By (2), there is a close connection between dot products and cosines. The following result shows that there is a similar connection between cross products and sines.

**8.10 Remark.** Let \( \mathbf{x}, \mathbf{y} \) be nonzero vectors in \( \mathbb{R}^2 \) and \( \theta \) be the angle between \( \mathbf{x} \) and \( \mathbf{y} \). Then

\[
\| \mathbf{x} \times \mathbf{y} \| = \| \mathbf{x} \| \| \mathbf{y} \| \sin \theta.
\]

**Proof.** By Theorem 8.9 and (2),

\[
\| \mathbf{x} \times \mathbf{y} \|^2 = (\| \mathbf{x} \| \| \mathbf{y} \|)^2 - (\| \mathbf{x} \| \| \mathbf{y} \| \cos \theta)^2
\]

\[
= (\| \mathbf{x} \| \| \mathbf{y} \|)^2 (1 - \cos^2 \theta) = (\| \mathbf{x} \| \| \mathbf{y} \|)^2 \sin^2 \theta. \tag*{\qed}
\]

This observation can be used to establish a connection between cross products and area or volume (see Exercise 8.2.7).

**EXERCISES**

**8.1.1.** Let \( \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n \).

a) If \( \| \mathbf{x} - \mathbf{z} \| < 2 \) and \( \| \mathbf{y} - \mathbf{z} \| < 3 \), prove that \( \| \mathbf{x} - \mathbf{y} \| < 5 \).

b) If \( \| \mathbf{x} \| < 2, \| \mathbf{y} \| < 3, \) and \( \| \mathbf{z} \| < 4 \), prove that \( \| \mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z} \| < 14 \).

c) If \( \| \mathbf{x} - \mathbf{y} \| < 2 \) and \( \| \mathbf{z} \| < 3 \), prove that \( \| \mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{z}) \| < 6 \).

d) If \( \| 2\mathbf{x} - \mathbf{y} \| < 2 \) and \( \| \mathbf{y} \| < 3 \), prove that \( \| \mathbf{x} - \mathbf{y} \|^2 - \mathbf{x} \cdot \mathbf{y} < 2 \).

e) If \( n = 3, \| \mathbf{x} - \mathbf{y} \| < 2 \), and \( \| \mathbf{z} \| < 3 \), prove that \( \| \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \times \mathbf{z} \| < 6 \).

f) If \( n = 3, \| \mathbf{x} \| < 1, \| \mathbf{y} \| < 2 \), and \( \| \mathbf{z} \| < 3 \), prove that \( \| \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) \| < 6 \).
8.1.2. Let $B := \{ x \in \mathbb{R}^n : \|x\| \leq 1 \}$.

a) If $a, b, c \in B$ and
$$v := \frac{(a \cdot b)c + (a \cdot c)b + (c \cdot b)a}{3},$$
prove that $v$ belongs to $B$.

b) If $a, b \in B$, prove that
$$|a \cdot c - b \cdot d| \leq \|b - c\| + \|a - d\|$$
for all $c, d \in \mathbb{R}^n$.

c) If $a, b, c \in B$ and $n = 3$, prove that
$$\sqrt{(a \cdot (b \times c))^2 + |a \cdot b|^2} \leq 1.$$

8.1.3. Use the proof of Theorem 8.5 to show that equality in the Cauchy–Schwarz Inequality holds if and only if $x = 0$, $y = 0$, or $x$ is parallel to $y$.

8.1.4. Let $a$ and $b$ be nonzero vectors in $\mathbb{R}^n$.

a) If $\phi(t) = a + tb$ for $t \in \mathbb{R}$, show that for each $t_0, t_1, t_2 \in \mathbb{R}$ with $t_1, t_2 \neq t_0$, the angle between $\phi(t_1) - \phi(t_0)$ and $\phi(t_2) - \phi(t_0)$ is $0$ or $\pi$.

b) If $\theta$ is the angle between $a$ and $b$, show that $a$ and $b$ are parallel according to Definition 8.4 if and only if $\theta = 0$ or $\pi$, and that $a$ and $b$ are orthogonal according to Definition 8.4 if and only if $\theta = \pi/2$.

8.1.5. The midpoint of a side of a triangle in $\mathbb{R}^3$ is the point that bisects that side (i.e., that divides it into two equal pieces). Let $\Delta$ be a triangle in $\mathbb{R}^3$ with sides $A$, $B$, and $C$ and let $L$ denote the line segment between the midpoints of $A$ and $B$. Prove that $L$ is parallel to $C$ and that the length of $L$ is one-half the length of $C$.

8.1.6. a) Prove that $(1, 2, 3), (4, 5, 6),$ and $(0, 4, 2)$ are vertices of a right triangle in $\mathbb{R}^3$.

b) Find all nonzero vectors orthogonal to $(1, -1, 0)$ which lie in the plane $z = x$.

c) Find all nonzero vectors orthogonal to the vector $(3, 2, -5)$ whose components sum to 4.

8.1.7. Let $a < b$ be real numbers. The Cartesian product $[a, b] \times [a, b]$ is obviously a square in $\mathbb{R}^2$. Define a cube $Q$ in $\mathbb{R}^n$ to be the $n$-fold Cartesian product of $[a, b]$ with itself, that is, $Q := [a, b] \times \cdots \times [a, b]$. Find a formula of the angle between the longest diagonal of $Q$ and any of its edges. Show that when $n = 3$, this angle is approximately 54.74 degrees.

8.1.8. a) Using Postulate 1 in Section 1.2 and Definition 8.1, prove Theorem 8.2.

b) Prove Theorem 8.9, parts i) through iii) and vi).

c) Prove that if $x, y \in \mathbb{R}^3$, then $\|x \times y\| \leq \|x\| \|y\|$. 
Suppose that \( \{a_k\} \) and \( \{b_k\} \) are sequences of real numbers which satisfy

\[
\sum_{k=1}^{\infty} a_k^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} b_k^2 < \infty.
\]

Prove that the infinite series \( \sum_{k=1}^{\infty} a_k b_k \) converges absolutely.

**8.1.10.** Prove that the \( \ell^1 \)-norm and the sup-norm also satisfy Theorem 8.6.

### 8.2 Planes and Linear Transformations

A plane \( \Pi \) in \( \mathbb{R}^3 \) is a set of points that is “flat” in some sense. What do we mean by flat? Any vector that lies in \( \Pi \) is orthogonal to a common direction, called the normal, which we will denote by \( b \). Fix a point \( a \in \Pi \). Since the vector \( x - a \) lies in \( \Pi \) for all \( x \in \Pi \) and since two vectors are orthogonal when their dot product is zero, we see that \( (x - a) \cdot b = 0 \) for all \( x \in \Pi \) (see Figure 8.4).

Using this three-dimensional case as a guide, for any \( a, b \in \mathbb{R}^n \) with \( b \neq 0 \), we call the set

\[
\Pi_b(a) := \{ x \in \mathbb{R}^n : (x - a) \cdot b = 0 \}
\]

the hyperplane in \( \mathbb{R}^n \) passing through a point \( a \in \mathbb{R}^n \) with normal \( b \). (We call it a plane when \( n = 3 \).) In particular, \( \Pi_b(a) \) is the set of all points \( x \) such that \( x - a \) is orthogonal to \( b \).

There is nothing unique about “the normal” of a hyperplane: Any nonzero vector \( c \) parallel to \( b \) will define the same hyperplane. Indeed, if \( b \) and \( c \) are parallel, then, by definition, \( b = tc \) for some nonzero \( t \in \mathbb{R} \); hence \( (x - a) \cdot b = 0 \) if and only if \( (x - a) \cdot c = 0 \). Nevertheless, many properties of hyperplane can be...
**8.18 Remark.** If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( U : \mathbb{R}^m \rightarrow \mathbb{R}^p \) are linear, then so is \( U \circ T \). In fact, if \( B \) is the \( m \times n \) matrix which represents \( T \), and \( C \) is the \( p \times m \) matrix which represents \( U \), then \( CB \) is the matrix which represents \( U \circ T \).

**Proof.** Let \( e_1, \ldots, e_n \) be the usual basis of \( \mathbb{R}^n \), \( u_1, \ldots, u_m \) be the usual basis of \( \mathbb{R}^m \), and \( w_1, \ldots, w_p \) be the usual basis of \( \mathbb{R}^p \). If \( B = [b_{ij}]_{m \times n} \) represents \( T \) and \( C = [c_{uk}]_{p \times m} \) represents \( U \), then, by Theorem 8.15,

\[
\sum_{k=1}^{m} b_{kj} u_k = (b_{1j}, \ldots, b_{mj}) = T(e_j), \quad j = 1, 2, \ldots, n,
\]

and

\[
\sum_{v=1}^{p} c_{vk} w_v = (c_{1k}, \ldots, c_{pk}) = U(u_k), \quad k = 1, 2, \ldots, m.
\]

Hence

\[
(U \circ T)(e_j) = U(T(e_j)) = U\left(\sum_{k=1}^{m} b_{kj} u_k\right) = \sum_{k=1}^{m} b_{kj} U(u_k) = \sum_{k=1}^{m} \sum_{v=1}^{p} b_{kj} c_{vk} w_v = \left(\sum_{k=1}^{m} b_{kj} c_{1k}, \ldots, \sum_{k=1}^{m} b_{kj} c_{pk}\right)
\]

for each \( 1 \leq j \leq n \). Since this last vector is the \( j \)th column of the matrix \( CB \), it follows that \( CB \) is the matrix which represents \( U \circ T \).

**EXERCISES**

**8.2.1.** Let \( a, b, c \in \mathbb{R}^3 \).

a) Prove that if \( a, b, \) and \( c \) do not all lie on the same line, then an equation of the plane through these points is given by \((x, y, z) \cdot d = a \cdot d\), where

\[d := (a - b) \times (a - c)\].

b) Prove that if \( c \) does not lie on the line \( \phi(t) = ta + b, \ t \in \mathbb{R} \), then an equation of the plane that contains this line and the point \( c \) is given by \((x, y, z) \cdot d = b \cdot d\), where \(d := a \times (b - c)\).

**8.2.2.** a) Find an equation of the hyperplane through the points \((1, 0, 0, 0), (2, 1, 0, 0), (0, 1, 1, 0), \) and \((0, 4, 0, 1)\).

b) Find an equation of the hyperplane that contains the lines \( \phi(t) = (t, t, t, 1) \) and \( \psi(t) = (1, t, 1 + t, t), \ t \in \mathbb{R} \).

c) Find an equation of the plane parallel to the hyperplane \(x_1 + \cdots + x_n = \pi\) passing through the point \((1, 2, \ldots, n)\).
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8.2.3. Find two lines in $\mathbb{R}^3$ which are not parallel but do not intersect.

8.2.4. Suppose that $T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ for some $n, m \in \mathbb{N}$.
   a) Find the matrix representative of $T$ if $T(x, y, z, w) = (0, x + y, x - z, x + y + w)$.
   b) Find the matrix representative of $T$ if $T(x, y, z) = x - y + z$.
   c) Find the matrix representative of $T$ if $T(x_1, x_2, \ldots, x_n) = (x_1 - x_n, x_n - x_1)$.

8.2.5. Suppose that $T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ for some $n, m \in \mathbb{N}$.
   a) If $T(1, 1) = (3, \pi, 0)$ and $T(0, 1) = (4, 0, 1)$, find the matrix representative of $T$.
   b) If $T(1, 1, 0) = (e, \pi, 0)$, $T(0, -1, 0) = (1, 0, 0)$, and $T(1, 1, -1) = (1, 2, 0)$, find the matrix representative of $T$.
   c) If $T(0, 1, 1, 0) = (3, 5, 0)$, $T(0, 1, -1, 0) = (5, 3, 0)$, and $T(0, 0, 0, -1) = (\pi, 3, -1)$, find all possible matrix representatives of $T$.
   d) If $T(1, 1, 0, 0) = (5, 4, 1, 0)$, $T(0, 1, 0, 0) = (1, 2, 0, 0)$, and $T(0, 0, 0, -1) = (\pi, 3, -1)$, find all possible matrix representatives of $T$.

8.2.6. Suppose that $a, b, c \in \mathbb{R}^3$ are three points which do not lie on the same straight line and that $\Pi$ is the plane which contains the points $a, b, c$.
   Prove that an equation of $\Pi$ is given by
   \[
   \det \begin{bmatrix}
   x - a_1 & y - a_2 & z - a_3 \\
   b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\
   c_1 - a_1 & c_2 - a_2 & c_3 - a_3
   \end{bmatrix} = 0.
   \]

8.2.7. This exercise is used in Appendix E. Recall that the area of a parallelogram with base $b$ and altitude $h$ is given by $bh$, and the volume of a parallelepiped is given by the area of its base times its altitude.
   a) Let $a, b \in \mathbb{R}^3$ be nonzero vectors and $P$ represent the parallelogram
      \[
      \{(x, y, z) = ua + vb : u, v \in [0, 1]\}.
      \]
      Prove that the area of $P$ is $\|a \times b\|$.
   b) Let $a, b, c \in \mathbb{R}^3$ be nonzero vectors and $P$ represent the parallelepiped
      \[
      \{(x, y, z) = ta + ub + vc : t, u, v \in [0, 1]\}.
      \]
      Prove that the volume of $P$ is $|(a \times b) \cdot c|$.

8.2.8. The distance from a point $x_0 = (x_0, y_0, z_0)$ to a plane $\Pi$ in $\mathbb{R}^3$ is defined to be
   \[
   \text{dist} (x_0, \Pi) := \begin{cases} 0 & x_0 \in \Pi \\ \|v\| & x_0 \notin \Pi, \end{cases}
   \]
where \( \mathbf{v} := (x_0 - x_1, y_0 - y_1, z_0 - z_1) \) for some \((x_1, y_1, z_1) \in \Pi\), and \( \mathbf{v} \) is orthogonal to \( \Pi \) (i.e., parallel to its normal). Sketch \( \Pi \) and \( x_0 \) for a typical plane \( \Pi \), and convince yourself that this is the correct definition. Prove that this definition does not depend on the choice of \( \mathbf{v} \), by showing that the distance from \( x_0 = (x_0, y_0, z_0) \) to the plane \( \Pi \) described by \( ax + by + cz = d \) is

\[
\text{dist} (x_0, \Pi) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.
\]

**8.2.9.** [Rotations in \( \mathbb{R}^2 \)]. This exercise is used in Section 8.15.1. Let

\[
B = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

for some \( \theta \in \mathbb{R} \).

a) Prove that \( \|B(x, y)\| = \|(x, y)\| \) for all \((x, y) \in \mathbb{R}^2\).

b) Let \((x, y) \in \mathbb{R}^2\) be a nonzero vector and \( \varphi \) represent the angle between \( B(x, y) \) and \((x, y)\). Prove that \( \cos \varphi = \cos \theta \). Thus, show that \( B \) rotates \( \mathbb{R}^2 \) through an angle \( \theta \). (When \( \theta > 0 \), we shall call \( B \) counterclockwise rotation about the origin through the angle \( \theta \).)

**8.2.10.** For each of the following functions \( f \), find the matrix representative of a linear transformation \( T \in \mathcal{L}(\mathbb{R}; \mathbb{R}^m) \) which satisfies

\[
\lim_{h \to 0} \frac{\|f(x + h) - f(x) - T(h)\|}{h} = 0.
\]

a) \( f(x) = (x^2, \sin x) \)

b) \( f(x) = (e^x, \sqrt{x}, 1 - x^2) \)

c) \( f(x) = (1, 2, 3, x^2 + x, x^2 - x) \)

**8.2.11.** Fix \( T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \). Set

\[
M_1 := \sup_{\|x\| = 1} \|T(x)\| \quad \text{and} \quad M_2 := \inf \{ C > 0 : \|T(x)\| \leq C \|x\| \text{ for all } x \in \mathbb{R}^n \}.
\]

a) Prove that \( M_1 \leq \|T\| \).

b) Using the linear property of \( T \), prove that if \( x \neq 0 \), then

\[
\frac{\|T(x)\|}{\|x\|} \leq M_1.
\]

c) Prove that \( M_1 = M_2 = \|T\| \).
or \( x_0 \in V \). We may suppose the former. Let \( y_k \in I_0 \) and suppose that \( y_k \to x_0 \)
as \( k \to \infty \). Since \( U \) is relatively open, there is an \( \epsilon > 0 \) such that \((x_0 - \epsilon, x_0 + \epsilon) \cap E \subset U \). Since \( y_k \in E \) and \( y_k \to x_0 \), it follows that \( y_k \in U \) for large \( k \). Hence \( f(y_k) = 0 = f(x_0) \) for large \( k \). Therefore, \( f \) is continuous at \( x_0 \) by the Sequential Characterization of Continuity.

We have proved that \( f \) is continuous on \( I_0 \). Hence by the Intermediate Value Theorem (Theorem 3.29), \( f \) must take on the value \( 1/2 \) somewhere on \( I_0 \). This is a contradiction, since by construction, \( f \) takes on only the values 0 or 1.

We shall use this result later to prove that a real function is continuous on a closed, bounded interval if and only if its graph is closed and connected (see Theorem 9.51).

**EXERCISES**

8.3.1. Sketch each of the following sets. Identify which of the following sets are open, which are closed, and which are neither. Also discuss the connectedness of each set.

a) \( E = \{ (x, y) : y \neq 0 \} \)
b) \( E = \{ (x, y) : x^2 + 4y^2 \leq 1 \} \)
c) \( E = \{ (x, y) : y \geq x^2, 0 \leq y < 1 \} \)
d) \( E = \{ (x, y) : x^2 - y^2 > 1, -1 < y < 1 \} \)
e) \( E = \{ (x, y) : x^2 - 2x + y^2 = 0 \} \cup \{ (x, 0) : x \in [2, 3] \} \)

8.3.2. Let \( n \in \mathbb{N} \), let \( a \in \mathbb{R}^n \), let \( s, r \in \mathbb{R} \) with \( s < r \), and set

\[
V = \{ x \in \mathbb{R}^n : s < \| x - a \| < r \} \quad \text{and} \quad E = \{ x \in \mathbb{R}^n : s \leq \| x - a \| \leq r \}.
\]

Prove that \( V \) is open and \( E \) is closed.

8.3.3. a) Let \( a \leq b \) and \( c \leq d \) be real numbers. Sketch a graph of the rectangle

\[
[a, b] \times [c, d] := \{ (x, y) : x \in [a, b], y \in [c, d] \},
\]

and decide whether this set is connected. Explain your answers.
b) Sketch a graph of set

\[
B_1(-2, 0) \cup B_1(2, 0) \cup \{ (x, 0) : -1 < x < 1 \},
\]

and decide whether this set is connected. Explain your answers.

8.3.4. a) Set \( E_1 := \{ (x, y) : y \geq 0 \} \) and \( E_2 := \{ (x, y) : x^2 + 2y^2 < 6 \} \), and sketch a graph of the set

\[
U := \{ (x, y) : x^2 + 2y^2 < 6 \quad \text{and} \quad y \geq 0 \}.
\]
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b) Decide whether $U$ is relatively open or relatively closed in $E_1$. Explain your answer.
c) Decide whether $U$ is relatively open or relatively closed in $E_2$. Explain your answer.

8.3.5.  a) Let $E_1$ denote the closed ball centered at $(0, 0)$ of radius $1$ and $E_2 := B_{\sqrt{2}}(2, 0)$, and sketch a graph of the set

\[ U := \{(x, y) : x^2 + y^2 \leq 1 \text{ and } x^2 - 4x + y^2 + 2 < 0\}. \]

b) Decide whether $U$ is relatively open or relatively closed in $E_1$. Explain your answer.
c) Decide whether $U$ is relatively open or relatively closed in $E_2$. Explain your answer.

8.3.6. Suppose that $E \subseteq \mathbb{R}^n$ and that $C$ is a subset of $E$.

a) Prove that if $E$ is closed, then $C$ is relatively closed in $E$ if and only if $C$ is (plain old vanilla) closed (in the usual sense).
b) Prove that $C$ is relatively closed in $E$ if and only if $E \setminus C$ is relatively open in $E$.

8.3.7.  a) If $A$ and $B$ are connected in $\mathbb{R}^n$ and $A \cap B \neq \emptyset$, prove that $A \cup B$ is connected.
b) If $\{E_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected sets in $\mathbb{R}^n$ and $\cap_{\alpha \in \Lambda} E_\alpha \neq \emptyset$, prove that

\[ E = \bigcup_{\alpha \in \Lambda} E_\alpha \]

is connected.
c) If $A$ and $B$ are connected in $\mathbb{R}$ and $A \cap B \neq \emptyset$, prove that $A \cap B$ is connected.
d) Show that part c) is no longer true if $\mathbb{R}^2$ replaces $\mathbb{R}$.

8.3.8. Let $V$ be a subset of $\mathbb{R}^n$.

a) Prove that $V$ is open if and only if there is a collection of open balls $\{B_\alpha : \alpha \in A\}$ such that

\[ V = \bigcup_{\alpha \in \Lambda} B_\alpha. \]

b) What happens to this result when open is replaced by closed?

8.3.9. Show that if $E$ is closed in $\mathbb{R}^n$ and $a \notin E$, then

\[ \inf_{x \in E} \|x - a\| > 0. \]

8.3.10. Graph generic open balls in $\mathbb{R}^2$ with respect to each of the "non-Euclidean" norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. What shape are they?
Next, we use (11) to construct the open set $B$. Set
\[
\delta_x := \inf \{ \| x - u \| : u \in \overline{U}, \ x \in V, \ \text{ and } \ B = \bigcup_{x \in V} B_{\delta_x/2}(x). \]

Clearly, $B$ is open in $\mathbb{R}^n$. Since $\delta_x > 0$ for each $x \notin \overline{U}$ (see Exercise 8.3.9), $B$ contains $V$; hence $B \cap E \supseteq V$. The reverse inequality also holds, since by construction $B \cap U = \emptyset$ and by hypothesis $E = U \cup V$. Therefore, $B \cap E = V$.

Similarly, we can construct an open set $A$ such that $A \cap E = U$ by setting
\[
\delta_y := \inf \{ \| v - y \| : v \in \overline{V}, \ y \in U \ \text{ and } \ A = \bigcup_{y \in \overline{U}} B_{\delta_y/2}(y). \]

In particular, $A$ and $B$ are nonempty open sets which satisfy $E \subseteq A \cup B$.

It remains to prove that $A \cap B = \emptyset$. Suppose, to the contrary, that there is a point $a \in A \cap B$. Then $a \in B_{\delta_x/2}(x)$ for some $x \in V$ and $a \in B_{\delta_y/2}(y)$ for some $y \in U$. We may suppose that $\delta_x \leq \delta_y$. Then
\[
\| x - y \| \leq \| x - a \| + \| a - y \| < \frac{\delta_x}{2} + \frac{\delta_y}{2} \leq \delta_y.
\]

Therefore, $\| x - y \| < \inf \{ \| v - y \| : v \in \overline{V} \}$. Since $x \in V$, this is impossible. We conclude that $A \cap B = \emptyset$.

**EXERCISES**

8.4.1. Find the interior, closure, and boundary of each of the following subsets of $\mathbb{R}$.

a) $E = \{ 1/n : n \in \mathbb{N} \}$
b) $E = \bigcup_{n=1}^{\infty} \left( \frac{1}{n+1}, \frac{1}{n} \right)$
c) $E = \bigcup_{n=1}^{\infty} (-n, n)$
d) $E = \mathbb{Q}$

8.4.2. For each of the following sets, sketch $E^o$, $\overline{E}$, and $\partial E$.

a) $E = \{(x, y) : x^2 + 4y^2 \leq 1\}$
b) $E = \{(x, y) : x^2 - 2x + y^2 = 0 \} \cup \{(x, 0) : x \in [2, 3]\}$
c) $E = \{(x, y) : y \geq x^2, 0 \leq y < 1\}$
d) $E = \{(x, y) : x^2 - y^2 < 1, -1 < y < 1\}$

8.4.3. This exercise is used in Section 12.1. Suppose that $A \subseteq B \subseteq \mathbb{R}^n$.

Prove that
\[
\overline{A} \subseteq \overline{B} \ \text{ and } \ A^o \subseteq B^o.
\]

8.4.4. Let $E$ be a subset of $\mathbb{R}^n$.

a) Prove that every subset $A \subseteq E$ contains a set $B$ which is the largest subset of $A$ that is relatively open in $E$. 

b) Prove that every subset $A \subseteq E$ is contained in a set $B$ which is the smallest closed set containing $A$ that is relatively closed in $E$.

8.4.5. Complete the proof of Theorem 8.36 by verifying (10).

8.4.6. Prove that if $E \subseteq R$ is connected, then $E^o$ is also connected. Show that this is false if "$R^2$" is replaced by "$R^2$ without the origin".

8.4.7. Suppose that $E \subseteq R^n$ is connected and that $E \subseteq A \subseteq \overline{E}$. Prove that $A$ is connected.

8.4.8. A set $A$ is called clopen if and only if it is both open and closed.

a) Prove that every Euclidean space has at least two clopen sets.

b) Prove that a proper subset $E$ of $R^n$ is connected if and only if it contains exactly two relatively clopen sets.

c) Prove that every nonempty proper subset of $R^n$ has a nonempty boundary.

8.4.9. Show that Theorem 8.37 is best possible in the following sense.

a) There exist sets $A$, $B$ in $R$ such that $(A \cup B)^o \neq A^o \cup B^o$.

b) There exist sets $A$, $B$ in $R$ such that $A \cap B \neq A \cap \overline{B}$.

c) There exist sets $A$, $B$ in $R$ such that $\partial(A \cup B) \neq \partial A \cup \partial B$ and $\partial(A \cap B) \neq \partial A \cup \partial B$.

8.4.10. Let $A$ and $B$ be subsets of $R^n$.

a) Show that $\partial(A \cap B) \cap (A^c \cup (\partial B)^c) \subseteq \partial A$.

b) Show that if $x \in \partial(A \cap B)$ and $x \notin (A \cap \partial B) \cup (B \cap \partial A)$, then $x \in \partial A \cap \partial B$.

c) Prove that $\partial(A \cap B) \subseteq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

d) Show that even in $R$, there exist sets $A$ and $B$ such that $\partial(A \cap B) \neq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B)$.

8.4.11. Let $E \subseteq R^n$ and $U$ be relatively open in $E$.

a) If $U \subseteq E^o$, then $U \cap \partial U = \emptyset$.

b) If $U \cap \partial E \neq \emptyset$, then $U \cap \partial U = U \cap \partial E$. 