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6. Let \( h : X \to Y \) be continuous, with \( h(x_0) = y_0 \) and \( h(x_1) = y_1 \). Let \( \alpha \) be a path in \( X \) from \( x_0 \) to \( x_1 \), and let \( \beta = h \circ \alpha \). Show that \( \beta \circ (h_{x_0})_* = (h_{x_1})_* \circ \tilde{\alpha} \).

Proof: By definition, for every \([f] \in \pi_1(X, x_0)\), \((h_{x_0})_*([f]) = [h \circ f] \). So, \((\beta \circ (h_{x_0})_*([f]) = \tilde{\beta}([h \circ f]) = \tilde{\beta} \circ \tilde{\alpha} \). Since \( \beta = h \circ \alpha \) is a path from \( y_0 \) to \( y_1 \), the reverse path \( \tilde{\beta} \) goes from \( y_1 \) to \( y_0 \) and is the composition \( h \circ \tilde{\alpha} \).

Thus, \((\beta \circ (h_{x_0})_*([f]) = [h \circ \tilde{\alpha}] \circ [h \circ f] \). Since \( \beta = h \circ \alpha \) is a path, \( [h \circ \tilde{\alpha}] \) is the composition \( h \circ \tilde{\alpha} \).

Thus, \( \beta \circ (h_{x_0})_* = (h_{x_1})_* \circ \tilde{\alpha} \).

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2. Let \( p : E \to B \) be continuous and surjective. Suppose that \( U \) is an open set of \( B \) that is evenly covered by \( p \). Show that if \( U \) is connected, then the partition of \( p^{-1}(U) \) into slices is unique.

Proof: By 4.2.1, let \( \{V_j\}_{j \in J} \) be a partition of \( p^{-1}(U) \) into slices. Since for every \( j \in J \), \( V_j \) is homeomorphic to \( U_j \), each \( V_j \) is connected.

Looking at the subspace \( P = p^{-1}(U) \) of \( E \), I claim that the collection \( \{V_j\}_{j \in J} \) is also the collection of connected components of \( P \). Proof of claim: Let \( C \) be a component of \( P \). Since \( U \cap C \neq \emptyset \). By 3.1.11, \( V_k \subseteq C \). To see the other inclusion, suppose \( c \in C \) and \( c \notin V_k \). Then \( c \in \bigcup_{j \neq k} V_j \), which is an open set is disjoint from \( V_k \). So, \( C = V_k \cup \bigcup_{j \neq k} V_j \cap C \) and this is a separation of \( C \). Since \( C \) is connected, this contradiction means that there exists no such \( c \). Hence, \( C = V_k \). So every component is one of the \( V_j \)'s, and since the slices are disjoint and their union is \( P \), every slice is also a component. Thus the claim is proved. By construction of the components as equivalence classes, the components of \( P \) are unique, and thus the partition of \( P \) into slices is unique.

3. Let \( p : E \to B \) be a covering map; let \( B \) be connected. Show that if \( p^{-1}(\{b_0\}) \) has \( k \) elements for some \( b_0 \in B \), then \( p^{-1}(\{b\}) \) has \( k \) elements for every \( b \in B \).

Proof: First, from set theory, recall that for a given set \( X \), \( |X| \) is the notation for the cardinality of the set \( X \). So, the hypothesis says that \( |p^{-1}(\{b_0\})| = k \).

Define two subsets \( B_k \) and \( B_h \) of \( B \) by \( B_k = \{ b \in B : |p^{-1}(\{b\})| = k \} \) and \( B_h = \{ b \in B : |p^{-1}(\{b\})| \neq k \} \). Then \( B_k \) and \( B_h \) are disjoint and their union is \( B \). Let, for every \( b \in B \), \( U_b \) be an open neighborhood of \( b \) which is evenly covered by \( p \). By definition of \( U_b \) being evenly covered (4.2.1), the number of slices that \( p^{-1}(U_b) \) is partitioned into is equal to \( |p^{-1}(\{b\})| \). Then define a set \( S = \bigcup_{b \in B_k} U_b \) and \( T = \bigcup_{b \in B_h} U_b \). Note that \( S \) and \( T \) are open sets in \( B \) and that \( S \cup T = B \). Since \( B \) is connected, \( S \) and \( T \) cannot separate \( B \), so either (1) \( S \) or \( T \) is empty, or (2) \( S \cap T \neq \emptyset \). First consider possibility (2). If \( S \cap T \neq \emptyset \), then there exists a \( x \in B_k \) and a \( y \in B_h \) such that \( U = U_x \cap U_y \neq \emptyset \). Let \( z \in U \). Since \( U \) is a subset of \( U_z \), \( p^{-1}(U) \) has \( k \) slices so \( |p^{-1}(\{z\})| = k \).
However, $U$ is also a subset of $U_y$, so by the same argument $|p^{-1}(\{y\})| \neq k$. This contradiction means possibility (2) cannot happen. So, (1) is true. Since $b_0 \in B_k$, $U_{b_0} \subset S \neq \emptyset$. So $T = \emptyset$. The only way that can be true is if $B_n = \emptyset$. Therefore $B = B_k$.


(a) If $B$ is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is $E$.

Proof:

1. Hausdorff: Let $s, t \in E$ and assume $s \neq t$. Then $p(s), p(t) \in B$ and we can divide the proof into two cases.

Case 1. Suppose $p(s) \neq p(t)$. Then since $B$ is Hausdorff, there exists open sets $U$ and $V$ such that $p(s) \in U$, $p(t) \in V$, and $U \cap V = \emptyset$. Since $p$ is continuous, $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint open sets in $E$ and $s \in p^{-1}(U)$ and $t \in p^{-1}(V)$.

Case 2. Suppose $p(s) = p(t)$, and call this element $b \in B$. Since $p$ is a covering map, by 4.2.1 choose an open set $b \in W \subset B$ that is evenly covered by $p$. Let $\{V_a\}_{a \in A}$ be a partition of $p^{-1}(W)$ into slices. There exists a $\beta \in A$ such that $s \in V_\beta$ and a $\gamma \in A$ such that $t \in V_\gamma$. Since $p$ restricted to $V_\beta$ or $V_\gamma$ is injective, $\beta \neq \gamma$. Thus $s \in V_\beta$ and $t \in V_\gamma$ and they are disjoint open sets in $E$.

Therefore, $E$ is Hausdorff.

2. Regular: Suppose $B$ is regular, $x \in E$, and $O$ is an open set in $E$ containing $x$. To show $E$ is regular it is sufficient to show that there exists an open set $W \subset E$ such that $x \in W \subset \overline{W} \subset O$.

Let $y = p(x)$ and let $U$ be an evenly covered open neighborhood of $y$. Let $\{V_\alpha\}_{\alpha \in A}$ be the partition of $p^{-1}(U)$ into slices and let $V_\beta$ be the slice that contains $x$. Then $V_\beta \cap O$ is an open subset of $V_\beta$ which is homeomorphic to an open subset $U^* \subset U$. Since $B$ is regular, for the given $y$ and open $U^*$ there exists an open $W$ such that $y \in W \subset \overline{W} \subset U^*$. Then $P = p^{-1}(W) \cap V_\beta$ is open in $E$, and using the hint in the book to make this faster, $C = p^{-1}(\overline{W}) \cap V_\beta$ is closed in $E$. Thus, $x \in P \subset C \subset C \subset \overline{V_\beta} \cap O \subset O$. Hence, $E$ is regular.

3. Completely Regular:

4. Locally Compact Hausdorff:

Suppose $B$ is locally compact Hausdorff. First, by part one we know that $E$ is Hausdorff. Let $x \in E$ and let $y = p(x)$. Since $p$ is a covering map, let $U$ be an evenly covered open neighborhood of $y$. Let $\{V_\alpha\}_{\alpha \in A}$ be the partition of $p^{-1}(U)$ into slices and let $V_\beta$ be the slice that contains $x$. Since $U$ is an open subset of $B$, it is also locally compact by 2.4.3. Since $V_\beta$ is homeomorphic to $U$ and local compactness is a topological property, $V_\beta$ is locally compact in the subspace topology. There exists an (relatively) open $S$ and a (relatively) compact $K$ in $V_\beta$ such that $x \in S \subset K$. Since $V_\beta$ is open, by 1.2.?, $S$ is open in $E$. Further, $K$ is compact in the whole space $E$ as well (using the convergent subnet definition of compactness and HW$ \#3$). Therefore, $E$ is locally compact by definition 2.4.1.

(b) If $B$ is compact and $p^{-1}(\{b\})$ is finite for each $b \in B$, then $E$ is compact.

Proof: Let $(x_A)$ be a net in $E$. Then $(p(x_A))$ is a net in $B$. Since $B$ is compact, by 2.1.1, there exists a subnet $(p(x_{A_b}))$ of the net $(p(x_A))$ that converges to a point $b \in B$. Let $U \subset B$ be an open neighborhood of $b$ that
is evenly covered by $p$. Note that by HW4 #3 we can consider the net in the subspace topology on $U$. For this convergent subnet, look at the corresponding net $(x_{\lambda_n})$ in $E$, or without loss of generality consider the net in the subspace $p^{-1}(U)$. By hypothesis, $p^{-1}([b])$ is finite, so $p^{-1}(U)$ can be partitioned into finitely many disjoint slices $V_1, V_2, ..., V_n$. There are finitely many disjoint slices, so there must be at least one slice, call it $V_m$, such that the net $(x_{\lambda_n})$ is frequently in $V_m$. Then let $(e_\beta)$ be the subnet of $(x_{\lambda_n})$ that is a subset of $V_m$. Now, $(p(e_\beta))$ is a subnet of the convergent net $(p(x_{\lambda_n}))$, so it also converges. Since a subnet of a subnet is a subnet, $(e_\beta)$ converges. Since a subnet of a subnet is a subnet, $(e_\beta)$ is a subnet of $(x_{\lambda_n})$ that converges. Therefore, $E$ is compact.

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8. Let $p: E \to B$ be a covering map with $E$ path connected. Show that if $B$ is simply connected, then $p$ is a homeomorphism.

Proof: Since $B$ is simply connected, $\pi_1(B, b)$ is the trivial one element group. Since the lifting correspondence $\phi: \pi_1(B, b) \to p^{-1}([b])$ is surjective by 4.2.5, $p^{-1}([b])$ is a singleton. Since $b$ was arbitrary, $p$ is 1-1. By 4.2.2, $p$ is open. Thus $p$ is a homeomorphism.

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2. Let $g: S^2 \to S^2$ be continuous. If for all $x \in S^2$, $g(x) \neq g(-x)$, then $g$ is surjective.

Proof by Contrapositive: Suppose $g$ is continuous and not surjective, and show that there exists a point $s \in S^2$ such that $g(s) = g(-s)$. Let $p \in S^2 - g(S^2)$. By 1.4.5, $g: S^2 \to S^2 - \{p\}$ is continuous. Since $S^2 - \{p\}$ is homeomorphic to $R^2$, let $h: S^2 - \{p\} \to R^2$ be the homeomorphism. Then, $h \circ g$ is a continuous map from $S^2$ to $R^2$. By 4.3.7 cor. 2, there exists a point $x \in S^2$ such that $(h \circ g)(x) = (h \circ g)(-x)$. Since $h$ is a homeomorphism, $h$ is injective and this means $g(x) = g(-x)$.

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3.(a) Show that $R^1$ and $R^n$ are not homeomorphic if $n > 1$.

Proof: This may be seen since $R^1$ with one point removed is disconnected, yet $R^n$ with one point removed is connected.

(b) Show that $R^2$ and $R^n$ are not homeomorphic if $n > 2$.

Proof: Let $n > 2$. Suppose there exists a homeomorphism $h : R^2 \to R^n$. Then $h$ restricted to the open set $R^2 \setminus \{0\}$ is also a homeomorphism between $R^2 \setminus \{0\}$ and $R^n \setminus \{h(0)\}$. This means that their respective fundamental groups are isomorphic. However, by 4.4.2, $\pi_1(R^2 \setminus \{0\}) \approx Z$ and $\pi_1(R^n \setminus \{h(0)\}) \approx 0$. This creates a contradiction, so $R^2$ and $R^n$ are not homeomorphic.