Q1.
(i) What is the definition of an orthonormal basis (o.n.b.) of an inner product space? 
(ii) Prove that if \( (v_1, \ldots, v_n) \) is an o.n.b. for \( V \) then for any \( v \in V \) we have \( v = \sum_{k=1}^{n} \langle v, v_k \rangle v_k \).
(iii) What does the Gram-Schmidt procedure do?
(iv) Show that any finite dimensional inner product space has an o.n.b.

Solution.
(i) It is a basis \( (v_1, \ldots, v_n) \), with \( v_i \perp v_j \) if \( i \neq j \) and \( \|v_i\| = 1 \) for all \( i \).
(ii) Write \( v = \sum_{k} c_k v_k \). We have \( \langle v, v_j \rangle = \langle \sum_k c_k v_k, v_j \rangle = \sum_k \langle c_k v_k, v_j \rangle = c_j \). 
(iii) It replaces any linearly independent \( (u_1, \ldots, u_n) \) by an orthonormal set \( (v_1, \ldots, v_n) \) with \( \text{Span}(v_1, \ldots, v_j) \subseteq \text{Span}(u_1, \ldots, u_j) \) for each \( j = 1, \ldots, n \).
(iv) If \( (u_1, \ldots, u_n) \) is a basis for \( V \) produce an orthonormal set \( (v_1, \ldots, v_n) \) by the Gram-Schmidt procedure. The new set is an o.n.b. since any orthonormal set is l.i., and \( \text{Span}(v_1, \ldots, v_n) = \text{Span}(u_1, \ldots, u_n) = V \).

Q2. Here \( V, W \) are inner product spaces.

(i) Write down the key formula satisfied by the adjoint \( T^* \) of \( T \in \mathcal{L}(V, W) \).
(ii) Find \( T^* \) if \( T : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathcal{P}_2(\mathbb{R}) \) is defined by \( T(a + bx + cx^2) = cx, \) and we use the inner product \( \langle a + bx + cx^2, d + ex + fx^2 \rangle = ad + be + cf \).
(iii) Prove that \( (S + T)^* = S^* + T^* \) if \( S, T \in \mathcal{L}(V, W) \).
(iv) Prove that the eigenvalues of a selfadjoint operator are real.

Solution.
(i) The key formula it satisfies is \( \langle T(v), w \rangle = \langle v, T^*(w) \rangle \) for all \( v \in V, w \in W \).
(ii) \( \langle T(a + bx + cx^2), d + ex + fx^2 \rangle = \langle cx, d + ex + fx^2 \rangle = c \langle x, e \rangle = (a + bx + cx^2, ex^2). \) Thus \( T^*(d + ex + fx^2) = ex^2 \).
(iii) \( \langle (S + T)v, w \rangle = \langle Sv + Tw, w \rangle = \langle Sv, w \rangle + \langle Tw, w \rangle = \langle v, S^*w \rangle + \langle v, T^*w \rangle = \langle v, S^*w + T^*w \rangle \).
From this we see \( (S + T)^*w = S^*w + T^*w \) so \( (S + T)^* = S^* + T^* \).
(iv) If \( \lambda \) is an evalue of \( T = T^* \), and \( \lambda \neq 0 \) with \( Tv \perp \lambda v \), then \( \lambda \langle v, v \rangle = \lambda \langle v, \lambda v \rangle = \langle T(v), \lambda v \rangle = \langle T(v), v \rangle \).
Dividing by \( \langle v, v \rangle \) we have \( \lambda = \lambda \) so \( \lambda \) is real.

Q3. In this question \( V \) is a finite dimensional complex inner product space.

(i) What is a normal operator on \( V \)?
(ii) State the spectral theorem for normal operators on \( V \).
(iii) Prove that if \( P \in \mathcal{L}(V) \) with \( P = P^2 \) and \( P = P^* \), then there is an o.n.b. for \( V \) with respect to which \( \mathcal{M}(T) \) is a diagonal matrix with only 1s and 0s on the diagonal.
(iv) Prove that if an \( n \times n \) matrix \( A \) satisfies \( AA^* = A^*A \), then \( A \) is unitarily equivalent to a diagonal matrix.

Solution.
(i) \( T \in \mathcal{L}(V) \) such that \( T^*T = TT^* \).
(ii) For \( T \in \mathcal{L}(V) \): if \( T^*T \), there is an o.n.b. for \( V \) with respect to which \( \mathcal{D}(T) \) is a diagonal matrix; if \( \mathcal{D}(T) \) is an o.n.b. for \( T \) consisting of eigenvectors of \( T \); if \( \mathcal{D}(T) \) is an o.n.b. for \( T \) consisting of orthogonal projections with \( P_i P_j = 0 \) if \( i \neq j \). The \( c_k \) may be taken to be the eigenvalues and \( P_k \) the projections onto the eigenspaces.
Q4. Let $V$ be a vector space, and $T \in \mathcal{L}(V)$.

(i) What is a generalized eigenvector of $T$?

(ii) Prove that if $(T - \lambda I)^j v = 0$ for a vector $v \neq 0$, and $j \in \{2, 3, \ldots \}$, then $\lambda$ is an eigenvalue of $T$.

(iii) Give a formula for the generalized eigenspace $G_\lambda$ involving $\dim V$.

(iv) If $V = \mathbb{C}^3$, find the generalized eigenspace $G_1$ if $T(x, y, z) = (y, -z, x - y + z)$. From this find the multiplicity of the evalve 1.

Solution.

(i) It is a vector $v$ such that $(T - \lambda I)^j v = 0$ where $j \in \{1, 2, 3, \ldots \}$, and $\lambda$ is an evalue of $T$.

(ii) If $j = 1$ we are done. Else, let $j$ be the smallest positive integer with $(T - \lambda I)^j v = 0$, and let $w = (T - \lambda I)^{j-1} v$, which will be nonzero. Then $(T - \lambda I) w = 0$, so $Tw = \lambda w$, so $\lambda$ is an eigenvalue of $T$.

(iii) $G_\lambda = \ker((T - \lambda I)^{\dim V})$.

(iv) Note $(T - I)(x, y, z) = (y-x, -y, x-y)$, so $(T - I)^2(x, y, z) = (T - I)^2(y-x, -y, x-y) = (-2y - z + x, -x + 2y + z, 2y + z - x)$. The kernel of this is $G_1$, and it is $\{(x, y, z) : 2y + z = x\}$. This is 2 dimensional so the multiplicity of the evalve 1 is 2.

Q5. Let $V$ be a vector space over $\mathbb{C}$, and $T \in \mathcal{L}(V)$.

(i) Give a formula for the dimension of the generalized eigenspace $G_\lambda$ involving the matrix of $T$.

(ii) Prove that $\dim V$ is the sum of the multiplicities of the eigenvalues of $T$.

(iii) Prove that if there is a basis for $V$ consisting of eigenvectors of $T$, then every generalized eigenvector of $T$ is an eigenvector of $T$.

Solution.

(i) It equals the number of times $\lambda$ appears on the main diagonal of the matrix of $T$ when the latter is upper triangular.

(ii) Suppose $n = \dim V$. Since the eigenvalues of $T$ are the $n$ numbers on the main diagonal of the $n \times n$ matrix of $T$ when the latter is upper triangular, and their multiplicity is the number of times each appears, it is clear that $\dim V$ is the sum of the multiplicities of the eigenvalues of $T$.

(iii) If $V$ has a basis $B$ of eigenvectors, and if $E_{\lambda_1}, \ldots, E_{\lambda_m}$ are the distinct eigenspaces, then by partitioning $B$ into subsets corresponding to the distinct evaules, each of these subsets is a basis for an eigenspace. So

$$\dim V = \dim E_{\lambda_1} + \cdots + \dim E_{\lambda_m} \leq \dim G_{\lambda_1} + \cdots + \dim G_{\lambda_m} = \dim V,$$

by CH8 Prop 4 and the fact that $E_\lambda \subset G_\lambda$. So $\dim E_{\lambda_k} = \dim G_{\lambda_k}$, and so $E_{\lambda_k} = G_{\lambda_k}$. So every generalized eigenvector is an eigenvector.