• **Basic Comparison Test:** Suppose that $0 \leq a_k \leq b_k$ for all $k$.
  1) If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.
  2) If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.

• **Limit Comparison Test:** If $0 < \lim_{k \to \infty} \frac{a_k}{b_k} < \infty$, then $\sum_k a_k$ converges if and only if $\sum_k b_k$ converges.

• In practice, most ‘Basic Comparison Test’ or ‘Limit Comparison Test’ examples can be done by a ‘winning term’ argument (explained in class, see the note at the end of this section).

11.3. **Root Test:** If $\sum_k a_k$ is a nonnegative series with $\lim_{k \to \infty} (a_k)^{\frac{1}{k}} = r$. If $0 \leq r < 1$ then $\sum_k a_k$ converges. If $1 < r \leq \infty$ then $\sum_k a_k$ diverges.

11.3. **Ratio Test:** If $\sum_k a_k$ is a nonnegative series with $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lambda$. If $0 \leq \lambda < 1$ then $\sum_k a_k$ converges. If $1 < \lambda \leq \infty$ then $\sum_k a_k$ diverges.

• Examples. Determine whether the following series converge or diverge: (a) $\sum_k \frac{1}{k^{1.3}}$; (b) $\sum_k \frac{1}{k^{1.5}}$; (c) $\sum_k \frac{1}{\sqrt{k^2 - k}}$; (d) $\sum_k \frac{k}{2^k}$; (e) $\sum_k \frac{k!}{(2k)!}$; (f) $\sum_k \frac{1}{(\ln k)^k}$.

**Solution:**
(a) The $k^3$ here is much more important that the $k$ here, so think of the series as being comparable to $\sum_k \frac{1}{k^{1.3}}$. This is a p-series with $p = \frac{3}{2} > 1$, so $\sum_k \frac{1}{k^{1.3}}$ converges by the p-series test above. Now we can use either the basic comparison test or the limit comparison test to see that $\sum_k \frac{1}{k^{1.3}}$ converges. For example, since $k^3 < k + k^3$ we have $\sqrt{k^3} < k + k^3$, so that $\frac{1}{\sqrt{k^3}} < \frac{1}{k + k^3}$. Since $\sum_k \frac{1}{\sqrt{k^3}}$ converges, our other series converges by the basic comparison test.

(b) We may start similarly to (a), compare with $\sum_k \frac{1}{k^2}$ which is divergent. Let's apply the limit comparison test, with $b_k = \frac{1}{k^{1.5}}$ and $a_k = \frac{1}{k^{1.3}}$. We have

$$\frac{a_k}{b_k} = \frac{k^{1.5}}{k^{1.3}} = 1 + \frac{0.5}{1} \to 1$$

as $k \to \infty$. Since this limit is $> 0$, the limit comparison test tells us that $\sum_k \frac{1}{k^{1.3}}$ diverges.

(c) Similar to (b), but compare with the convergent p-series $\sum_k \frac{1}{\sqrt{k}}$ (see (a)), using the limit comparison test. We let $a_k = \frac{1}{\sqrt{k}}$ and $b_k = \frac{1}{\sqrt{k^3}}$, then

$$\frac{a_k}{b_k} = \frac{\sqrt{k^3}}{\sqrt{k^3}} = \sqrt{1 - \frac{k^3}{k^{1.5}}} \to 1 > 0$$

as $k \to \infty$. Thus the limit comparison test tells us that $\sum_k \frac{1}{\sqrt{k^3}}$ converges.

(d) You could use the ratio or the root test here. If we use the ratio test with $a_k = k^3$, then $a_{k+1} = \frac{(k+1)^3}{2^k}$, so that

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^3}{2 \cdot 2^k} \cdot \frac{2^k}{k^3} = \left(1 + \frac{1}{k}\right)^3 = \frac{1}{2} \left(1 + \frac{1}{k}\right)^3 \to \frac{1}{2}$$

as $k \to \infty$. Since this limit is less than 1, the ratio test tells us that the series converges.
(e) We use the ratio test with \( a_k = \frac{k!}{(2k+1)!} \). Then \( a_{k+1} = \frac{(k+1)!}{(2k+3)(2k+1)!} \). But \( (k+1)! \) may be written as \((k+1)k(k-1) \cdots 3 \cdot 2 \cdot 1 = (k+1)k!\), and similarly, \((2k+2)! = (2k+2)(2k+1)(2k)!\). Thus
\[
\frac{a_{k+1}}{a_k} = \frac{(k+1)k!}{(2k+2)(2k+1)(2k)!} \times \frac{(2k)!}{k!} = \frac{(k+1)}{(2k+2)(2k+1)(2k)!} = \frac{1}{2(2k+1)} \to 0
\]
as \( k \to \infty \). Since this limit is less than 1, the ratio test tells us that the series converges.

(f) Converges by the root test with \( a_k = \frac{1}{(\ln k)^k} \), since \( (a_k)^{\frac{1}{k}} = \frac{1}{(\ln k)^k} \to 0 \) as \( k \to \infty \).

- **Examples.** Determine whether the following series converge or diverge: (a) \( \sum_{k=1}^{\infty} \frac{k}{\sqrt{1+k^2}} \); (b) \( \sum_{k=10}^{\infty} \frac{1}{\sqrt{k-3}} \); (c) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}} \); (d) \( \sum_{k=1}^{\infty} \frac{1}{3^k+2} \); (e) \( \sum_{k=1}^{\infty} \frac{k!}{2^k} \); (f) \( \sum_{k=1}^{\infty} \frac{\sin(\frac{1}{k})}{\sqrt{k}} \); (g) \( \sum_{k=10}^{\infty} \frac{1}{3k^2-4k+5} \); (h) \( \sum_{k=10}^{\infty} \frac{k^2+10}{4k^3-k^2+7} \); (i) \( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}+1} \); (j) \( \sum_{k=1}^{\infty} \frac{2k}{2k^2} \); (k) \( \sum_{k=1}^{\infty} \frac{(2k+1)^k}{3k} \); (l) \( \sum_{k=1}^{\infty} \frac{k^2}{3k} \).

These are for extra practice. Nearly all were done in class; all worked in Pam B’s online notes.

Items (a)–(d), (g)–(i) can also be answered very quickly by the ‘winning term’ trick. For example, in (h) the ‘winning terms’ in numerator and denominator give \( \frac{k^2}{4k^3} = \frac{1}{4k} \). So our series behaves like \( \sum_k \frac{1}{4k} = \frac{1}{4} \sum_k \frac{1}{k} \), which diverges (basically is the harmonic series). So the series in (h) diverges too. That is the trick. To fully justify this though, if pressed for a proof, one would use the limit comparison test, with \( b_n = \frac{1}{4k} \).

In (f), use the limit comparison test, with \( a_n = \frac{\sin(\frac{1}{k})}{\sqrt{k}} \) and \( b_n = \frac{1}{k^{1/2}} \). Then \( \frac{a_n}{b_n} = \frac{\sin(\frac{1}{k})}{\sqrt{k}} \to 1 \) as \( k \to \infty \). Since \( \sum_k b_k \) is a convergent p-series, the series in (f) converges by the limit comparison test.

Order of tests for nonnegative series: If you don’t recognize it as a geometric or p-series, etc, I’d use the following order: divergence, limit comparison, root, comparison, integral, ratio. Use the ratio test if you have factorials, the root test if you have powers, limit comparison test if you have ‘winning terms’, integral if terms are decreasing.

### 11.4. Absolute and conditional convergence

A series \( \sum_k a_k \) is called **absolutely convergent** if \( \sum_k |a_k| \) converges.

- Example. \( 1+\frac{1}{4} - \frac{1}{9} + \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} + \cdots \) is absolutely convergent, because \( 1+\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \) converges (by the p-series test with \( p = 2 \)).

- **Key Fact in 11.4:** Any absolutely convergent series is convergent.

- We shall see that the converse is false, a series may be convergent, but not absolutely convergent. Such a series is called **conditionally convergent**.

- (Proof of Key Fact: If \( \sum_k |a_k| \) converges, then so does \( \sum_k 2|a_k| \). However \( 0 \leq a_k + |a_k| \leq 2|a_k| \). So by the basic comparison test, \( \sum_k (a_k + |a_k|) \) converges. By the ‘difference rule’ (4th ‘bullet’ on page 5 of these notes), \( \sum_k a_k = \sum_k (a_k + |a_k|) - \sum_k |a_k| \) converges.)
Example. Does the series \( \sum \frac{\sin(k\pi^2)}{k^2} \) converge or diverge?

Solution. The series \( \sum \frac{\sin(k\pi^2)}{k^2} \) converges by the comparison test, since \( \frac{\sin(k\pi^2)}{k^2} \leq \frac{1}{k^2} \), and \( \sum \frac{1}{k^2} \) is a convergent p-series. So the series \( \sum \frac{\sin(k\pi^2)}{k^2} \) converges absolutely. So by the Key Fact above it is convergent.

Example. Does series \( 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots \) converge or diverge?

Solution. It converges, since as we saw in the previous example, this series is absolutely convergent. So by the Key Fact above it is convergent.

Example. Does series \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \) converge or diverge?

Solution. The series \( \sum \frac{\sin(k\pi^2)}{k^2} \) converges by the comparison test, since \( \frac{\sin(k\pi^2)}{k^2} \leq \frac{1}{k^2} \), and \( \sum \frac{1}{k^2} \) is a convergent p-series. So the series \( \sum \frac{\sin(k\pi^2)}{k^2} \) converges absolutely. So by the Key Fact above it is convergent.

The Alternating Series Test: Suppose that \( a_0 > a_1 > a_2 > \cdots \), and that \( \lim_{k \to \infty} a_k = 0 \). Then \( a_0 - a_1 + a_2 - a_3 + \cdots \) (which in sigma notation is \( \sum_k (-1)^k a_k \)) converges.

[Proof: The 2nth partial sum is
\[
2n = a_0 - a_1 + a_2 - a_3 + \cdots - a_{2n-1} = (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1}).
\]
Each bracketed term is nonnegative, so that \( s_2, s_4, s_6, \cdots \) is an increasing sequence, so it has a limit \( s \). Similarly
\[
s_{2n+1} = a_0 - (a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{2n-1} - a_{2n})
\]
and each bracketed term is nonnegative, so that \( s_1, s_3, s_5, \cdots \) is a decreasing sequence, and so has a limit \( t \). But \( s_{2n+1} - s_{2n} = a_{2n} \) which has limit 0 as \( n \to \infty \). So \( s = t \) and this is a finite number. Thus \( \{s_n\} \) converges.]

Example. Determine whether the series \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \) is convergent, converges absolutely, converges conditionally, or diverges.

Solution: The series \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \) is convergent by the Alternating Series Test. But it is not absolutely convergent, because \( 1 + \frac{1}{2} + \frac{1}{3} + \cdots \) is the divergent harmonic series (which we met close to the start of Section 11.2). So \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \) is conditionally convergent. We’ll see later that its sum is 1/2.

Example. Determine whether the following series are convergent, converges absolutely, converges conditionally. (a) \( \sum_k \frac{(-1)^k}{\sqrt{k}} \), (b) \( \sum_k \frac{(-1)^k}{2^k} \), (c) \( \sum_k (-1)^k (\sqrt{k+1} - \sqrt{k}) \).

Solution: Both series are convergent by the Alternating Series Test. But (a) is not absolutely convergent, because \( \sum_k \frac{1}{\sqrt{k}} \) is divergent by the p-series test of Section 11.2. So (a) is conditionally convergent. Series (b) is a convergent geometric series, and so is \( \sum_k \frac{1}{2^k} \), so series (b) is absolutely convergent and hence not conditionally convergent. Series (c) is not absolutely convergent, conditionally convergent, convergent, and not divergent (this example was worked on the review).

A much more difficult fact to prove is that any ‘rearrangement’ of an absolutely convergent series is convergent and has the same sum.

Example. Determine whether the series \( 1 - \frac{1}{6} + \frac{1}{10} - \frac{1}{30} + \frac{1}{16} - \frac{1}{80} + \frac{1}{250} + \cdots \) converges or diverges.

Solution: The series converges, since it is a ‘rearrangement’ of the absolutely convergent series considered at the beginning of this section.
Consider the series \( \sum_k \frac{x^k}{k!} \). We saw this fact towards the end of Section 10.4.

We are now ready to finish the Calculus 2 syllabus with a discussion of Taylor and Power Series.

Example. Approximate the sum of \( \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!} \) by its first six terms, and estimate the error in your approximation.

Solution. \( \sum_{k=1}^{6} (-1)^k \frac{1}{k!} \approx 0.63194 \) (calculator). The error in this approximation is less than \( |a_7| = \frac{1}{7!} = 0.00019 \) (calculator).

Example. Approximate the sum of \( \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} \) with an error of less than 0.001.

Solution. The error in using \( s_n \) to approximate the sum is \( < |a_{n+1}| = \frac{1}{(n+1)!} \). Now \( \frac{1}{(n+1)!} < 0.001 \) if \( (n+1)^4 > 1000 \). Choosing \( n = 5 \) will work. So an approximation with an error of less than 0.001 is \( s_5 = \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k!} = 0.94754 \) (calculator).

11.5 and 11.6. Taylor Series.

We are now ready to finish the Calculus 2 syllabus with a discussion of Taylor and Power Series. Up until now we have pretty much proved everything in these notes for Chapter 11 (although you are not expected to read most of these proofs). However from now on the proofs become too lengthy to include.

The \( n \text{th} \) Taylor polynomial of a function \( f(x) \) is defined to be

\[
P_n(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(n)}(0)}{n!} x^n
\]

which in sigma notation is \( \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \).

The Taylor series or MacLaurin series of a function \( f(x) \) is defined to be

\[
f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots
\]

which in sigma notation is \( \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \).

Example. Find the 7th Taylor polynomial of \( \sin x \). Also, find the Taylor series of \( \sin x \).

Solution: If \( f(x) = \sin(x) \) then \( f'(x) = \cos x \), \( f''(x) = -\sin(x) \), \( f'''(x) = -\cos x \), and then it starts repeating, \( f^{(4)}(x) = f(x) = \sin(x) \), \( f^{(5)}(x) = f'(x) = \cos x \), and so on. Thus we have \( f(0) = 0 = f^{(4)}(0) \), \( f'(0) = 1 = f^{(5)}(0) \), \( f''(0) = 0 = f^{(6)}(0) \), \( f'''(0) = 0 = f^{(7)}(0) \), and so on. Thus

\[
P_7(x) = 0 + 1 \cdot x + 0 \cdot x^2 + \frac{-1}{3!} x^3 + 0 \cdot x^4 + \frac{1}{5!} x^5 + 0 \cdot x^6 + \frac{-1}{7!} x^7 = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}
\]

\[\text{1We saw this fact towards the end of Section 10.4}\]
and the Taylor series of \( \sin x \) is
\[
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.
\]

- **Example.** Find the Taylor polynomial \( P_n(x) \) for \( e^x \). Also, find the Taylor series of \( e^x \).

**Solution:** If \( f(x) = e^x \) then \( e^x = f'(x) = f''(x) = f'''(x) = \cdots \). Thus we have \( 1 = f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0) \), and so on. Thus
\[
P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.
\]
The Taylor series of \( e^x \) is
\[
1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.
\]

- **Example.** Find the \( n \)th Taylor polynomial, and the Taylor series, of \( \ln(1-x) \).

**Solution:** If \( f(x) = \ln(1-x) \) then \( f'(x) = -(1-x)^{-1} \), and
\[
f''(x) = -(-1) \cdot (1-x)^{-2} \cdot (-1) = -(1-x)^{-2}
\]
Similarly, \( f'''(x) = -(2)(1-x)^{-3} \cdot (-1) = -(1-x)^{-3} \), and
\[
f^{(4)}(x) = -(-3) \cdot 2(1-x)^{-4} \cdot (-1) = -(1-x)^{-4}
\]
and so on. In general, the pattern is \( f^{(k)}(x) = \frac{(-1)^{k+1}k!}{(1-x)^k} \). Also \( f(0) = \ln(1) = 0, f'(0) = -1, f''(0) = -1, f'''(0) = -2, f^{(4)}(0) = -3 \cdot 2 \), and in general \( f^{(k)}(0) = -(k-1)! \). Therefore
\[
\frac{f^{(k)}(0)}{k!} = \frac{-1}{(1-x)^k} = \frac{-1}{k}
\]
for \( k = 1, 2, 3, \ldots \). So the \( n \)th Taylor polynomial of \( \ln(1-x) \) is
\[
P_n(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}.
\]
The Taylor series of \( \ln(1-x) \) is
\[
-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = \sum_{k=1}^{\infty} \frac{x^k}{k}.
\]

- We define the **Taylor remainder** to be
\[
R_n(x) = f(x) - P_n(x).
\]
This is the error in approximating \( f(x) \) by \( P_n(x) \). Note
\[
R_n(x) = f(x) - P_n(x).
\]

- **Example.** Find the Taylor remainder \( R_5(x) \) for \( f(x) = \sin x \).

**Solution:** \( R_5(x) = \sin x - P_3(x) = \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} \), by the example above.

- **Taylor's Theorem.** If \( f^{(n+1)} \) is continuous on an open interval \( I \) containing 0, then for every \( x \in I \) we have
\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)
\]
where \( R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n \, dt \). There exists a number \( c \) between 0 and \( x \) such that

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.
\]

This is the Lagrange formula, or the Lagrange form of the remainder. Thus if \( |f^{(n+1)}(t)| \leq M \) for all \( t \) between 0 and \( x \) then

\[
|R_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!}.
\]

We can take \( M \) to be the maximum of \( |f^{(n+1)}(t)| \) on the interval \([0, x]\).

- **Example.** Estimate the error for the approximation of \( \sin x \) by \( P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \) for an arbitrary value of \( x \) between 0 and \( \frac{\pi}{4} \).

**Solution:** We saw

\[
P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.
\]

By the Lagrange formula

\[
|\sin x - P_5(x)| = \left| \frac{f^{(6)}(c)}{6!} x^6 \right| \leq \frac{1}{720} \left( \frac{\pi}{4} \right)^6 = \frac{\pi^6}{720 \cdot 4096} = 0.000326.
\]

That is, the error in this approximation is smaller than 0.000326.

- **Example.** Assume that \( f \) is a function such that \( |f^{(4)}(x)| \leq 2 \) for all \( x \). Find the maximal possible error if \( P_3(\frac{1}{2}) \) is used to approximate \( f(\frac{1}{2}) \).

**Solution:** We may take \( M = 2 \). The error in using \( P_3(\frac{1}{2}) \) to approximate \( f(\frac{1}{2}) \) is \( R_3(\frac{1}{2}) \), and have

\[
|R_3(\frac{1}{2})| \leq \frac{M(\frac{1}{2})^4}{4!} \leq \frac{1}{2^4 \cdot 4!} = \frac{1}{192} = 0.0052.
\]

So the maximal possible error is less than 0.0052.

- **Example.** Assume that \( f \) is a function such that \( |f^{(4)}(x)| \leq 2 \) for all \( x \). Find a small \( n \) such that the maximal possible error is smaller than 0.001, if \( P_3(\frac{1}{2}) \) is used to approximate \( f(\frac{1}{2}) \).

**Solution:** Just as in the previous example, \( M = 2 \). The error in using \( P_n(\frac{1}{2}) \) to approximate \( f(\frac{1}{2}) \) is \( R_n(\frac{1}{2}) \), and have

\[
|R_n(\frac{1}{2})| \leq \frac{M(\frac{1}{2})^{n+1}}{(n+1)!} = \frac{1}{2^n(n+1)!}.
\]

If we want the error smaller than 0.001 we can solve \( \frac{1}{2^n(n+1)!} < 0.001 \). That is, \( 1000 < 2^n \cdot (n+1)! \). Clearly \( n = 4 \) will do that job: \( 1000 < 2^4 5! = 16 \cdot 120 \). So if \( n = 4 \) then the maximal possible error (if \( P_3(\frac{1}{2}) \) is used to approximate \( f(\frac{1}{2}) \)) is smaller than 0.001.

- **Example.** Find the Lagrange form of the Taylor remainder \( R_n \) for the function \( f(x) = e^{2x} \) and \( n = 3 \).

**Solution:** We have \( f'(x) = 2e^{2x}, f''(x) = 2^2 e^{2x}, f'''(x) = 2^3 e^{2x}, f^4(x) = 2^4 e^{2x} \). So the Lagrange form is

\[
R_3(x) = \frac{f^{(4)}(c)}{4!} x^4 = \frac{2^4 e^{2c} x^4}{4!} = \frac{2^4 e^{2c} x^4}{3}.
\]
The Mean Value Theorem from Calculus I is essentially the case $n = 0$ of Taylor’s Theorem.

**KEY POINT:** Up until now, we have not considered at all the question of whether the Taylor series of $f(x)$ converges. Clearly it does converge when $x = 0$ to $f(0)$ though. The partial sums of the Taylor series of $f(x)$ are just the $P_n(x)$, so it follows that the Taylor series converges to $f(x)$ if and only if $P_n(x) \to f(x)$, or equivalently, if and only if $\lim_{n \to \infty} R_n(x) = 0$.

To check if $\lim_{n \to \infty} R_n(x) = 0$, the last fact in Taylor’s theorem is very useful, as we shall now see in many examples:

**Example 1.** Show that the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ converges to $\sin x$ for every number $x$. Thus for every number $x$,

$$
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
$$

**Solution:** The series in this example is the Taylor series of $f(x) = \sin x$, as we saw on the previous page. To show that the series converges to $\sin x$ for all $x$, by the KEY POINT above, we need to show that $\lim_{n \to \infty} R_n(x) = 0$. Note that $|f^{(n+1)}(x)| = |\sin x|$ or $|\cos x|$, so that $|f^{(n+1)}(x)| \leq 1$ for every number $x$. So we can take $M = 1$, and by the last part of Taylor’s theorem,

$$
|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}
$$

But the right hand side here converges to 0 as we saw close to the end of Section 10.4. So by the squeezing or pinching rule $\lim_{n \to \infty} R_n(x) = 0$.

**Example 2.** An exactly similar argument shows that for every $x$,

$$
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
$$

$$
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}
$$

**Example 3.** (a). If $f(x) = \ln(1 - x)$, find an $M$ so that $|f^{(n+1)}(t)| \leq M$ for all $x$ between $0$ and $-1$. Use this to find an estimate for $|R_n(-1)|$.

(b). Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ has sum equal to $\ln 2$.

(c). Approximate $\ln 2$ with error less than $0.05$, using a ‘Taylor approximation’.

(d). Find a value of $n$ so that $P_n(x)$ approximates $\ln 2$ with error less than $0.001$.

**Solution:** (a): Let $f(x) = \ln(1 - x)$. We saw in an example early in this section that $f^{(n+1)}(t) = -\frac{n!}{(1-t)^{n+1}}$, so if $-1 \leq t \leq 0$ then $1 - t \geq 1$, so that $|1 - t|^{n+1} \geq 1$, so that we have

$$
|f^{(n+1)}(t)| = \frac{n!}{|1 - t|^{n+1}} \leq n!.
$$

So we can take $M = n!$ and $x = -1$ in the last line of Taylor’s theorem and get

$$
|R_n(-1)| \leq \frac{n!}{(n + 1)!} = \frac{1}{n + 1}.
$$
To get (b), in the example early in this section we saw that the Taylor series of \( \ln(1-x) \) is \(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\). By (a)

\[
|R_n(-1)| \leq \frac{1}{n+1} \to 0
\]
as \( n \to \infty \). Thus by the KEY POINT above, the Taylor series of \( \ln(1-x) \) when \( x = -1 \), which is \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \), converges to \( f(-1) = \ln(2) \).

(c) we want the Taylor remainder \( |R_n(-1)| \), which is the error in approximating \( f(-1) \) by \( P_n(-1) \), to be less than 0.05. We saw in (a) that \( |R_n(-1)| \leq \frac{1}{n+1} \). So if \( \frac{1}{n+1} < 0.05 \) we will be done. But \( \frac{1}{n+1} < 0.05 \) if \( n \geq 20 \). Thus \( P_{20}(-1) \), which equals \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{20} \), is a Taylor approximation to \( \ln 2 \) with error less than 0.05.

Item (d) is just like (c), we want \( |R_n(-1)| < 0.001 \). In (a) we saw that \( |R_n(-1)| \leq \frac{1}{n+1} \). So if \( \frac{1}{n+1} < 0.001 \) we will be done. But \( \frac{1}{n+1} < 0.001 \) if \( n + 1 > 1000 \). So choose \( n = 1000 \).

11.6. The Taylor series of a function \( f(x) \) about a number \( a \), is the series

\[
f(a) + f'(a) (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots
\]

which in sigma notation is \( \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \).

- The \( n \)th \emph{Taylor polynomial} of a function \( f(x) \) about a number \( a \), is defined to be

\[
P_n(x) = f(a) + f'(a) (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

and the Taylor remainder again is \( R_n(x) = f(x) - P_n(x) \).

- Example. Suppose that \( g \) is a function which has continuous derivatives, and that \( g(2) = 3, g'(2) = -4, g''(2) = 7, g'''(2) = -5 \). Find the Taylor polynomial of degree 3 for \( g \) centered at \( x = 2 \).

Solution: \( P_3(x) = 3 - 4(x-2) + \frac{7}{2}(x-2)^2 - \frac{5}{6}(x-2)^3 \).

- Example. Find the Taylor series of \( f(x) = x^2 + 3x - 1 \) about \( x = 1 \).

Solution: \( f(1) = 3, f'(1) = 2, f''(1) = 5, f'''(x) = 2, f''''(x) = 0, \) and so on. So the Taylor series about \( x = 1 \) is

\[
f(1) + f'(1) (x - 1) + \frac{f''(1)}{2!} (x - 1)^2 + 0 + 0 + \cdots = 3 + 5(x - 1) + (x - 1)^2.
\]

- Example. Find the Taylor series of \( \ln x \) about \( x = 1 \).

Solution: Let \( f(x) = \ln x \). Then \( f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = +2x^{-3}, f^{(4)}(x) = -3 \cdot 2x^{-4} \). In general \( f^{(k)}(x) = (-1)^{k-1} \cdot (k-1)! x^{-k} \). Thus \( f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2, f^{(4)}(1) = -3 \cdot 2, \) and in general \( f^{(k)}(1) = (-1)^{k-1} \cdot (k-1)! \). Thus \( \frac{f^{(k)}(1)}{k!} = (-1)^{k-1} \cdot \frac{1}{k} \). So the Taylor series about \( x = 1 \) is

\[
(x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots
\]
• Example. Expand \( \ln x \) in powers of \((x - 1)\).

**Solution:** This is just another way to ask the last question. The answer was \((x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \cdots \).

• Just as in 11.5 we have **Taylor’s Theorem:** If \( f^{(n+1)} \) is continuous on an open interval \( I \) containing \( a \), then for every \( x \in I \) we have

\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)
\]

and \( R_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n \, dt \), and indeed

\[
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}
\]

for some number \( c \) between \( a \) and \( x \). This is the **Lagrange formula**, or the **Lagrange form of the remainder**. Thus if \( |f^{(n+1)}(t)| \leq M \) for all \( t \) between \( a \) and \( x \) then

\[
|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!}.
\]

Again we can take \( M \) to be the maximum of \( |f^{(n+1)}(t)| \) for \( t \) between \( a \) and \( x \).

• Example. Use Taylor polynomials to approximate \( \ln(1.1) \) with error less than 0.001.

**Solution:** Let \( a = 1 \), \( x = 1.1 \) in Taylor’s Theorem above. In the previous example we saw that \( f^{(n+1)}(t) = (-1)^n \cdot n! t^{-n-1} \), so that \( |f^{(n+1)}(t)| = n! t^{-n-1} \leq n! \), if \( 1 \leq t \leq 1.1 \).

From the last line of Taylor’s Theorem,

\[
|R_n(1.1)| \leq \frac{n! \cdot (0.1)^{n+1}}{(n+1)!} = \frac{(0.1)^{n+1}}{n+1}.
\]

Notice if \( n = 2 \) then \( |R_2(1.1)| \leq \frac{0.001}{3} < 0.001 \). So the approximation we want is the 2nd Taylor polynomial, which by what we did in the previous example will be

\[
P_2(x) = (x - 1) - \frac{(x-1)^2}{2}.
\]

Thus \( \ln(1.1) \) is approximately \( P_2(1.1) = (0.1) - \frac{(0.1)^2}{2} = 0.1 - 0.005 = 0.095 \).

• Example. Find the Lagrange form of the Taylor remainder \( R_n \) for the function \( f(x) = \ln x \) and \( n = 3 \) and \( a = 1 \).

**Solution:** We saw above that \( f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = +2x^{-3}, f^{(4)}(x) = -3 \cdot 2x^{-4} \). So \( f^{(4)}(c) = -\frac{6}{c^4}, \) and so the Lagrange form is

\[
R_3(x) = \frac{f^{(4)}(c)}{4!}(x-1)^4 = -\frac{6}{c^4 \cdot 4!} (x-1)^4 = -\frac{(x-1)^4}{3c^4}.
\]

11.7. Power Series.

• A **power series** is a series of the form \( \sum_{k=0}^{\infty} c_k x^k \), or in longhand,

\[
c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots
\]

where \( c_0, c_1, c_2, \cdots \) are constants.
• Example 1. $1 + x + x^2 + x^3 + \cdots$.
• Example 2. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.
• Example 3. $\sum_{k=1}^{\infty} \frac{x^k}{k}$.

• KEY RESULT IN 11.7: For any power series $\sum_{k=0}^{\infty} c_k x^k$, there is a constant $R$, called the radius of convergence of the power series. We have $0 \leq R \leq \infty$ and
  (a) If $R = 0$ then the power series only converges when $x = 0$.
  (b) If $R = \infty$ then the power series converges for every number $x$.
  (c) If $0 < R < \infty$ then the power series converges if $-R < x < R$, and it diverges if $|x| > R$.
  Thus the set of numbers $x$ for which the power series converges is an interval centered at the origin on the number line. This interval is called the interval of convergence. [Picture drawn in class].

• Examples. Find the radius of convergence and the interval of convergence for each of Examples 1, 2, 3 above.

  Solutions: Example 1 above is a geometric series, and from the geometric series test, we know it converges only for $x \in (-1, 1)$. This is its interval of convergence. Its radius of convergence is therefore 1.

  Example 2: we said in 11.5 that this series converges for all $x$ to cosine. So its interval of convergence is $(-\infty, \infty)$, and its radius of convergence is therefore $\infty$.

  Put $x = 1$ in Example 3, and one gets the divergent harmonic series (see Section 11.2). Put $x = -1$ in Example 3, and one gets a convergent alternating series (this was a major example in Section 11.4). Thus the interval of convergence must be $[-1, 1)$, and the radius of convergence is 1.

• A formula for the radius of convergence $R$:
  \[
  R = \lim_{k \to \infty} \frac{|c_k|}{|c_{k+1}|}
  \]
  if this limit exists.

• Example 4. Find the radius of convergence and the interval of convergence for the power series $\sum_{k=1}^{\infty} k x^k$.

  Solution: The radius of convergence of the power series $\sum_{k=1}^{\infty} k x^k$ is 1 since
  \[
  \lim_{k \to \infty} \frac{k}{k+1} = \lim_{k \to \infty} \frac{1}{1 + \frac{1}{k}} = 1.
  \]
  Since this series diverges when $x = 1$ or $x = -1$, but the radius of convergence is 1, the interval of convergence must be $(-1, 1)$.

• If the limit in the ‘formula for $R$’ above does not exist, try
  \[
  R = \lim_{k \to \infty} \frac{1}{|c_k|^k}
  \]
if this limit exists.

[These two ‘formulae for $R$’ are easily proved using the ‘ratio test’ and ‘root test’.]

- **The sum function of a power series:** Suppose that $\sum_{k=0}^{\infty} c_k x^k$ is a power series, and suppose that $I$ is its interval of convergence. For $x$ in $I$, define $f(x)$ to be the sum of the series; that is $f(x) = \sum_{k=0}^{\infty} c_k x^k$ (in the sense of Meaning #2 of the sum of a series (see 2nd page of these typed notes on Chapter 11)). Then $f : I \to (-\infty, \infty)$ is a function defined on the interval of convergence. We call $f$ the **sum function**. It is actually a continuous function on $I$, and differentiable in the interior of $I$ as we shall see in the next section.

- Examples. In Example 1 above, the sum function is, by the geometric series formula, $f(x) = \frac{1}{1-x}$. In Example 2, the sum function is $\cos x$.

  In Example 3, the sum function $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ is defined on the interval of convergence. On the last page we saw that the interval of convergence is $[-1, 1)$. So $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is a continuous function on $[-1, 1)$, and it is differentiable on $(-1, 1)$.

  The results in this section above have variants for power series $\sum_{k=0}^{\infty} c_k (x-a)^k$ about a number $a$. The radius of convergence $R$ still has the formula $R = \lim_{k \to \infty} \frac{1}{|c_k|^{1/k}}$ if this limit exists (or $\lim_{k \to \infty} \frac{|c_k|}{|c_{k+1}|}$). The interval of convergence of this power series about $x = a$ will be either $[a-R, a+R], (a-R, a+R], [a-R, a+R)$, or $(a-R, a+R)$ [Picture drawn in class]. Outside of the interval of convergence it diverges, but inside its sum function is continuous.

- Example 4. Find the radius of convergence and the interval of convergence for the power series $\sum_{k=1}^{\infty} k (x-1)^k$.

  **Solution:** The radius of convergence of the power series $\sum_{k=1}^{\infty} k (x-1)^k$ is 1 since

  $$\lim_{k \to \infty} \frac{k}{k+1} = \lim_{k \to \infty} \frac{1}{1 + \frac{1}{k}} = 1.$$ 

  Since this series diverges when $x = 0$ or $x = 2$, but the radius of convergence is 1, the interval of convergence must be $(0, 2)$. 


11.8. Differentiation and integration of power series.

- **The differentiated power series** of a power series \( c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \), is the power series
  \[
  c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots
  \]

- Example. The differentiated power series of the power series \( 1 + x + x^2 + x^3 + \cdots \) is the power series \( 1 + 2x + 3x^2 + 4x^3 + \cdots \).

- **KEY RESULTS ON DIFFERENTIATING POWER SERIES**: Suppose that a power series \( \sum_{k=0}^{\infty} c_k x^k \) has a radius of convergence \( R > 0 \). Then:
  
  A) the differentiated power series has the same radius of convergence \( R \).
  
  Let \( f(x) \) be the sum function of the power series (defined on the previous page), and let \( g(x) \) be the sum function of the differentiated power series.
  
  B) \( f(x) \) is differentiable on \((-R, R)\), and \( f'(x) = g(x) \) for all \( x \) in \((-R, R)\). That is, if \(-R < x < R\) then
  
  \[
  f'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}.
  \]

  C) We can iterate this process, and look at the ‘second differentiated power series’ (i.e., the differentiated power series of differentiated power series) \( \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} \). By A) and B), this has the same radius of convergence \( R \), and its sum function equals \( f''(x) \) on \((-R, R)\). Thus if \(-R < x < R\) then
  
  \[
  f''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}.
  \]

  Similarly for \( f'''(x) \), \( f^{(4)}(x) \), and so on.

  D) Putting \( x = 0 \) in \( f(x) \), \( f'(x) \), \( f''(x) \), \( f'''(x) \) and so on, we find from B) and C) that \( f(0) = c_0, f'(0) = c_1, f''(0) = 2c_2, \) and more generally, \( f^{(k)}(0) = k! c_k \), or \( c_k = \frac{f^{(k)}(0)}{k!} \).

  E) Because of D), the Taylor series of \( f(x) \) is our original power series \( \sum_{k=0}^{\infty} c_k x^k \).

- Part E is saying that the sum function \( f(x) \) of a power series on its interval of convergence, has Taylor series equal to the original power series.

- Example: Find a simple formula for the sum of the series \( x + 2 x^2 + 3 x^3 + 4 x^4 + \cdots \).

  **Solution**: The geometric series \( 1 + x + x^2 + \cdots \) converges to \( \frac{1}{1-x} \) for \( x \in (-1, 1) \).

  Differentiating this, we have by B) above that \( 1 + 2x + 3x^2 + \cdots \) converges to \( \frac{d}{dx} \left( \frac{1}{1-x} \right) = (1-x)^{-2} \), for all \( x \in (-1, 1) \). Multiplying by \( x \) shows that \( x + 2 x^2 + 3 x^3 + 4 x^4 + \cdots \) converges to \( x (1-x)^{-2} \) for \( x \in (-1, 1) \). This is the desired sum function.

- **Equality of power series**. Suppose that \( \sum_{k=0}^{\infty} a_k x^k \) and \( \sum_{k=0}^{\infty} b_k x^k \) are two power series whose sums are equal (i.e., \( \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x^k \)) for all \( x \) in an open interval containing 0. Then \( a_0 = b_0, a_1 = b_1, \cdots, a_k = b_k \) for every \( k \).
Consider the series 1

By the previous example, we know that

Note that the power series here converges when

Example 1. Show that \( \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \), for \(-1 < x < 1\).

**Solution:** Consider the series \( 1 - x^2 + x^4 - x^6 + \cdots \). This is a geometric series which converges for \(-1 < x < 1\), with sum \( \frac{1}{1+x^2} \). By the last bullet,

\[
x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \int_0^x \frac{dt}{1+t^2} = \tan^{-1} x - \tan^{-1} 0 = \tan^{-1} x
\]

Example 2. Show that \( \tan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \), for \(-1 \leq x \leq 1\).

**Solution:** Note that the power series here converges when \( x = 1 \) and when \( x = -1 \), by the Alternating Series Test (from Section 11.4). So the sum function \( f(x) \) of this power series is continuous on \([-1, 1]\) by the second last bullet or so in Section 11.7 above. Hence

\[
f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \tan^{-1} x = \tan^{-1} 1.
\]

That is, \( \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \). Similarly \( f(-1) = \tan^{-1}(-1) \).

Example 3. Show that \( \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots) \).

**Solution:** By the previous example, \( \frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \).

Example. Note \( \cosh x = \frac{1}{2}(e^x + e^{-x}) \). But

\[
e^x + e^{-x} = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) + (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots) = 2(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots).
\]

Thus \( \cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \). Similarly,

\[
\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.
\]

Example. Expand \( x^2 \cos x^3 \) in powers of \( x \).

**Solution:** We know that

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.
\]
Thus
\[ \cos x^3 = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots. \]

Thus
\[ x^2 \cos x^3 = x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!} + \cdots. \]

• Read Abel’s theorem, bottom of page 693 in text.

• Similar results about differentiation and integration hold for power series about a point \( x = a \). For example: If \( c_0 + c_1 (x - a) + c_2 (x - a)^2 + c_3 (x - a)^3 + c_4 (x - a)^4 + \cdots \) is a power series with radius of convergence \( R > 0 \), then the ‘differentiated power series’ is the power series
\[ c_1 + 2c_2 (x - a) + 3c_3 (x - a)^2 + 4c_4 (x - a)^3 + \cdots, \]
and this has the same radius of convergence \( R \). If \( f(x) \) is the sum function of the power series, and if \( g(x) \) be the sum function of the differentiated power series, then \( f(x) \) is differentiable on \((a - R, a + R)\), and \( f'(x) = g(x) \) for all \( x \) in \((a - R, a + R)\). That is, if \( a - R < x < a + R \) then
\[ f'(x) = \sum_{k=1}^{\infty} k c_k (x - a)^{k-1}. \]

The integrated power series is the power series
\[ c_0 (x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + c_3 \frac{(x - a)^4}{4} + \cdots. \]

The integrated power series has the same radius of convergence \( R \). If \( f(x) \) is the sum function of \( \sum_{k=0}^{\infty} c_k \ x^k \), and if \( F(x) \) is the sum function of the integrated power series, then \( \int_{a}^{x} f(t) \ dt = F(x) \), for \( |x - a| < R \).

END OF COURSE