Topology, Spring 2005

Homework 1 [Flagg/Blecher]

1. We show that the quotient topology on R/Q is the 'indiscrete topology' of 1.2.1. Indeed suppose that U was a nonempty open set in the quotient topology on R/Q, and let $p : R \to R/Q$ be the 'quotient map', i.e. the map taking $x \in R$ to its equivalence class $[x] \in R/Q$. Let $[x] \in U$. Then $p^{-1}(U)$ is an open set in R containing x. Thus there is an open interval (a, b) inside $p^{-1}(U)$. Clearly for every $x \in p^{-1}(U)$ and $r \in Q$, we have $x+r \in p^{-1}(U)$ (since $p(x + r) = p(x)$). Thus $p^{-1}(U)$ contains $\cup_{r \in Q}(a + r, b + r)$. Thus $p^{-1}(U) = R$, so that $U = p(p^{-1}(U)) = p(R) = R/Q$.

2. Text p. 144, 2(a): If $p \circ f = I_Y$ then p is onto. Since p is continuous, if U is open then $p^{-1}(U)$ is open. Conversely, if $p^{-1}(U)$ is open then $f^{-1}(p^{-1}(U))$ is open. But $f^{-1}(p^{-1}(U)) = (p \circ f)^{-1}(U) = U$. Thus p satisfies the definition of a quotient map.

Text p. 144, 2(b): Follows from 2(a), taking $p = r$, and f the inclusion map from A into X.

Text p. 144, 3: Certainly q is onto, and continuous (since it is the restriction of a continuous function). Let $f : R \to A$ be the function $f(x) = (x, 0)$; clearly f is continuous, and $p \circ f = I_R$. So by 2(a) above, q is a quotient map. Let $U = \{(x, y) : y > 0\} \cap A$. This is open in A, but $q(U) = [0, \infty)$ which is not open in R. Let $C = \{(x, y) : y = \frac{1}{x}\}$. This is closed in A, but $q(C)$ is not closed.

3-5. Discussed in 'workshop'—I have notes from the workshop if you want to zerox them. Be sure you know why, for example, the torus in R^3 is homeomorphic to $S^1 \times S^1 \subset R^4$.

6. Let R be the real line with its usual topology, and define an equivalence relation on R by $x \sim y$ if and only if $x = y$ or x and y are both integers. Show that the projection of R onto the quotient topological space $R/\sim$ is closed, but that $R/\sim$ is not locally compact, nor first or second countable.

Proof: Let us write Z for the integers, and β for the equivalence class in $R/\sim$ consisting of the integers. Let $R_0 = R \setminus Z$. We may identify $R/\sim$ with the set $R_0 \cup \{\beta\}$, and then the quotient map $q : R \to R/\sim$ is the function taking Z to β, and otherwise is the 'identity map' on R_0. We will write this function as p. The quotient topology on $R/\sim$ then corresponds to the following topology on $R_0 \cup \{\beta\}$, namely the usual open sets in R_0, together with sets of the form $(U \setminus Z) \cup \{\beta\}$, for an open set U in R which contains Z. This is easy to see (divide the open sets V in $R_0 \cup \{\beta\}$ into two classes, the ones containing β and the ones not containing β. An open set in $R_0 \cup \{\beta\}$ not containing β is just an open set in R which contains no integers. If V is an open set in $R_0 \cup \{\beta\}$ containing β, then $U = p^{-1}(V)$ is open set in R containing Z. Write $U = (U \setminus Z) \cup Z$, then $V = p(p^{-1}(V)) = p((U \setminus Z) \cup Z) = p(U \setminus Z) \cup \{\beta\} = (U \setminus Z) \cup \{\beta\}$.)

It is now easy to argue that p is closed: let C be a closed subset of R. Case 1: $C \cap Z = \emptyset$. In this case, $p(C) = C$, and the complement of C in $R_0 \cup \{\beta\}$ is $\{\beta\} \cup (R_0 \setminus C) = \{\beta\} \cup ((R \setminus C) \setminus Z)$, which is open according to the last paragraph. Case 2: $C \cap Z \neq \emptyset$. In this case, $p(C) = (C \setminus Z) \cup \{\beta\}$, whose complement is easily seen to be $(R \setminus C) \setminus Z$, which is an open set in the usual sense in R_0. In either case, $p(C)$ is closed in $R_0 \cup \{\beta\}$.

To see that $R_0 \cup \{\beta\}$ is not locally compact, suppose that V was an open set in $R_0 \cup \{\beta\}$ containing β, and that K is a compact set in $R_0 \cup \{\beta\}$ containing V. By the facts above, $V = (U \setminus Z) \cup \{\beta\}$, for an open set U in R containing Z. For each integer n, we can pick $t_n \in (0,1)$ such that $(n - t_n, n + t_n) \subset U$. Let $U_1 = \cup_{n \in Z}(n - \frac{t_n}{2}, n + \frac{t_n}{2})$, then $U_0 = (U_1 \setminus Z) \cup \{\beta\}$ is open in $R_0 \cup \{\beta\}$. Also $\{U_0\} \cup \{(n, n+1) : n \in Z\}$ is an open cover of K. Since K is compact, there is a finite subcover, $\{U_0\} \cup \{(n, n+1) : n \in Z, |n| \leq M\}$ say. But the latter does not cover V, and hence cannot cover K. This is a contradiction.

To see that $R_0 \cup \{\beta\}$ is not second countable, it is enough to show it is not first countable, since any second countable space is first countable (see 2.3.1). By way of contradiction, suppose that $\{B_1, B_2, \cdots\}$ was a countable set of open sets in $R_0 \cup \{\beta\}$ containing β such that for any open set V in $R_0 \cup \{\beta\}$ containing β, there exists an n with $\beta \in B_n \subset V$ (or equivalently, that $Z \subset p^{-1}(B_n) \subset p^{-1}(V)$). By intersecting each B_n with $p(\cup_{n \in Z}(n - \frac{1}{2}, n + \frac{1}{2}))$, we may assume without loss of generality that $p^{-1}(B_n)$ contains no subinterval of length $\geq \frac{1}{2}$. Fix $n \in Z$. As in the last paragraph, for any positive integer k, let t_n^k be the supremum of the numbers $t \in (0,1)$ such that $(n - t, n + t) \subset p^{-1}(B_k)$. In particular, $(n - 2t_n^n, n + 2t_n^n)$ is not contained in $p^{-1}(B_n)$ for any positive integer n. Let $U = \cup_{n \in Z}(n - 2t^{|}n|_n, n + 2t^{|}n|_n)$. Then $V = (U \setminus Z) \cup \{\beta\}$ is an open neighborhood of β in $R_0 \cup \{\beta\}$. However V is not contained in B_n since $p^{-1}(V) = U$ contains $(n - 2t_n^n, n + 2t_n^n)$. This contradicts the hypothesis towards the start of this paragraph. Thus $R_0 \cup \{\beta\}$ is not first countable.

7. Let B^2 be the closed unit disk in R^2, with boundary the unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Show that the unit sphere S^2 is homeomorphic to the attachment space $B^2 \cup_f X$, if either (a) X is a singleton set and $f : S^1 \to X$ is constant; or (b) $X = [-1, 1]$ and $f : S^1 \to X$ is the function $f(x, y) = x$.

Proof (Linsenmann): By the Proposition after the definition of $X \cup_f Y$ in the notes, it suffices to define a continuous surjective $g : B^2 \cup X \to S^2$ such that for every $w \in S^2$, $g^{-1}(\{w\})$ is one of the equivalence classes in $B^2 \cup_f X$. In (a) it is easy to see that there exists such a function by drawing pictures. Namely, let g take X to the north pole $(0, 0, 1)$; and on B^2 let g be a function taking the center of the disk, the origin $(0, 0)$, to the south pole $(0, 0, -1)$, and taking the circle of radius r center the origin $(0, 0)$ to the circle which is the intersection of the plane $z = c$ with the sphere $x^2 + y^2 + z^2 = 1$. This should be done in such a way that c increases as r increases, and that $c = 1$ when $r = 1$. It is easy to see that this can be done in such a way that g is continuous. (If you try a bit harder one may explicitly write down the function:

$$g(x,y) = (\frac{x\sin(\pi\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}, \frac{y\sin(\pi\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}, -\cos(\pi\sqrt{x^2+y^2})),$$

for $(x,y) \in B^2, (x,y) \neq (0,0)$, and $g(0,0) = (0,0,-1)$ and $g = (0,0,1)$ on X.)

(b) is similar to (a) so we omit most details. As in (a) one can see that there exists such a function by drawing pictures. Basically it is the function that behaves like a zipper, the one half of the zipper sewed to the top semicircle of B^2, the other half of the zipper sewed to the bottom semicircle, and then zipping it up. (Again, if you try a bit harder one may explicitly write such a function, as in (a), but I won't take the trouble to do so).

8. Prove that a locally Euclidean space is locally compact, locally connected, and locally path connected.

Proof: Let X be locally Euclidean and let $x \in X$. Choose an open neighborhood U of x such that U is homeomorphic to an open set $N \subset R^m$. Let $g : N \to U$ be the homeomorphism, and choose $p \in N$ such that $g(p) = x$. Since N is locally compact (why?), there exists an open V and compact K in R^m with $p \in V \subset K \subset N$. Since g is a homeomorphism, $g(V)$ is open, $g(K)$ is compact, and $x \in g(V) \subset g(K) \subset X$. So X is locally compact.

To see that X is locally connected (resp. locally path connected) at x, note that for every $T \in \tau$ with $x \in T$, $T \cap U$ is an open subset of U, so $g^{-1}(T \cap U)$ is open in N. There exists an open ball B with $p = g^{-1}(x) \in B \subset g^{-1}(T \cap U) \subset N$. The image of B under g is an open subset of T which is connected and path connected by 3.1.6. Therefore, X is locally (path) connected by definition.

9. Show that every compact m-manifold is the topological sum of a finite number of connected compact m-manifolds.

Proof: If X is a compact m-manifold then X is locally connected by the previous question. By 3.1.13 Prop 1, every component of X is clopen. Since X is compact, there must therefore be a finite number of components. Since each component of X is clopen, it is easy to see that X is homeomorphic to the topological sum of these components. Each component is clearly connected and compact, and is Hausdorff and second countable since these properties are hereditary. If x is a point in a component C of X, and if U is an open neighborhood of x in X homeomorphic via a function φ to an open set in R^m, then $U \cap C$ is an open neighborhood of x in C which is easily checked to be homeomorphic via $\varphi_{|C}$ to an open set in R^m. So C is an m-manifold. Thus X is the topological sum of a finite number of connected compact m-manifolds.

10. Text page 227

1. Prove that every manifold is regular and hence metrizable.

Proof: An m-manifold X is (by definition 3.2.5) second countable, Hausdorff, and m-locally Euclidean. By Question 5 above, X is locally compact, and hence is regular by 2.4.2. By 2.3.2, X is metrizable.

2. Let X be a compact Hausdorff space. Suppose that for each $x \in X$, there is a neighborhood U of x and a positive integer k such that U can be imbedded in R^k. Show that X can be imbedded in R^N for some positive integer N.

Proof: Since X is compact, we can cover X by a finite number of open sets $U_1, U_2, .., U_n$ such that each U_i can be embedded in R^{k_i}. Then use the proof of 3.3.3.

3. Let X be a Hausdorff space such that each point of X has a neighborhood that is homeomorphic with an open subset of R^m. Show that if X is compact, then X is an m-manifold.

Proof: The space X is a compact Hausdorff space that is m-locally Euclidean by hypothesis. To be a manifold, X needs to be second countable. Also, note that X satisfies the hypothesis for question 2 above, so it can be imbedded in R^n for some integer n. This means that X is homeomorphic to a subset of R^n. By last semester's homework 7 problem 5 we know that R^n is second countable and a subspace of a second countable space is also second countable. So, X is homeomorphic to a second countable space. Since second countability is a topological property, X is second countable. Hence, X is an m-manifold.