MATH 4389–TOPOLOGY AND METRIC SPACES–BLECHER

- A *topological space* is a set with a collection of subsets of it which we have decided to call 'open sets'. This collection of open sets is called a 'topology'. The union of any number of open sets is open, and the intersection of two open sets is open. We can generalize much of Analysis to such spaces, as we shall see briefly here.

Our interest in this part of the survey course/test preparation is really in using a common *language* that applies to $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$, metric spaces, and topology (which are successively more general). Thus we should know, for example, things like 'a union of open sets is open and the intersection of two open sets is open'. Or interior, neighborhood, boundary, limit/accumulation points, closure, compact sets , etc. We will visit these below, but the point is that you know such things from 3333 and elsewhere, the results should all be quite familiar sounding. There are often a few questions on topology or the associated notion of metrics on the Field test.

IF YOU GET LOST BELOW WHEN YOU HEAR THE WORD TOPOLOGY OR METRIC SPACE, JUST TAKE THAT SPACE TO BE THE REAL NUMBERS, OR $\mathbb{R}^n$. (THAT PROBABLY GOES FOR THE RELATED QUESTIONS ON THE FIELD TEST TOO.)

A *neighborhood* of a point x is an open set U containing x. The *interior* int S of a set S are the points x which have a (open) neighborhood which is a subset of S. The interior of a set S is the biggest open set contained in S. And S is open iff int $S = S$. The boundary of a set S are the points x such that every (open) neighborhood of x contains points in S and in S^c. An open set is a set that contains none of its boundary points. An *accumulation* (or limit or cluster) point of a set A is an element x such that every neighborhood of x contains at least one point in $A \setminus \{x\}$ (or equivalently contains infinitely many points in A. If $x \notin A$ then x is an accumulation point iff it is a boundary point.

A *metric space* is a set X, with a function $d : X \times X \to [0, \infty)$, called a *metric*, satisfying the following properties:

(i) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(ii) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

(iii) $d(x, y) = 0$ if and only if $x = y$.

A normed (vector) space is a vector space X over the real or complex scalars, with a function $\|\cdot\| : X \to [0, \infty)$, called a *norm*, satisfying the following properties:

(i) $\|x\| = 0$ iff $x = 0$,
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all scalars λ and $x \in X$,
(iii) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

Proposition 1 Every normed space is a metric space with metric $d(x, y) = ||x - y||$. *Proof.* (i) $d(x, y) = 0$ iff $||x - y|| = 0$ iff $x - y = 0$ iff $x = y$.
(ii) $d(x, y) = ||x - y|| = ||(-1)(y - x)|| = ||y - x|| = d(y, x)$.
(iii) $d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \leq ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$.
□

In a metric space (X, d), an *open ball* is $B(x, r) = \{y \in X : d(x, y) < r\}$, for some $r > 0$. An *interior point* of a set S is a point x for which $B(x, r) \subset S$ for some $r > 0$.

Proposition 2 Open balls in a metric space are open.

Open sets in a metric space are unions of open balls. Indeed

Proposition 3 A set U is open iff $\forall x \in U\ \exists r > 0$ s.t. $B(x, r) \subset U$.

Summary: Every metric space (X, d) has an associated topology, called the *metric topology.* Putting this together with Proposition 1, we see that every normed space, $(X, || \cdot ||)$, has an associated topology, called the *norm topology.*

A closed set is the complement U^c of an open set U, and also can be defined as a set that contains all its boundary (or limit/accumulation) points. A set is clopen iff it is both open and closed. Both the empty set and its complement (the universe) are clopen. Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.

The *closure*, $\overline{A}$, of A, is the intersection of all closed subsets of X, containing A. We take $\overline{\phi} = \phi, \overline{X} = X$. Note that $\overline{A}$ is the smallest closed subset of X containing A. Also A is closed iff $A = \overline{A}$. Test: $x \in \overline{A}$ iff $U \cap A \neq \phi$ for all open sets U with $x \in U$. In a metric space (X, d), $x \in \overline{A}$ iff $B(x, r) \cap A \neq \phi\ \forall r > 0$.

The boundary of A equals $\bar{A} \setminus \text{int}(A)$; and $\bar{A}$ is the union of A and its boundary (or the union of the interior of A and its boundary).

We say that a sequence (x_n) in a metric space (X, d) converges to x if $d(x_n, x) \to 0$. Then a set A in a metric space is closed iff it contains the limits of all convergent sequences with terms in A. Also $\bar{A}$ is the set of limits of all convergent sequences with terms in A.

An *open cover* of a subset A of a topological space X, is a collection $\mathcal{C}$ of open sets whose union contains A. A *subcover* is a subset of this collection $\mathcal{C}$ whose union also contains A. A set is *compact* if every open cover of it has a finite subcover.

A topological space X is *connected* if it does not have a clopen set besides $\emptyset$ and X. A subset of a topological space X is connected if it is connected in the subspace topology. The connected sets in $\mathbb{R}$ are just the intervals.

Let $f : X \to Y$ be a function between topological spaces (we sometimes call a function a 'map'). We say that f is *continuous* if $f^{-1}(V)$ is open in X for all open V in Y. (We may take V to be an open ball here in the metric space case.) We say that f is *continuous at a point* $x \in X$ if for every open neighborhood V of $f(x)$ in Y, there exists an open neighborhood U of x in X such that $f(U) \subset V$. If A is a subset of X then we say that f is *continuous on* A if f is continuous at every point in A.

Theorem Let $f : X \to Y$ be a function between topological spaces. TFAE:
(i) f is continuous.
(ii) $f^{-1}(C)$ is closed in X $\forall$ closed $C \subset Y$.
(iii) f is continuous at every $x \in X$.

We have the usual results, such as compositions of continuous functions are continuous, etc.

Theorem Let $f : X \to Y$ be a function between metric spaces, and $x \in X$. TFAE:
(i) f is continuous at x.
(ii) whenever $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.
(iii) Given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(y), f(x)) < \epsilon$ whenever $d(y, x) < \delta$.

A function $f : X \to Y$ between topological spaces is called *open* if $f(U)$ is open in Y for all open U in X. We say that f is a *homeomorphism* if f is one-to-one, onto, continuous, and open (note that a one-to-one, onto function is open iff f^{-1} is continuous). Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism between them.

Theorem (Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Theorem (Bolzano-Weierstrass) Every nonempty bounded set in $\mathbb{R}^n$ has an accumulation point. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Theorem If $f : X \to Y$ is continuous, and X is compact, then $f(X)$ is compact.

Corollary (Extreme value/Min-Max theorem) If $f : X \to \mathbb{R}$ is continuous on a topological space X, and given a compact subset A of X, then f achieves a minimum and a maximum on A. That is, there exist $x_1, x_2 \in X$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in A$. In particular, f is bounded.

[Proof: Exercise using last theorem, and results from 3333 like the Heine-Borel theorem.]

For sets in $\mathbb{R}^n$, path connectedness or convexity is often more important than connectedness, and if you are taking the math subject GRE you could look up these on wikipedia, but we will treat connectedness here.

Theorem If $f : X \to Y$ is continuous, and X is connected, then $f(X)$ is connected.

Corollary (Intermediate Value Theorem (IVT)) If $f : X \to \mathbb{R}$ is continuous, X is connected, then $f(X)$ is an interval. Hence if c is a number between $f(x_1)$ and $f(x_2)$, for $x_1, x_2 \in X$, then $\exists z \in X$ such that $f(z) = c$.

If, in addition, X is compact, then $f(X) = [a, b]$, some $a \leq b$ in $\mathbb{R}$.

Proof. Use the last theorem and the fact that connected sets in the $\mathbb{R}$ are intervals. So $f(X)$ is an interval. By the Extreme value/Min-Max theorem above (or the Theorem stated before that), it is a compact interval. But the only compact intervals are $[c, d]$, some c, d in R. Since Range f equals $[c, d]$, if $f(x_1) < t < f(x_2)$ for $x_1, x_2 \in X$ then $t \in [c, d] = f(X)$, so there exists $z \in X$ with $f(z) = t$. $\square$

As time permits, read as much as you can of the online notes on Analysis, and the two "Multivariable" pdf's on the course notes link under the analysis tab. You could also watch Almus' topology video on `https://www.math.uh.edu/~almus/4389_notes.html` for a presentation on some part of the above.