1. Find the following limits:

\[(i) \lim_{x \to 0} \frac{3x}{\sin(2x)}, \quad (ii) \lim_{x \to 1^+} \frac{1}{1-x^2}, \quad (iii) \lim_{x \to 1^-} \frac{1}{1-x^2}, \quad (iv) \lim_{w \to 0} \frac{\sqrt{1+3w}-\sqrt{1-3w}}{w}\]

\[= 3/2 \quad 4 \quad 3 \quad \text{Answer only is OK.}\]

\[= -\infty \quad 4 \quad 3\]

\[= +\infty \quad 3\]

\[= \lim_{w \to 0} \frac{(\sqrt{1+3w}-\sqrt{1-3w})(\sqrt{1+3w}+\sqrt{1-3w})}{w(\sqrt{1+3w}+\sqrt{1-3w})} \quad 4\]

\[= \lim_{w \to 0} \frac{6w}{6(\sqrt{1+3w}+\sqrt{1-3w})} = \frac{6}{2} = 3 \quad 2\]

2. Suppose \(f(x) = \begin{cases} x^2, & x \leq 1 \\ x, & x > 1 \end{cases}\).

(i) Evaluate \(\lim_{x \to 1^+} f(x)\).

(ii) Evaluate \(\lim_{x \to 1^-} f(x)\).

(iii) For what values of \(x\) is \(f(x)\) continuous? \(\text{All values of } x \quad 4\)

(iv) Evaluate \(\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}\).

(v) Evaluate \(\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}\).

(vi) Where is \(f(x)\) differentiable? \(\text{Everywhere except at } x=1 \quad 2\)

(vii) Draw a sketch of \(f(x)\) and write an explanation of what the above information tells you about the graph.

\[\text{The graph is smooth (no corners or sharp angles) except at } x=1 \quad 2\]
3. From the definition of the derivative find \( f'(x) \) if \( f(x) = \frac{1}{2x-3} \).

\[
\lim_{x \to x} \frac{f(x) - f(x)}{x - x} = \lim_{x \to x} \frac{1}{2x-3} - \frac{1}{2x-3} = \lim_{x \to x} \frac{2x-x - (2x-3)}{(2x-3)(2x-3)(2x-3)}
\]

\[
= \lim_{x \to x} \frac{-2(x-x)}{(2x-3)(2x-3)(2x-3)} = -\frac{2}{(2x-3)^2}
\]

If only one

4. How fast is \( y = \cos^2(3x) \) changing when \( x = \frac{\pi}{3} \)?

\[
y' = 2\cos(3x) (-\sin(3x))(3) = -6\cos(\frac{3\pi}{4})\sin(\frac{3\pi}{4}) = \frac{3}{2}
\]

5. Find the equation of the tangent line to the graph of \( y = 1 + 3x + x^4 \) at \( x = 1 \).

\[
y' = 3x^2 + 4x^3 = 7 \quad \text{at} \quad x = 1 \quad \Rightarrow \quad y = 1 + 3(1) + 5 \quad \text{at} \quad x = 1
\]

\[
\text{get} \quad y - 5 = 7(x-1)
\]

6. Suppose that \((x - y)^2 + 2 = 2x\). Find \( \frac{dy}{dx} \) at the point \((3, 1)\).

\[
2(x-y)(1-\frac{dy}{dx}) = 2 \quad \Rightarrow \quad 2\cdot 2(1-\frac{dy}{dx}) = 2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \quad \text{at} \quad (3, 1)
\]

7. Suppose that \( f(x) = \frac{15x^2+25x}{3x^2-12x-15} \).

(i) Find the vertical asymptotes of the graph of \( f \), if any.

(ii) Find the horizontal asymptotes of the graph of \( f \), if any.

\( f(x) = \frac{5x(3x+5)}{(3x+5)(x-3)} \) so a vertical asymptote at \( x = 3 \)

\[
\lim_{x \to \pm\infty} \frac{15+25x}{3-\frac{9}{2}-15x^2} = \frac{15}{5} = 5 \quad \text{so} \quad y = 5 \quad \text{is a vertical asymptote}
\]

8. Compute the derivative of \( \sec(1 + x^3) \).

\[
= \sec(1+x^3)\tan(1+x^3) \cdot (3x^2)
\]

\[
3 \quad 3 \quad 2
\]
9. Use the tangent line approximation to approximate \( f(2.2) \) when \( f(x) = \sqrt{x^2 + 5} \).

\[
\begin{align*}
f(2.2) & \approx f(2) + 0.2 f'(2) \\ &= 3 + 0.2 \left( \frac{2 \cdot 12}{2 \cdot (2^2 + 5)} \right) \\ &= 3 + \frac{2}{15} \\ &= 3\frac{2}{15}
\end{align*}
\]

10. A girl who is 5 feet tall, is walking away from a 20 foot tall lamppost. She is walking at a speed of 6 feet per second. How fast is the length of her shadow changing?

By similar triangles, \( \frac{d}{5} = \frac{d+x}{20} \rightarrow d+x = 4d \rightarrow x = 3d \)

\( \frac{d}{dt} = \frac{3dl}{dt} \rightarrow \frac{dl}{dt} = \frac{1}{3} (6) = 2 \)

Her shadow is lengthening at 2 feet/sec.

11. A rectangular sheet of metal 20 inches wide and 100 inches long is folded along the center to form a triangular trough. Two extra pieces of metal are attached to the ends of the trough, and it is filled with water. How deep should the trough be to maximize its capacity? (See diagram.)

Volume \( V = 100h \sqrt{100-h^2} \)

\( \frac{dV}{dh} = 100 \sqrt{100-h^2} + 100h \left( -\frac{2h}{2\sqrt{100-h^2}} \right) = 0 \)

\( 100-h^2 = h^2 \rightarrow h^2 = 50 \rightarrow h = \sqrt{50} = 5\sqrt{2} \)

The trough should be \( 5\sqrt{2} \) inches deep.

12. Let \( f(x) = 2x^3 - 3x^2 + 1 \).

(a) Find one x-intercept. Use synthetic division to find the other x-intercepts.
(b) Find the y-intercept.
(c) Find where the function is decreasing and increasing.
(d) Find the critical points of \( f \) and classify these as local maxima, local minima, or neither. What is the slope of \( f(x) \) at \( x = 1? \)
(e) Find \( \lim_{x \to \infty} f(x) \).
(f) Find all the inflection points of \( f(x) \).
(g) Sketch the graph of \( f(x) \), clearly showing and labelling the details found above.
(h) On the same sketch draw \( g(x) = 5x + 1 \). Find where \( f \) and \( g \) cross.
(i) Find the area of the region between the graphs of \( f \) and \( g \).
13. Let \( f(x) = x^2 - x \) for \( 0 \leq x \leq 2 \) and let \( P = \{0, 1, 2\} \). Find the lower and upper sums \( L_f(P) \) and \( U_f(P) \).

\[
\begin{align*}
  f(\frac{1}{2}) &= -\frac{1}{4} \\
  L(f, P) &= -\frac{1}{4}(1) = -\frac{1}{4} \\
  U(f, P) &= 2.1 = 2
\end{align*}
\]

14. Calculate the following: \([\text{approximately 7 points each}]\)

(a) \( \int_0^\frac{\pi}{3} 7x \, dt \), (b) \( \int_0^\sqrt{8} x \sqrt{1+x^2} \, dx \), (c) \( \frac{d}{dx} \left( \int_0^1 \frac{u^2}{\sqrt{1+u^2}} \, du \right) \).

(a) \( \frac{7\pi}{12} \)

(b) \( \frac{1}{2} \left. u^{3/2} \right|_0^1 = \frac{1}{2} \left. \frac{26}{3} \right| = \frac{26}{3} \)

(c) \( \int \left( \frac{u^2}{1+u^2} \right) \, du = -\int g'(x) \, dx = -\int \left( \frac{x}{1+x} \right)' \, dx = -\int \left( \frac{1}{1+x} \right) \, dx = -\frac{\sqrt{2}}{2(1+x)} \)

15. Find the indefinite integral \([\text{approximately 5 points each}]\)

(a) \( \int (x^5 - x^3) \, dx \), (b) \( \int \frac{1 + \sqrt{x} + 1}{\sqrt{x} + 1} \, dx \), (c) \( \int \sin^3 t \cos t \, dt \).

(a) \( \frac{5}{8} x^{8/5} - \frac{3}{8} x^{8/3} + C \)

(b) \( \int 2u \, du = u^2 + C = (1 + \sqrt{x} + 1)^2 + C \), where \( u = 1 + \sqrt{x} + 1 \), \( du = \frac{1}{2 \sqrt{x} + 1} \)

(c) \( \int u^3 \, du = \frac{u^4}{4} + C = \frac{\sin^4 t}{4} + C \)

16. What is the volume of the body that you get when you rotate the region between \( y = x \) and \( y = x^2 \) for \( 0 \leq x \leq 1 \) about the (a) x-axis? (b) the y-axis? (c) the line \( y = -1 \)? \([20 + 18]\)

(a) \( \pi \int_0^1 (x^2 - x^4) \, dx = \pi \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{8\pi}{15} \)

(b) \( \pi \int_0^1 ((x+1)^2 - (x+1)^2) \, dx = \pi \int_0^1 (x^2 + 2x + 1 - x^2 + 1) \, dx \)

(c) \( \pi \int_0^1 (bx^2 + ax - x^4) \, dx = \pi \left( \frac{1}{1} - \frac{1}{5} \right) = 14 + \pi \)