phi[-.63258206] \approx 1 - \Phi[.6326] \approx 1 - \{(.74)\,\Phi[.63] + (.26)\,\Phi[.64]\} =$$
$$1 - \{(.74)\,(.7357) + (.26)\,(.7389)\} = .263468.$$

From these calculations, it appears that the impact of the correction for continuity is greater when $n$ is small. From part a, we see that the impact of the continuity correction on the argument of $\Phi$ is $.50/\sqrt{.2499\,n}$, which tends to 0 as $n \to \infty$. Hence the correction for continuity is most important when $n$ is small.

22. Let $X$ be the number of heads obtained in 1000 tosses of the selected coin and let $I$ be an indicator of the fairness of the coin, i.e.,

$$I = \begin{cases} 1 & \text{if selected coin is fair,} \\ 0 & \text{if selected coin is biased.} \end{cases}$$

Since the gambler concludes that the coin is biased if $X \ge 525$ and concludes that it is fair otherwise, the probability that the gambler reaches a false conclusion is, by the law of total probability,

$$\Pr[X \geq 525 \mid I = 1] \Pr[I = 1] + \Pr[X < 525 \mid I = 0] \Pr[I = 0].$$

Consider first the quantity $\Pr[X \geq 525 \mid I = 1]$. This is the probability of reaching a false conclusion when the coin being tossed is known to be fair. Note that the distribution of $X \mid I = 1$ is binomial with parameters $n = 1000$ and $p = .50$. (The total number of tosses is 1000 and since the coin is fair, the probability of heads on a single toss of the coin is .50.) Hence

$$\Pr[X \geq 525 \mid I = 1] = \sum_{x=525}^{1000} \binom{1000}{x} (.50)^{1000}.$$

Alternatively, we can evaluate the probability using a normal approximation with continuity correction. When we do this, we obtain

$$\Pr[X \geq 525 \mid I = 1] = \Pr[X \geq 524.5 \mid I = 1] =$$
$$\Pr\left[ \frac{X - (1000)(.50)}{\sqrt{(1000)(.50)(.50)}} \geq \frac{524.5 - (1000)(.50)}{\sqrt{(1000)(.50)(.50)}} \,\middle|\, I = 1 \right] \approx \Pr[Z \geq 1.5495].$$

From Appendix E of the textbook and using linear interpolation as appropriate we have

$$\Phi[1.5495] \approx (.05)\,\Phi[1.54] + (.95)\,\Phi[1.55] = (.05)(.9382) + (.95)(.9394) = .93934.$$

Hence

$$\Pr[X \geq 525 \mid I = 1] \approx \Pr[Z \geq 1.5495] \approx 1 - \Phi[1.5495] \approx .06066,$$

which is close to the value .0606071 calculated directly.

Now consider the quantity $\Pr[X < 525 \mid I = 0]$. This is the probability of reaching a false conclusion when the coin being tossed is known to be the biased one. Since the probability of heads for the coin known to be biased is 55% by assumption, the distribution of $X \mid I = 0$ is binomial with parameters $n = 1000$ and $p = .55$. Hence

$$\Pr[X < 525 \mid I = 0] = \sum_{x=0}^{524} \binom{1000}{x} (.55)^x (.45)^{1000-x}.$$

Hence

$$\Pr[X < 525 \mid I = 0] \approx .0526817.$$

Alternatively, we can use a normal approximation with continuity correction:

$$\Pr[X < 525 \mid I = 0] = \Pr[X \leq 524.5 \mid I = 0] =$$
$$\Pr\left[\frac{X - (1000)(.55)}{\sqrt{(1000)(.55)(.45)}} \leq \frac{524.5 - (1000)(.55)}{\sqrt{(1000)(.55)(.45)}} \;\middle|\; I = 0\right] \approx \Pr[Z \leq -1.6209] =$$
$$\Phi[-1.6209] = 1 - \Phi[1.6209].$$

From Appendix E of the textbook and using linear interpolation as appropriate we have

$$\Phi[1.6209] \approx (.91)\,\Phi[1.62] + (.09)\,\Phi[1.63] \approx (.91)(.9474) + (.09)(.9484) = .94749.$$

Hence

$$\Pr[X < 525 \mid I = 0] \approx 1 - \Phi[1.6209] \approx .05251,$$

which is close to the value .0526817 calculated directly.

The only remaining probabilities to consider are $\Pr[I = 0]$ and $\Pr[I = 1]$. Since the gambler has one coin of each type and selects the coin to flip at random, we must have

$$\Pr[I = 0] = \frac{1}{2} \quad \text{and} \quad \Pr[I = 1] = \frac{1}{2}.$$

Putting this together, we find that the probability of reaching a false conclusion is

$$\Pr[X \geq 525 \mid I = 1]\Pr[I = 1] + \Pr[X < 525 \mid I = 0]\Pr[I = 0] =$$
$$(.0606071)\left(\frac{1}{2}\right) + (.0526817)\left(\frac{1}{2}\right) = .0566444.$$

30. Let $X$ be the insurer's payment in dollars for a randomly selected policy and let $I$ be an indicator of a claim for this policy. Then according to the assumptions,

$$I = \begin{cases} 1 & \text{with probability } .25, \\ 0 & \text{with probability } .75, \end{cases}$$

and

$$(X \mid I = 1) \sim \text{Pareto}(3,\ 100).$$

Hence

$$S_{X|I=1}[x] = \left(\frac{100}{100 + x}\right)^3 \quad \text{for } x > 0.$$

a. The desired probability is $\Pr[X > 50]$. By the law of total probability we have

$$\Pr[X > 50] = \Pr[X > 50 \mid I = 1] \Pr[I = 1] + \Pr[X > 50 \mid I = 0] \Pr[I = 0].$$

Clearly $\Pr[X > 50 \mid I = 0] = 0$ since no payment is made if no claim is submitted. From the formula for $S_{X|I=1}$ stated earlier we also have

$$\Pr[X > 50 \mid I = 1] = S_{X|I=1}[50] = \left(\frac{100}{100+50}\right)^3 = \left(\frac{2}{3}\right)^3.$$

Consequently,

$$\Pr[X > 50] = \left(\frac{2}{3}\right)^3 (.25) + (0)\,(.75) = \frac{2}{27}.$$

b. The desired probability is $\Pr[X > 10]$. (If the question had stated that a claim is known to occur then the required probability would be $\Pr[X > 10 \mid I = 1]$. However, tl question does not make this assumption.) Arguing as in part a we have

$$\Pr[X > 10] = \Pr[X > 10 \mid I = 1]\Pr[I = 1] + \Pr[X > 10 \mid I = 0]\Pr[I = 0] =$$
$$S_{X|I=1}[10] \cdot \Pr[I = 1] + 0 \cdot \Pr[I = 0] = \left(\frac{100}{100+10}\right)^3 (.25) = \left(\frac{10}{11}\right)^3 \left(\frac{1}{4}\right) \approx .18782870.$$

c. Applying the law of total probability as in parts a and b we have for $x \geq 0$,

$$S_X[x] = \Pr[X > x] = \Pr[X > x \mid I = 1]\Pr[I = 1] + \Pr[X > x \mid I = 0]\Pr[I = 0] =$$
$$S_{X|I=1}[x] \cdot \Pr[I = 1] + 0 \cdot \Pr[I = 0] = \left(\frac{100}{100+x}\right)^3 (.25).$$

Since the payment on a given policy cannot be negative we must also have

$S_X[x] = \Pr[X > x] = 1 \quad$ for $x < 0$.

Consequently, the survival function of $X$ is given by

$$S_X[x] = \left(\frac{100}{100+x}\right)^3 (.25) \qquad \text{for } x \geq 0,$$
$$S_X[x] = 1 \qquad \text{for } x < 0.$$

It follows that the distribution function $F_X$ is given by

$$F_X[x] = 1 - \left(\frac{100}{100+x}\right)^3 (.25) \qquad \text{for } x \geq 0,$$
$$F_X[x] = 0 \qquad \text{for } x < 0.$$

Note that

$\Pr[X = 0] =$
$\Pr[X = 0 \mid I = 1] \Pr[I = 1] + \Pr[X = 0 \mid I = 0] \Pr[I = 0] = 0 \cdot \Pr[I = 1] + 1 \cdot \Pr[I = 0] = .75.$

This also follows from the formula for $F_X$. Hence we see that $X$ has a mixed distribution with a probability mass of size .75 at $x = 0$ (representing the event that no claim is submitted) and a continuous distribution of probability on $x > 0$.

d. Recall that for nonnegative random variables $X$ we have

$$E[X] = \int_0^\infty S_X[x]\, dx.$$

Hence using the formula for $S_X$ derived in part c we have

$E[X] =$

$$\int_0^\infty \left(\frac{100}{100 + x}\right)^3 (.25)\, dx = (.25)\,(100)^3\, \frac{(100 + x)^{-2}}{-2} \Big|_0^\infty = (.25)\,(100)^3\, \frac{100^{-2}}{2} = 12.5.$$

To determine the variance of $X$ we need to consider the density function $f_X$. From part c, it follows that the continuous part of the distribution has density function

$$f_X[x] = -S_X'[x] = (.25)\, \frac{3}{100} \left(1 + \frac{x}{100}\right)^{-4} \quad \text{for } x > 0.$$

The discrete part consists of a probability mass of size .75 at $x = 0$. Hence

$$E\left[X^2\right] = 0^2 \cdot \Pr[X = 0] + \int_0^\infty x^2\, f_X[x]\, dx = 0^2 \cdot (.75) + (.25) \int_0^\infty x^2 \cdot \frac{3}{100} \left(1 + \frac{x}{100}\right)^{-4} dx.$$

The integral $\int_0^\infty x^2 \cdot \frac{3}{100} \left(1 + \frac{x}{100}\right)^{-4} dx$ can be determined by recursively applying integration by parts. Alternatively, one could recognize this integral as the second moment of a Pareto distribution with parameter $s = 3$, $\beta = 100$, and use the formula for the second moment stated in section 6.1.3. Taking the latter approach we have

$$\int_0^\infty x^2 \cdot \frac{3}{100} \left(1 + \frac{x}{100}\right)^{-4} dx = \frac{100^2 \cdot 2}{(3 - 1)\,(3 - 2)} = 100^2.$$

Consequently, the second moment of $X$ is

$$E\left[X^2\right] = (.25)\int_0^\infty x^2 \cdot \frac{3}{100}\left(1+\frac{x}{100}\right)^{-4} dx = \left(\frac{1}{4}\right)100^2 = 2500.$$

It follows that

$$\text{Var}(X) = E\left[X^2\right] - E[X]^2 = 2500 - (12.5)^2 = 2343.75.$$

34. Let $X$ be the number of claims received in the next day. From the information given in the question, it is reasonable to model $X$ in the following way: $(X \mid \Lambda = \lambda) \sim \text{Poisson}(\lambda)$ and $\Lambda \sim \text{Gamma}(5,\ 1)$. From section 5.3, it follows that the unconditional distribution of $X$ is negative binomial with parameters $r = 5$ and $p = \frac{1}{2}$. One can also show this directly using the law of total probability and the techniques illustrated in the solutions to exercises 29, 32, and 33. Hence the desired probability is

$$\Pr[X > 4] = 1 - \sum_{x=0}^{4}\binom{r+x-1}{r-1}p^r(1-p)^x =$$

$$1 - \sum_{x=0}^{4}\binom{4+x}{4}\left(\frac{1}{2}\right)^{5+x} = 1 - \left\{\left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^6 + 15\left(\frac{1}{2}\right)^7 + 35\left(\frac{1}{2}\right)^8 + 70\left(\frac{1}{2}\right)^9\right\} = \frac{1}{2}.$$

36. Let $X$ be the size of a randomly selected claim. We are given that $(X \mid \Lambda = \lambda) \sim \text{Exponential}(\lambda)$ and $\Lambda \sim \text{Gamma}(2,\ 100)$. Hence from section 6.1.3 (or exercise 29), the unconditional distribution of $X$ is Pareto(2, 100); in particular,

$$S_X[x] = \left(\frac{100}{100+x}\right)^2 \quad \text{for } x \ge 0.$$

It follows that the desired probability is

$$\Pr[X > 100] = S_X[100] = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

47. Let $X$ be the diameter of a randomly selected bolt and let $Y$ be the diameter of the hole in a randomly selected nut. We are given that

$X \sim \text{Normal}(2,\ 0.03)$

and

$Y \sim \text{Normal}(2.02,\ 0.04)$.

From the given information, this bolt and nut will fit together if $X < Y$ and $Y - X \le 0.05$. Hence the probability that a bolt and nut will fit together is $\Pr[0 < Y - X \le 0.05]$.

Since $X$ and $Y$ are independent, it follows from section 6.3.1 that $Y - X \sim \text{Normal}(0.02,\ 0.05)$. (Note that $\mu_{Y-X} = \mu_Y - \mu_X = 2.02 - 2$ and $\sigma_{Y-X} = \sqrt{(.03)^2 + (.04)^2} = .05$.) Hence the desired probability is

$$\Pr[0 < Y - X \le 0.05] = \Pr\left[\frac{0 - 0.02}{0.05} < \frac{Y - X - 0.02}{0.05} \le \frac{0.05 - 0.02}{0.05}\right] =$$
$$\Pr[-0.4 < Z \le 0.6] = \Phi[0.6] - \Phi[-0.4] = \Phi[0.6] + \Phi[0.4] - 1$$

where $Z \sim \text{Normal}(0,\ 1)$ and $\Phi$ is the standard normal distribution function. From Appendix E of the textbook, $\Phi[0.6] \approx .7257$ and $\Phi[0.4] \approx .6554$. Consequently the desired probability is

$$\Pr[0 < Y - X \le 0.05] \approx .7257 + .6554 - 1 = .3811.$$