1. We show that the quotient topology on \( R/Q \) is the ‘indiscrete topology’ of 1.2.1. Indeed suppose that \( U \) was a nonempty open set in the quotient topology on \( R/Q \), and let \( p : R \to R/Q \) be the ‘quotient map’, i.e. the map taking \( x \in R \) to its equivalence class \([x] \in R/Q\). Let \([x] \in U\). Then \( p^{-1}(U) \) is an open set in \( R \) containing \( x \). Thus there is an open interval \((a,b)\) inside \( p^{-1}(U) \). Clearly for every \( x \in p^{-1}(U) \) and \( r \in Q \), we have \( x + r \in p^{-1}(U) \) (since \( p(x + r) = p(x) \)). Thus \( p^{-1}(U) \) contains \( \cup_{r \in Q} (a + r, b + r) \). Thus \( p^{-1}(U) = R \), so that \( U = p(p^{-1}(U)) = p(R) = R/Q \).

2. Text p. 144, 2(a): If \( p \circ f = I_Y \) then \( p \) is onto. Since \( p \) is continuous, if \( U \) is open then \( p^{-1}(U) \) is open. Conversely, if \( p^{-1}(U) \) is open then \( f^{-1}(p^{-1}(U)) \) is open. But \( f^{-1}(p^{-1}(U)) = (p \circ f)^{-1}(U) = U \). Thus \( p \) satisfies the definition of a quotient map.

Text p. 144, 2(b): Follows from 2(a), taking \( p = r \), and \( f \) the inclusion map from \( A \) into \( X \).

Text p. 144, 3: Certainly \( q \) is onto, and continuous (since it is the restriction of a continuous function). Let \( f : R \to A \) be the function \( f(x) = (x,0) \); clearly \( f \) is continuous, and \( p \circ f = I_R \). So by 2(a) above, \( q \) is a quotient map. Let \( U = \{(x,y) : y > 0\} \cap A \). This is open in \( A \), but \( q(U) = [0, \infty) \) which is not open in \( R \). Let \( C = \{(x,y) : y = \frac{1}{2}\} \). This is closed in \( A \), but \( q(C) \) is not closed.

3-5. Discussed in ‘workshop’—I have notes from the workshop if you want to xerox them. Be sure you know why, for example, the torus in \( R^3 \) is homeomorphic to \( S^1 \times S^1 \subset R^4 \).

6. Let \( R \) be the real line with its usual topology, and define an equivalence relation on \( R \) by \( x \sim y \) if and only if \( x = y \) or \( x \) and \( y \) are both integers. Show that the projection of \( R \) onto the quotient topological space \( R/\sim \) is closed, but that \( R/\sim \) is not locally compact, nor first or second countable.

Proof: Let us write \( Z \) for the integers, and \( \beta \) for the equivalence class in \( R/\sim \) consisting of the integers. Let \( R_0 = R \setminus Z \). We may identify \( R/\sim \) with the set \( R_0 \cup \{\beta\} \), and then the quotient map \( q : R \to R/\sim \) is the function taking \( Z \) to \( \beta \), and otherwise is the ‘identity map’ on \( R_0 \). We will write this function as \( p \). The quotient topology on \( R/\sim \) then corresponds to the following topology on \( R_0 \cup \{\beta\} \), namely the usual open sets in \( R_0 \), together with sets of the form \((U \setminus Z) \cup \{\beta\}\), for an open set \( U \) in \( R \) which contains \( Z \). This is easy to see (divide the open sets \( V \) in \( R_0 \cup \{\beta\} \) into two classes, the ones containing \( \beta \) and the ones not containing \( \beta \). An open set in \( R_0 \cup \{\beta\} \) not containing \( \beta \) is just an open set
in $R$ which contains no integers. If $V$ is an open set in $R_0 \cup \{\beta\}$ containing $\beta$, then $U = p^{-1}(V)$ is open set in $R$ containing $Z$. Write $U = (U \setminus Z) \cup Z$, then $V = p(p^{-1}(V)) = p((U \setminus Z) \cup Z) = p(U \setminus Z) \cup \{\beta\} = (U \setminus Z) \cup \{\beta\}$.

It is now easy to argue that $p$ is closed: let $C$ be a closed subset of $R$. Case 1: $C \cap Z = \emptyset$. In this case, $p(C) = C$, and the complement of $C$ in $R_0 \cup \{\beta\}$ is $\{\beta\} \cup (R_0 \setminus C) = \{\beta\} \cup ((R \setminus C) \setminus Z)$, which is open according to the last paragraph. Case 2: $C \cap Z \neq \emptyset$. In this case, $p(C) = (C \setminus Z) \cup \{\beta\}$, whose complement is easily seen to be $(R \setminus C) \setminus Z$, which is an open set in the usual sense in $R_0$. In either case, $p(C)$ is closed in $R_0 \cup \{\beta\}$.

To see that $R_0 \cup \{\beta\}$ is not locally compact, suppose that $V$ was an open set in $R_0 \cup \{\beta\}$ containing $\beta$, and that $K$ is a compact set in $R_0 \cup \{\beta\}$ containing $V$. By the facts above, $V = (U \setminus Z) \cup \{\beta\}$, for an open set $U$ in $R$ containing $Z$. For each integer $n$, we can pick $t_n \in (0,1)$ such that $(n - t_n, n + t_n) \subset U$. Let $U_1 = \cup_{n \in Z}(n - \frac{1}{n}, n + \frac{1}{n})$, and $U_0 = (U_1 \setminus Z) \cup \{\beta\}$ is open in $R_0 \cup \{\beta\}$. Also $(U_0) \cup \{n, n + 1 + n Z \setminus Z\} \cup \{(n, n + 1): n \in Z\}$ is an open cover of $K$. Since $K$ is compact, there is a finite subcover, $(\{U_0\} \cup \{(n, n + 1): n \in Z, |n| \leq M\})$. But the latter does not cover $V$, and hence cannot cover $K$. This is a contradiction.

To see that $R_0 \cup \{\beta\}$ is not second countable, it is enough to show it is not first countable, since any second countable space is first countable (see 2.3.1). By way of contradiction, suppose that $\{B_1, B_2, \ldots\}$ was a countable set of open sets in $R_0 \cup \{\beta\}$ containing $\beta$ such that for any open set $V$ in $R_0 \cup \{\beta\}$ containing $\beta$, there exists an $n$ with $\beta \in B_n \subset V$ (or equivalently, that $Z \subset p^{-1}(B_n) \subset p^{-1}(V)$). By intersecting each $B_n$ with $p(n \setminus Z(n - \frac{1}{n}, n + \frac{1}{n}))$, we may assume without loss of generality that $p^{-1}(B_n)$ contains no subinterval of length $\geq \frac{1}{2}$. Fix $n \in Z$. As in the last paragraph, for any positive integer $k$, let $t^n_k$ be the supremum of the numbers $t \in (0,1)$ such that $(n - t, n + t) \subset p^{-1}(B_k)$. In particular, $(n - 2t^n_k, n + 2t^n_k)$ is not contained in $p^{-1}(B_n)$ for any positive integer $n$. Let $U = \cup_{n \in Z}(n - 2t^n_k, n + 2t^n_k)$. Then $V = (U \setminus Z) \cup \{\beta\}$ is an open neighborhood of $\beta$ in $R_0 \cup \{\beta\}$. However $V$ is not contained in $B_n$ since $p^{-1}(V) = U$ contains $(n - 2t^n_k, n + 2t^n_k)$, contradicting the hypothesis towards the start of this paragraph. Thus $R_0 \cup \{\beta\}$ is not first countable.

7. Let $B^2$ be the closed unit disk in $R^2$, with boundary the unit circle $S^1 = \{(x, y): x^2 + y^2 = 1\}$. Show that the unit sphere $S^2$ is homeomorphic to the attachment space $B^2 \cup_f X$, if either (a) $X$ is a singleton set and $f : S^1 \rightarrow X$ is constant; or (b) $X = [-1, 1]$ and $f : S^1 \rightarrow X$ is the function $f(x, y) = x$.

Proof (Linsenmann): By the Proposition after the definition of $X \cup_f Y$ in the notes, it suffices to define a continuous surjective $g : B^2 \cup X \rightarrow S^2$ such that for every $w \in S^2$, $g^{-1}\{w\}$ is one of the equivalence classes in $B^2 \cup_f X$. In (a) it is easy to see that there exists such a function by drawing pictures. Namely, let $g$ take $X$ to the north pole $(0, 0, 1)$; and on $B^2$ let $g$ be a function taking the center of the disk, the origin $(0, 0)$, to the south pole $(0, 0, -1)$, and the circle of radius $r$ center the origin $(0, 0)$ to the circle which is the intersection of the plane $z = c$ with the sphere $x^2 + y^2 + z^2 = 1$. This should be done in such a way that $c$ increases as $r$ increases, and that $c = 1$ when $r = 1$. It is easy to
see that this can be done in such a way that \( g \) is continuous. (If you try a bit harder one may explicitly write down the function:

\[
g(x, y) = \left( \frac{x \sin(\pi \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, \frac{y \sin(\pi \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, -\cos(\pi \sqrt{x^2 + y^2}) \right),
\]

for \((x, y) \in B^2, (x, y) \neq (0, 0), \) and \( g(0, 0) = (0, 0, -1) \) and \( g = (0, 0, 1) \) on \( X. \) )

(b) is similar to (a) so we omit most details. As in (a) one can see that there exists such a function by drawing pictures. Basically it is the function that behaves like a zipper, the one half of the zipper sewed to the top semicircle of \( B^2, \) the other half of the zipper sewed to the bottom semicircle, and then zipping it up. (Again, if you try a bit harder one may explicitly write such a function, as in (a), but I won’t take the trouble to do so).

8. Prove that a locally Euclidean space is locally compact, locally connected, and locally path connected.

Proof: Let \( X \) be locally Euclidean and let \( x \in X. \) Choose an open neighborhood \( U \) of \( x \) such that \( U \) is homeomorphic to an open set \( N \subset \mathbb{R}^m. \) Let \( g : N \to U \) be the homeomorphism, and choose \( p \in N \) such that \( g(p) = x. \) Since \( N \) is locally compact (why?), there exists an open \( V \) and compact \( K \) in \( \mathbb{R}^m \) with \( p \in V \subset K \subset N. \) Since \( g \) is a homeomorphism, \( g(V) \) is open, \( g(K) \) is compact, and \( x \in g(V) \subset g(K) \subset X. \) So \( X \) is locally compact.

To see that \( X \) is locally connected (resp. locally path connected) at \( x, \) note that for every \( T \in \tau \) with \( x \in T, \) \( T \cap U \) is an open subset of \( U, \) so \( g^{-1}(T \cap U) \) is open in \( N. \) There exists an open ball \( B \) with \( p = g^{-1}(x) \in B \subset g^{-1}(T \cap U) \subset N. \) The image of \( B \) under \( g \) is an open subset of \( T \) which is connected and path connected by 3.1.6. Therefore, \( X \) is locally (path) connected by definition.

9. Show that every compact \( m \)-manifold is the topological sum of a finite number of connected compact \( m \)-manifolds.

Proof: If \( X \) is a compact \( m \)-manifold then \( X \) is locally connected by the previous question. By 3.1.13 Prop 1, every component of \( X \) is clopen. Since \( X \) is compact, there must therefore be a finite number of components. Since each component of \( X \) is clopen, it is easy to see that \( X \) is homeomorphic to the topological sum of these components. Each component is clearly connected and compact, and is Hausdorff and second countable since these properties are hereditary. If \( x \) is a point in a component \( C \) of \( X, \) and if \( U \) is an open neighborhood of \( x \) in \( X \) homeomorphic via a function \( \varphi \) to an open set in \( \mathbb{R}^m, \) then \( U \cap C \) is an open neighborhood of \( x \) in \( C \) which is easily checked to be homeomorphic via \( \varphi|_C \) to an open set in \( \mathbb{R}^m. \) So \( C \) is an \( m \)-manifold. Thus \( X \) is the topological sum of a finite number of connected compact \( m \)-manifolds.

10. Text page 227

1. Prove that every manifold is regular and hence metrizable.
Proof: An m-manifold $X$ is (by definition 3.2.5) second countable, Hausdorff, and m-locally Euclidean. By Question 5 above, $X$ is locally compact, and hence is regular by 2.4.2. By 2.3.2, $X$ is metrizable.

2. Let $X$ be a compact Hausdorff space. Suppose that for each $x \in X$, there is a neighborhood $U$ of $x$ and a positive integer $k$ such that $U$ can be imbedded in $R^k$. Show that $X$ can be imbedded in $R^N$ for some positive integer $N$.

Proof: Since $X$ is compact, we can cover $X$ by a finite number of open sets $U_1, U_2, ..., U_n$ such that each $U_i$ can be embedded in $R^{k_i}$. Then use the proof of 3.3.3.

3. Let $X$ be a Hausdorff space such that each point of $X$ has a neighborhood that is homeomorphic with an open subset of $R^m$. Show that if $X$ is compact, then $X$ is an m-manifold.

Proof: The space $X$ is a compact Hausdorff space that is m-locally Euclidean by hypothesis. To be a manifold, $X$ needs to be second countable. Also, note that $X$ satisfies the hypothesis for question 2 above, so it can be imbedded in $R^n$ for some integer $n$. This means that $X$ is homeomorphic to a subset of $R^n$. By last semester’s homework 7 problem 5 we know that $R^n$ is second countable and a subspace of a second countable space is also second countable. So, $X$ is homeomorphic to a second countable space. Since second countability is a topological property, $X$ is second countable. Hence, $X$ is an m-manifold.