Instructions. Show all working and reasoning, the points are almost all for logical, complete reasoning. Time = 90 mins. [Approximate point values]

1. (a) What does it mean for a sequence \((s_n)\) of real numbers to converge to a number \(s\)? [5]
   (b) Prove that if \((s_n)\) and \((t_n)\) are convergent sequences, then \((s_n t_n)\) is a convergent sequence. [15]

2. Prove that a decreasing bounded sequence \((a_n)\) converges to \(\inf_n a_n\).

3. (a) What is the definition of a Cauchy sequence?
   (b) Suppose that \((s_n)\) is a sequence with \(|s_{n+1} - s_n| \leq \frac{1}{2^n}\) for all \(n \in \mathbb{N}\). Show that \((s_n)\) is a Cauchy sequence.
   (c) Is the sequence in (b) convergent? Why?

4. Here \(f: D \to \mathbb{R}\), and \(c\) is an accumulation point of \(D\). Mark each statement True or False. If it is true, give a simple reason. If it is false, give a counterexample (you don’t need to show that it is a counterexample).
   (a) Every sequence of real numbers has a convergent subsequence.
   (b) If \(\lim_{x \to c} f(x) \neq L\) then there is a sequence \((s_n)\) in \(D\) which converges to \(c\), but \((f(s_n))\) does not converge to \(L\).
   (c) If \(f: D \to \mathbb{R}\) is continuous and bounded on \(D\), then \(f(x)\) has a maximum and a minimum value on \(D\).

5. Suppose that \(f: (a, b) \to \mathbb{R}\), \(g: (a, b) \to \mathbb{R}\), \(L \in \mathbb{R}\), and \(a < c < b\).
   (a) Prove that \(\lim_{x \to c} f(x) = L\) iff whenever \((s_n)\) is a sequence in \((a, b)\) \(\setminus \{c\}\) with \(\lim_n s_n = c\), then \(\lim_n f(s_n) = L\).
   (b) Prove that if \(\lim_{x \to c} f(x) = L\) and \(\lim_{x \to c} g(x) = M\), then \(\lim_{x \to c} f(x)g(x) = LM\).
   (c) Using (a) show that if \(g(x) \leq f(x) \leq h(x)\) for all \(x \in (a, b)\), and if \(\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L\), then \(\lim_{x \to c} f(x) = L\).
   (d) Show that if \(f\) is continuous, and \(f(r) = 0\) for all rational numbers \(r \in (a, b)\), then \(f(x) = 0\) for all \(x \in (a, b)\).

6. (a) Give the \(\epsilon - \delta\) definition for a function \(f: (a, b) \to \mathbb{R}\) to be continuous at a point \(c \in (a, b)\).
   (b) List as many other conditions as you know that are equivalent to \(f: (a, b) \to \mathbb{R}\) being continuous at \(c \in (a, b)\).
   (c) Using the \(\epsilon - \delta\) definition, show that the function \(x^3 - x\) is continuous at \(x = -1\).

7. State and prove the ‘min-max theorem’.