Chapter 10. Sequences, etc.

These typed notes have pictures. Thus you are reminded that it is crucial that you copy good pictures from class. The most important technique in 10.1–10.4 is to look at the 'pattern' you see emerging in the picture. Actually in 10.2–10.4 the most important technique is to (1) write the sequence as a long list of numbers, and (2) draw the numbers in (1) as dots on a number line, and seeing the pattern that is emerging with those dots on the number line.

10.1: Least upper bounds and greatest lower bounds.

- Draw a set S of numbers as a subset of the real number line [picture drawn in class]. An *upper bound* of S is a number to the right of S in my picture. [Picture drawn in class.] That is, an upper bound of S is a number α which is greater than or equal to every number in S. That is, an upper bound of S is a number α such that $x \leq \alpha$ for all x in S.

  A *lower bound* of S is a number to the left of S in my picture. [Picture drawn in class.] That is, a lower bound of S is a number β which is less than or equal to every number in S. That is, a lower bound of S is a number β such that $x \geq \beta$ for all x in S.

- We say that the set S is *bounded above*, if it has an upper bound. We say that the set S is *bounded below* if it has a lower bound. If it is not bounded above, or if it is not bounded below, then we say that S is *unbounded*. If it is both bounded above and bounded below, then we say that S is *bounded*.

- A *maximum* of S is an upper bound of S that is in S. A *minimum* of S is a lower bound of S that is in S.

- We are mainly interested in this section in the *least upper bound* of S, written LUB(S) (or in some books sup(S), and called the supremum). And in the *greatest lower bound* of S, written GLB(S) (or in some books inf(S), and called the infimum).

- Viewed in the picture of S on the real number line [picture drawn in class], to find LUB(S) start at any upper bound to the right of S in the picture, then walk towards S until you are forced by S to stop. That stopping point is LUB(S). Similarly, to find GLB(S) start at any lower bound to the left of S in the picture, then walk towards S until you are forced by S to stop. That stopping point is GLB(S). [Picture drawn in class.]

- **Example 1.** Find three upper bounds of $S = (-\infty, 0)$, and identify the least upper bound LUB(S).

  Solution. [Picture of $(-\infty, 0)$ on the real number line drawn in class.] Three upper bounds are $2, 1, 0$. LUB(S) $= 0$.

- **Example 2.** Find three lower bounds of $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\}$, and identify the greatest lower bound GLB(S).

  Solution. [Picture of S as dots on the real number line bunching up at 1 drawn in class.] Three lower bounds are $0, -1, \frac{1}{2}$. Also, GLB(S) $= \frac{1}{2}$.

- **Example 3.** Determine if the set $S = [0, 4]$ is bounded above. If it is, determine the least upper bound LUB(S).

  Solution. [Picture of S as an interval on the real number line drawn in class.] Yes, S is bounded above, and LUB(S) $= 4$.

- **Example 4.** Determine if the set S in Example 2 is bounded above. If it is, determine the least upper bound LUB(S).

  Solution. Look at the picture of S drawn in Example 2. Yes, S is bounded above, and LUB(S) $= 4$.

- **Example 5.** Determine if the set $S = \{x : x^2 < 9\}$ is bounded above. If it is, determine the least upper bound LUB(S).

  Solution. [Picture of S as the interval $(-3, 3)$ on the real number line drawn in class.] Yes, S is bounded above, and LUB(S) $= 3$.

- **Example 6.** Determine if the set $S = \{x : |x-2| < 3\}$ is bounded above. If it is, determine the least upper bound LUB(S).

  Solution. [Picture of S as the interval $(-1, 5)$ on the real number line drawn in class.] Yes, S is bounded above, and LUB(S) $= 5$.

- **Example 7.** Determine if the set $S = \{x : \ln x < 2\}$ is bounded above. If it is, determine the least upper bound LUB(S).

  Solution. $\ln x < 2$ is the same as $x < e^2$. Note that $x > 0$ since $\ln x$ is not defined for negative numbers. [Picture of S as the interval $(0, e^2)$ on the real number line drawn in class.] Yes, S is bounded above, and LUB(S) $= e^2$.

- **Example.** Redo Examples 1–7, but with 'bounded above' replaced by 'bounded below' and least upper bound LUB replaced by greatest lower bound GLB.

  Solutions. 1) is not bounded above, so no greatest lower bound or GLB. We did 2) and 4) already. The set in 3) is bounded below and GLB(S) $= 0$. The set in 5) is bounded below and GLB(S) $= -3$. The set in 6) is bounded below and GLB(S) $= -1$. The set in 7) is bounded below and GLB(S) $= 0$.

10.2: Sequences.

- A sequence is a numbered string of objects $a_1, a_2, a_3, \cdots$. We often write the sequence as $(a_n)_{n=1}^{\infty}$ or simply as (a_n). The nth *term* is a_n. In this course our sequences are infinite sequences of real numbers.

- Example 1. List $(n^2 - 2n + 1)_{n=1}^{\infty}$ as a string of numbers.

  Solution. $0, 1, 4, 9, \cdots$.

- Example 2. List $(n^2 - 2n + 1)_{n=2}^{\infty}$ and $(n^2 - 2n + 1)_{n=2}^{4}$, as strings of numbers (the second is a finite sequence.

  Solution. $1, 4, 9, 16, \cdots$; and $1, 4, 9$.

- Example 3. List $(1 + \frac{(-1)^n}{n})$ as a string of numbers, and draw them on the real number line noting the pattern that emerges.

  Solution. $0, 1 + \frac{-1}{1}, 1 + \frac{1}{2}, 1 + \frac{-1}{3}, 1 + \frac{1}{4}, \cdots$. This is $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \cdots$. [Picture drawn in class on the real number line of dots alternating between being to the left and to the right of 1, but always getting closer and closer to 1. These dots each get a number above them: 1 above 0, write 2 above $\frac{3}{2}$, write 3 above $\frac{2}{3}$, and so on.]

The technique in Examples 1–3 you will use in almost every problem: listing a sequence as a string of numbers, and drawing them on the real number line noting the pattern that emerges.

A question you will often encounter in the quizzes and tests is to do the opposite of the procedure we used in Examples 1–3. In each of the examples below the trick I taught in class is to write 1 above the first term in the sequence, 2 above the second term in the sequence, 3 above the third term in the sequence, and so on, and then find how each number $1, 2, 3, \cdots$ is connected to the number below it.

- Example 4. Assuming that the pattern continues as indicated, find an explicit formula for the nth term a_n: Here the sequence is $-1, 1, -1, 1, 1, 1, -1, \cdots$.

  Solution. $a_n = (-1)^n$. So the sequence is $((-1)^n)$.

- Example 5. Assuming that the pattern continues as indicated, find an explicit formula for the nth term a_n: Here the sequence is $1, 8, 27, \cdots$.

  Solution. $a_n = n^3$. So the sequence is (n^3).

- Example 6. Assuming that the pattern continues as indicated, find an explicit formula for the nth term a_n: Here the sequence is $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$.

  Solution. $a_n = \frac{n}{n+1}$. So the sequence is $(\frac{n}{n+1})$.

- Example 7. Assuming that the pattern continues as indicated, find an explicit formula for the nth term a_n: Here the sequence is $0, \frac{4}{3}, \frac{4}{5}, \frac{8}{7}, \frac{8}{9}, \frac{12}{11}, \frac{12}{13}, \cdots$.

  Solution. $a_n = \frac{2n-1+(-1)^n}{2n-1} = 1 + \frac{(-1)^n}{2n-1}$. So the sequence is $(1 + \frac{(-1)^n}{2n-1})$.

- Example 8. Assuming that the pattern continues as indicated, find an explicit formula for the nth term a_n: Here the sequence is $\frac{3}{2}, -\frac{9}{4}, \frac{27}{8}, -\frac{81}{16}, \cdots$.

  Solution. $a_n = (-1)^{n+1}\frac{3^n}{2^n} = -(-\frac{3}{2})^n$. So the sequence is $(-(-\frac{3}{2})^n)$.

A sequence is called *monotone* if it is either increasing or decreasing. The *boundedness* and *monotonicity* of a sequence can almost always be discovered by the procedure we did above of listing a sequence as a string of numbers, and drawing them on the real number line noting the pattern that emerges.

Example. Determine the boundedness and monotonicity of each of the following sequences: (1) the sequence $(n^2 - 2n + 1)_{n=1}^{\infty}$, (2) the sequence $(n^2 - 2n + 1)_{n=2}^{\infty}$, (3) the sequence $(1 + \frac{(-1)^n}{n})$, (4) the sequence $((-1)^n)$, (5) the sequence (n^3), (6) the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$, (7) the sequence $(1 + \frac{(-1)^n}{2n-1})$, (8) the sequence $(-(-\frac{3}{2})^n)$.

Solution. These are just the 8 sequences we looked at a few moments ago. The sequence in Example (1) is $0, 1, 4, 9, \cdots$. When one looks at this as dots on the number line at the points $0, 1, 4, 9$, and so on, [Picture drawn in class], one sees that Example (1) is bounded below, but not above (if you are asked for it, the GLB is 0). It is increasing.

The sequence in Example (2) is $1, 4, 9, \cdots$. When one looks at this as dots on the number line at the points $1, 4, 9$, and so on, one sees that Example (2) is bounded below, but not above (if you are asked for it, the GLB is 1). It is increasing.

The sequence in Example (3) we saw above is the same as $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \cdots$. [Picture drawn in class on the real number line of dots alternating between being to the left and to the right of 1, but always getting closer and closer to 1. These dots each get a number above them: 1 above 0, write 2 above $\frac{3}{2}$, write 3 above $\frac{2}{3}$, and so on.] This sequence is bounded below and bounded above, so it is bounded (if you are asked for it, the GLB is 0 and the LUB is $\frac{3}{2}$). This sequence is not monotonic.

The sequence in Example (4) is the same as $-1, 1, -1, 1, 1, 1, -1, \cdots$. This sequence is bounded below and bounded above, so it is bounded (if you are asked for it, the GLB is -1 and the LUB is 1). This sequence is not monotonic.

The sequence in Example (5) is $1, 8, 27, \cdots$. When one looks at this as dots on the number line at the points $1, 8, 27$, and so on, one sees that Example (5) is bounded below, but not above (if you are asked for it, the GLB is 1). It is increasing.

The sequence in Example (6) is $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$. [Picture drawn in class on the real number line of dots between $\frac{1}{2}$ and 1 but getting closer and closer to 1. These dots each get a number above them: 1 above $\frac{1}{2}$, write 2 above $\frac{2}{3}$, write 3 above $\frac{4}{5}$, and so on.] This sequence is bounded below and bounded above, so it is bounded. It is increasing.

The sequence in Example (7) is $0, \frac{4}{3}, \frac{4}{5}, \frac{8}{7}, \frac{8}{9}, \frac{12}{11}, \frac{12}{13}, \cdots$. [Picture drawn in class on the real number line of dots between 0 and 2, alternating between being to the left and to the right of 1, but always getting closer and closer to 1. These dots each get a number above them: 1 above 0, write 2 above $\frac{4}{3}$, write 3 above $\frac{4}{5}$, and so on.] This sequence is bounded below and bounded above, so it is bounded. This sequence is not monotonic.

The sequence in Example (8) is $\frac{3}{2}, -\frac{9}{4}, \frac{27}{8}, -\frac{81}{16}, \cdots$. [Picture drawn in class on the real number line of dots alternating between being to the left and to the right of 0, but always larger and larger in absolute value. These dots each get a number above them: 1 above $\frac{3}{2}$, write 2 above $-\frac{9}{4}$, write 3 above $\frac{27}{8}$, and so on.] This sequence is neither bounded below nor bounded above. This sequence is not monotonic.

The following are done like the last examples, but are a bit harder (except for the first):

Example 9. Determine the boundedness and monotonicity of the sequence $(\sqrt{1 - \frac{1}{n}})$.

Solution. This is the sequence $\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \cdots$, which is 'the square root' of the sequence in Example (6) above. The picture will be similar to the picture drawn in class on the real number line for Example (6), but with the dots slightly to the right of where they were before. Namely, dots between $\frac{1}{2}$ and 1 but getting closer and closer to 1. These dots each get a number above them: 1 above $\sqrt{\frac{1}{2}}$, write 2 above $\sqrt{\frac{2}{3}}$, and so on.] This sequence is bounded below and bounded above, so it is bounded. It is increasing.

Example 10. a) Write down the first 6 terms, and then the general term of the sequence defined recursively (or inductively) by $a_1 = 1, a_{n+1} = \frac{a_n}{2} + 1$.

b) Determine the boundedness and monotonicity of the sequence in (a).

Solution. (a) This is the sequence $a_1 = 1, a_2 = \frac{a_1}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$, and

$$a_3 = \frac{a_2}{2} + 1 = \frac{\frac{3}{2}}{2} + 1 = \frac{7}{4},$$

$$a_4 = \frac{a_3}{2} + 1 = \frac{\frac{7}{4}}{2} + 1 = \frac{15}{8},$$

$$a_5 = \frac{a_4}{2} + 1 = \frac{\frac{15}{8}}{2} + 1 = \frac{31}{16},$$

$$a_6 = \frac{a_5}{2} + 1 = \frac{\frac{31}{16}}{2} + 1 = \frac{63}{32}.$$

The general term $a_n = \frac{2^n-1}{2^{n-1}}$, as one can see similarly to how we proceeded when we first met Examples 4–8 at the start of Section 10.2 above.

(b) [Picture drawn in class on the real number line of dots between 1 and 2, These dots each get a number above them: 1 above 1, write 2 above $\frac{3}{2}$, write 3 above $\frac{7}{4}$, and so on.] This sequence is bounded below (by 1) and is bounded above (by 2). This sequence is increasing.

Example 11. Determine the boundedness and monotonicity of the sequence $(\frac{n}{e^n})$.

Solution. Method 1: On a calculator compute $\frac{1}{e^1}, \frac{2}{e^2}, \frac{3}{e^3}, \cdots$, draw these on the real number line, and note the pattern that emerges. One sees that the sequence is decreasing and bounded below (by 0) and bounded above (by $\frac{1}{e}$).

Method 2: Let $f(x) = \frac{x}{e^x}$. We sketched the graph of this at the end of Section 7.4 (see typed notes), and saw that f is decreasing for $x \geq 1$. So $(\frac{n}{e^n})$ is decreasing. These numbers are positive, so the sequence is bounded below by 0 and bounded above its first term $\frac{1}{e}$ since it is decreasing [Picture drawn in class on the real number line of dots on the graph of $y = \frac{x}{e^x}$ decreasing in height.]

Example 12. Determine the boundedness and monotonicity of the sequence $(n^{\frac{1}{n}})$.

Solution. This is the sequence $1, 2^{\frac{1}{2}}, 3^{\frac{1}{3}}, \cdots$, which is $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \cdots$.

Method 1: On a calculator compute these numbers, draw these on the real number line, and note the pattern that emerges. The first two or three terms are increasing, but then the sequence starts to steadily decrease, but is always bigger than 1. So it is not monotone, is bounded below (by 1), and is bounded above. [Picture drawn in class on the real number line of dots]

Method 2: Let $f(x) = x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$. If one computes $f'(x)$ by logarithmic differentiation one gets $f'(x) = x^{\frac{1}{x}} \frac{1-\ln x}{x^2}$. This is negative if $x > e$. So f is decreasing for $x \geq e$. So $\sqrt[3]{3}, \sqrt[4]{4}, \cdots$ is decreasing. But $1, \sqrt{2}$ is increasing, so the sequence is not monotone. These numbers are positive, so the sequence is bounded below by 0. The sequence is bounded above, because after the 3rd term it is decreasing and bigger than 0 [Draw a picture].

Example 13. Determine the boundedness and monotonicity of the sequence $(\frac{n^2}{n+3})$.

Solution. Divide top and bottom by n^2: $\frac{n^2}{n+3} = \frac{1}{\frac{1}{n}+\frac{3}{n^2}}$. Both $\frac{1}{n}$ and $\frac{3}{n^2}$ are decreasing, so the denominator $\frac{1}{n} + \frac{3}{n^2}$ is decreasing and positive. So the sequence is increasing. These numbers are positive, so the sequence is bounded below by 0. The denominator $\frac{1}{n} + \frac{3}{n^2}$ gets arbitrarily close to 0, so the sequence gets arbitrarily large. So the sequence is unbounded above.

Another way to see that the sequence is unbounded above is to look at the 'winning term' in the numerator and 'winning term' in the denominator and divide them. So, for example $\frac{n^2}{n+3}$ acts like $\frac{n^2}{n} = n$, and the latter is unbounded above. Another way to see that the sequence is monotone: let $f(x) = \frac{x^2}{x+3}$, then $f'(x) = \frac{2x(x+3)-x^2}{(x+3)^2} = \frac{x^2+6x}{(x+3)^2} > 0$ for $x > 0$. So f is increasing, and hence so is $f(n) = \frac{n^2}{n+3}$.

10.3: Limits of sequences.

- We say that a sequence (a_n) *converges* to a real number a if the numbers a_n are getting closer to closer to a, as close as we like, as n gets huge. In this case, we write $a = \lim_{n\to\infty} a_n$, or $a_n \to a$ as $n \to \infty$. We also say a_n *approaches* a as $n \to \infty$.

- The technical definition: $a = \lim_{n\to\infty} a_n$ if for all numbers $\epsilon > 0$, there exists a number N such that $|a_n - a| < \epsilon$ whenever $n \geq N$.

- Note that $a = \lim_{n\to\infty} a_n$ if and only if $\lim_{n\to\infty}(a_n - a) = 0$.

- If (a_n) does not converge to any number then we say that it *diverges*.

- If the terms in the sequence are all positive and are getting huge without any bound, we write $\lim_{n\to\infty} a_n = \infty$. Note that if the terms in the sequence are all positive then $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Similarly, if the terms in a sequence are all negative then $\lim_{n\to\infty} a_n = -\infty$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

- Suppose that (a_n) is increasing. Then $\lim_{n\to\infty} a_n$ is the same as the LUB (and equals ∞ if (a_n) is not bounded).

  Suppose that (a_n) is decreasing. Then $\lim_{n\to\infty} a_n$ is the same as the GLB (and equals $-\infty$ if (a_n) is not bounded).

- FACT: A convergent sequence is bounded. Hence:

  An unbounded sequence must be divergent.

- (Warning: a bounded sequence does not HAVE to be convergent. See, for example, Example 4 in Section 10.2.)

- Example. State if the following sequences converge or diverge. If the sequence converges, find its limit. If it diverges, explain why. (1) the sequence $(n^2 - 2n + 1)_{n=1}^{\infty}$, (2) the sequence $(n^2 - 2n + 1)_{n=2}^{\infty}$, (3) the sequence $(1 + \frac{(-1)^n}{n})$, (4) the sequence $((-1)^n)$, (5) the sequence (n^3), (6) the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$, (7) the sequence $(1 + \frac{(-1)^n}{2n-1})$, (8) the sequence $(-(-\frac{3}{2})^n)$, (9) the sequence $(\sqrt{1 - \frac{1}{n}})$.

  Solution. These are just the 9 sequences we looked at at the start of Section 10.2. The sequence in Example (1) is $0, 1, 4, 9, \cdots$. Since this is not bounded, but is increasing, the limit of the sequence is ∞, by the third last bullet above. The sequence diverges because it is unbounded (see second last bullet above). Similarly, the limit of the sequence in Example (2) is ∞, and this sequence diverges.

  The sequence in Example (3) converges, and its limit is 1 (since we said when we looked at Example (3) at the start of Section 10.2, or its picture that we drew there, that these numbers are getting closer and closer to 1.

  The sequence in Example (4) diverges, since these numbers are not getting closer and closer to any one number.

  The sequence in Example (5) diverges (the reasoning is identical to that of Example 1 a few lines above).

  The sequences in Example (6), (7), and (9) converge, and their limit is 1 (since we said when we looked at those examples at the start of Section 10.2, or their pictures that we drew there, that these numbers are getting closer and closer to 1.

  The sequence in Example (8) diverges, since as we said when we looked at Example (8) at the start of Section 10.2, it is unbounded. But we said recently that every unbounded sequence diverges.

- Example. Consider the sequence defined recursively by $a_1 = 1, a_{n+1} = \frac{a_n}{2} + 1$. State if the sequences converges or diverges. If it converges, find $\lim_{n\to\infty} a_n$.

Solution. This is Example (10) at the start of Section 10.2, and we said there, or by the pictures that we drew there, this is an increasing sequence which is getting closer and closer to 2. So $\lim_{n\to\infty} a_n = 2$.

- Example. State if the sequence $(\sin(\frac{1}{n}))$ converges or diverges. If it converges, find its limit.

  Solution. It converges with limit 0. To see this, note that as n gets huge, $\frac{1}{n} \to 0$. Because sin is continuous, $\sin(\frac{1}{n})$ is approaching $\sin 0 = 0$.

- Example. State if the sequence $(\cos(\frac{n\pi}{n+1}))$ converges or diverges. If it converges, find its limit.

  Solution. It converges with limit -1. To see this, note that as n gets huge, $\frac{n\pi}{n+1} \to \pi$. Because cos is continuous, $\cos(\frac{n\pi}{n+1})$ is approaching $\cos \pi = -1$.

- Example. Does the following sequence converge:

$$\frac{1}{2}, \quad \frac{3}{4}\cdot\frac{1}{2}, \quad \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}, \quad \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}, \quad \cdots .$$

  Solution. This is a nonnegative sequence, and notice that each term in this sequence is equal to its predecessor multiplied by a positive number less than 1. Thus this sequence is decreasing. By the bullet before the FACT on the last page, this sequence converges to its greatest lower bound.

- A very common trick to find limits of sequences is to divide numerator and denominator through by the highest power in the denominator. We do several examples of this:

- Example. State if the sequence $(\frac{5n+1}{2n-3})$ converges or diverges. If it converges, find its limit.

  Solution. Divide numerator and denominator through by n. We get $\frac{5n+1}{2n-3} = \frac{5+\frac{1}{n}}{2-\frac{3}{n}}$, which has limit $\frac{5+0}{2-0} = \frac{5}{2}$. So the sequence converges with limit $\frac{5}{2}$.

- Example. State if the sequence $(\frac{3n^2+1}{n^3-1})$ converges or diverges. If it converges, find its limit.

  Solution. Divide numerator and denominator through by n^3. We get $\frac{3n^2+1}{n^3-1} = \frac{\frac{3}{n}+\frac{1}{n^3}}{1-\frac{1}{n^3}}$, which has limit $\frac{0+0}{1-0} = 0$. So the sequence converges with limit 0.

- Example. State if the sequence $(\frac{n}{\sqrt{n+1}})$ converges or diverges. If it converges, find its limit.

  Solution. Divide numerator and denominator through by $\sqrt{n}$. We get

$$\frac{n}{\sqrt{n+1}} = \frac{\frac{n}{\sqrt{n}}}{\frac{\sqrt{n+1}}{\sqrt{n}}} = \frac{\sqrt{n}}{\sqrt{1+\frac{1}{n}}}.$$

  The numerator here goes to ∞ and the denominator goes to $\sqrt{1+0} = 1$. So the sequence diverges, and its limit is ∞.

- 'Cheat' in the previous examples: look at 'winning term' in the numerator and 'winning term' in the denominator and divide them. So, for example $\frac{5n+1}{2n-3}$ acts like $\frac{5n}{2n} = \frac{5}{2}$; so the limit is $\frac{5}{2}$. And $\frac{3n^2+1}{n^3-1}$ acts like $\frac{3n^2}{n^3} = \frac{3}{n}$ which has limit 0. And $\frac{n}{\sqrt{n+1}}$ acts like $\frac{n}{\sqrt{n}} = \sqrt{n}$ which has limit ∞; so the limit is ∞.

- One can turn many limits (including most of the ones above) into a Calculus I problem by the following principle: If $\lim_{x\to\infty} f(x) = L$ in the Calculus I sense, then $\lim_{n\to\infty} a_n = L$, if $a_n = f(n)$.

- Example. State if the sequence $((\frac{2}{3})^n)$ converges or diverges. If it converges, find its limit.

  Solution. Let $f(x) = (\frac{2}{3})^x$. We saw the graph of f in Chapter 7 [Picture drawn in class.] So $\lim_{x\to\infty} f(x) = 0$, hence $\lim_{n\to\infty} (\frac{2}{3})^n = 0$. So the sequence converges with limit 0.

- Example. State if the sequence $(\frac{\ln n}{n})$ converges or diverges. If it converges, find its limit.

  Solution. The answer will be the same as $\lim_{x\to\infty} \frac{\ln x}{x}$. This limit is of form $\frac{\infty}{\infty}$. By L'Hospitals rule (later),
$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$
  So the sequence converges with limit 0.

- Example. State if the sequence $(\frac{2^n}{3^n+1})$ converges or diverges. If it converges, find its limit.

  Solution. Divide numerator and denominator through by 3^n. We get $\frac{2^n}{3^n+1} = \frac{(\frac{2}{3})^n}{3+\frac{1}{3^n}}$, which has limit $\frac{0}{3+0} = 0$ (see previous problem). So the sequence converges with limit 0. Quicker: look at the 'winning term' in the numerator and 'winning term' in the denominator and divide them. One gets $(\frac{2}{3})^n$, which has limit 0.

- Example. State if the sequence $(\ln(n+1) - \ln n)$ converges or diverges. If it converges, find its limit.

  Solution. $\ln(n+1) - \ln n = \ln \frac{n+1}{n} = \ln(1 + \frac{1}{n})$. This has limit $\ln 1 = 0$. So the sequence converges with limit 0.

- It is sometimes helpful to remember that whether a sequence converges or diverges, has nothing to do with its first few terms. Thus, for example, the sequence $500, 1000, 1500, 2000, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots,$ converges to 0.

- The 'squeezing' or 'pinching rule': uppose that $(s_n), (x_n)$, and (t_n) are sequences with $s_n \leq x_n \leq t_n$, for every $n \geq 1$. If $\lim_n s_n = s$ and $\lim_n t_n = s$, then $\lim_n x_n = s$.

- If $\lim_n s_n = s$ and $\lim_n t_n = t$, then:
  (1) $\lim_n s_n + t_n = s + t$;
  (2) $\lim_n s_n - t_n = s - t$;
  (3) $\lim_n s_n t_n = st$;
  (4) $\lim_n Cs_n = Cs$, if C is a constant;
  (5) $\lim_n \frac{s_n}{t_n} = \frac{s}{t}$, if $t \neq 0$;
  (6) $\lim_n |s_n| = |s|$;
  (7) $\lim_n \sqrt{t_n} = \sqrt{t}$, if $t_n \geq 0$ for all $n \in \mathbb{N}$.

10.4: Some important limits.

- $x^{\frac{1}{n}} \to 1$ as $n \to \infty$ if $x > 0$ (since $x^{\frac{1}{n}} = e^{\frac{1}{n}\ln x}$, and $\frac{\ln x}{n} \to 0$, so $e^{\frac{1}{n}\ln x} \to e^0 = 1$.)

- $x^n \to 0$ as $n \to \infty$ if $|x| < 1$ (this is proved like in the example in Section 10.3 of $(\frac{2}{3})^n$).

- (x^n) diverges if $x > 1$ or $x \leq -1$ (since these cases are unbounded, or is Example (4) in Section 10.3).

- $\frac{1}{n^p} \to 0$ as $n \to \infty$ if $p > 0$ (since $\frac{1}{n^p} = (\frac{1}{n})^p$ and $\frac{1}{n} \to 0$).

- Variant of the 'squeezing' or 'pinching' rule: Suppose that $0 \le a_n \le b_n$. If $b_n \to 0$ then $a_n \to 0$. If $a_n \to \infty$ then $b_n \to \infty$.

- Example. State if the sequence $(\frac{\sin n}{n})$ converges or diverges. If the sequence converges, find its limit.

  Solution: Note that $-1 \le \sin n \le 1$, so $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$. But $\frac{1}{n} \to 0$, so by 'squeezing' $(\frac{\sin n}{n})$ converges with limit 0.

- We give another proof, using 'squeezing', that $(\frac{\ln n}{n})$ converges to 0. We first notice that $\ln x \le \sqrt{x}$ for $x \ge 4$, since by Calculus if $f(x) = x^{\frac{1}{2}} - \ln x$ then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{x} = \frac{\sqrt{x}-2}{2x} > 0$ if $x > 4$. So $f(x)$ is strictly increasing on $[4, \infty)$, and since $f(4) = 2 - \ln 4 > 0$, we must have $f(x) = \sqrt{x} - \ln x \ge 0$ for $x \ge 4$. So $\sqrt{x} \ge \ln x$ for $x \ge 4$. Thus $\frac{\ln n}{n} \le \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Since $\frac{1}{\sqrt{n}} \to 0$ by 'squeezing' we must have $\frac{\ln n}{n} \to 0$.

- In the last item we showed that $\ln x \le \sqrt{x}$. In fact $\ln x \le x^p$ for any $p > 0$ (the proof is similar). Similarly, $x^p \le a^x$ for any p and any $a > 1$, at least for large x.

- Example. State if the sequence $(n^{\frac{1}{n}})$ converges or diverges. If the sequence converges, find its limit.

  Solution: $n^{\frac{1}{n}} == e^{\frac{1}{n}\ln n} \to e^0 = 1$ as $n \to \infty$, since we just saw that $\frac{\ln n}{n} \to 0$.

- Example. Show that $\frac{2^n}{n!} \to 0$ as $n \to \infty$.

  Solution: $0 \le \frac{2^n}{n!} = \frac{2}{n}\frac{2}{n-1}\frac{2}{n-2}\cdots\frac{2}{2}\frac{2}{1} \le \frac{2}{n} \cdot 1 \cdot 1 \cdots \cdot 1 \cdot 2 = \frac{4}{n}$. But $\frac{4}{n} \to 0$, so by by 'squeezing' we must have $\frac{2^n}{n!} \to 0$.

- Similarly, $\frac{x^n}{n!} \to 0$ as $n \to \infty$.

- $(1 + \frac{x}{n})^n \to e^x$ as $n \to \infty$. We shall do this as an application of L'Hospitals rule in the next section.

10.5–10.6: L'Hospitals rule.

- We have already used L'Hospitals rule twice in this class. First we looked at $\lim_{x\to\infty} \frac{x}{e^x}$, which is called an *indeterminant form of type* $\frac{\infty}{\infty}$. We saw that

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

  Similarly, $\lim_{x\to\infty} \frac{\ln x}{x}$ is an indeterminant form of type $\frac{\infty}{\infty}$, and

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

- Example. Find $\lim_{x\to 0} \frac{\sin x}{x}$.

  Solution. This is an indeterminant form of type $\frac{0}{0}$. By L'Hospitals rule

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

- Example. Find $\lim_{x\to\infty} \frac{e^x}{x}$.

Solution. By L'Hospitals rule $= \lim_{x\to\infty} \frac{e^x}{1} = \infty$.

- Example. $\lim_{x\to 1} \frac{\ln x}{x-1}$ is an indeterminant form of type $\frac{0}{0}$. By L'Hospitals rule

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1.$$

- Example. Find $\lim_{x\to 0^+} \frac{\ln x}{e^{\frac{1}{x}}}$.

  Solution. This is an indeterminant form of type $\frac{-\infty}{\infty}$ (actually we still call this an indeterminant form of type $\frac{\infty}{\infty}$), and by L'Hospitals rule it equals

$$\lim_{x\to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(e^{\frac{1}{x}})} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x}}(-\frac{1}{x^2})} = \lim_{x\to 0^+} \frac{-x}{e^{\frac{1}{x}}} = -(\lim_{x\to 0^+} x)\,(\lim_{x\to 0^+} \frac{1}{e^{\frac{1}{x}}} = 0.$$

- These kinds of examples can always be applied to sequences (as we did in Section 10.3 when we showed $\lim_{n\to\infty} \frac{\ln n}{n} = 0$). Here is another example of this:

- Example. Find $\lim_{n\to\infty} \frac{\ln n}{e^n}$.

  Solution. Set $x = \frac{1}{n}$. As $n \to \infty$ we have $x \to 0^+$, so by the previous example

$$\frac{-\ln n}{e^n} = \frac{\ln \frac{1}{n}}{e^n} = \frac{\ln x}{e^{\frac{1}{x}}} \to 0.$$

  So $\lim_{n\to\infty} \frac{\ln n}{e^n} = 0$.

  Example. Find $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

  Solution. This is an indeterminant form of type $\frac{0}{0}$. By L'Hospitals rule it equals $\lim_{x\to 0} \frac{\cos x - 1}{3x^2}$, which is again an indeterminant form of type $\frac{0}{0}$. By L'Hospitals rule it equals $\lim_{x\to 0} \frac{-\sin x}{6x} = -\frac{1}{6}$.

- Example. Find $\lim_{x\to 0^+} x \ln x$.

  Solution. $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$, and the latter is an indeterminant form of type $\frac{-\infty}{\infty}$ (actually we still call this an indeterminant form of type $\frac{\infty}{\infty}$). By L'Hospitals rule it equals $\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+} x = 0$.

- Example. Find $\lim_{x\to 0^+} x^x$.

  Solution. This is called an indeterminant form of type 0^0. Note that $x^x = e^{x\ln x} \to e^0 = 1$ as $x \to 0^+$, since we just saw that $\lim_{x\to 0^+} x \ln x = 0$.

- Example. Find $\lim_{x\to\infty} \frac{1}{x} \int_0^x \frac{1}{1+t}\, dt$.

  Solution. By L'Hospitals rule it equals

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\int_0^x \frac{1}{1+t}}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{1+x}}{1} = 0.$$

- Example. Show that $\lim_{x\to\infty} (1 + \frac{a}{x})^x = e^a$, and that $\lim_{x\to\infty} (1 + \frac{1}{x})^x = e$.

Solution. $(1+\frac{a}{x})^x = e^{x\ln(1+\frac{a}{x})}$. Now $x\ln(1+\frac{a}{x}) = \frac{\ln(1+\frac{a}{x})}{\frac{1}{x}}$, and $\lim_{x\to\infty} \frac{\ln(1+\frac{a}{x})}{\frac{1}{x}}$ is an indeterminant form of type $\frac{0}{0}$. By L'Hospitals rule

$$\lim_{x\to\infty} \frac{\ln(1+\frac{a}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}}(-\frac{a}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}} = a.$$

Thus as $x \to \infty$ we have that $x\ln(1+\frac{a}{x}) \to a$, and so $(1+\frac{a}{x})^x = e^{x\ln(1+\frac{a}{x})} \to e^a$.

For the last assertion set $a = 1$.

- Proof of L'Hospitals rule: We just do the case that $\lim_{x\to c} \frac{f(x)}{g(x)}$ is an indeterminant form of type $\frac{0}{0}$. So $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$. We assume that f and g, and f' and g', are continuous at c. Then $f(c) = g(c) = 0$ and

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} = \frac{f'(c)}{g'(c)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

- L'Hospitals rule 'does not work' if the limit is not of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

10.7: Improper integrals.

- There are three basic kinds of improper integrals. To describe them recall that in an integral $\int_a^b f(x)\,dx$, we call a and b the *lower and upper limits of integration*, and $f(x)$ is the *integrand*. The first kind of improper integral is when $a = -\infty$ or $b = \infty$ or both. The second kind of improper integral is when a or b are numbers that $f(x)$ 'blows up' at. For example, $\int_0^1 \frac{1}{\sqrt{x}}dx$, note that $\frac{1}{\sqrt{x}}$ is not defined at 0, and goes to infinity as $x \to 0^+$. The third kind of improper integral $\int_a^b f(x)\,dx$ is when $f(x)$ 'blows up' at a number between a and b. For example, $\int_{-2}^1 \frac{1}{x^{\frac{4}{5}}}\,dx$. Here $\frac{1}{x^{\frac{4}{5}}}$ 'blows up' at 0, and 0 is between -2 and 1.

  An integral may be more than one of the above types, like $\int_{-\infty}^{\infty} \frac{dx}{x^2}\,dx$.

- Example. Determine whether or not the integral $\int_1^\infty \frac{dx}{x^2}\,dx$ is improper. Give a reason for your answer. Also, by taking an appropriate limit, compute the value of the integral.

  Solution. This is an improper integral of the first kind, since one of the limits of integration is ∞. Also,

$$\int_1^\infty \frac{dx}{x^2}\,dx = \lim_{C\to\infty} \int_1^C \frac{dx}{x^2}\,dx.$$

  Now

$$\int_1^C \frac{dx}{x^2} = -x^{-1}]_1^C = -\frac{1}{C} + 1 \to 1$$

  as $C \to \infty$. We say that this improper integral *converges* and $\int_1^\infty \frac{dx}{x^2}\,dx = 1$.

  The last example is really saying that the area underneath the graph of $y = \frac{1}{x^2}$ for $x \geq 1$, is 1. [Picture drawn in class.]

- Example. Determine whether or not the integral $\int_1^\infty \frac{dx}{x}\,dx$ is improper. Give a reason for your answer. Also, by taking an appropriate limit, compute the value of the integral.

  Solution. This is an improper integral of the first kind, since one of the limits of integration is ∞. Also, $\int_1^\infty \frac{dx}{x}\,dx = \lim_{C\to\infty} \int_1^C \frac{dx}{x}\,dx$. Now

$$\int_1^C \frac{dx}{x} = \ln x]_1^C = \ln C - \ln 1 = \ln C \to \infty$$

as $C \to \infty$. We say that this improper integral *diverges* and $\int_1^\infty \frac{dx}{x}\,dx = \infty$.

The last example is really saying that the area underneath the graph of $y = \frac{1}{x}$ for $x \geq 1$, is 1. [Picture drawn in class.]

- Similarly $\int_1^\infty \frac{dx}{x^p}$ diverges ($= \infty$) if $0 \leq p \leq 1$, and $\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$ (it converges) if $p > 1$.

- Compute the following improper integrals if they converge: (a) $\int_{-\infty}^0 \frac{dx}{x^2+1}$, and (b) $\int_{-\infty}^\infty \frac{dx}{x^2+1}$.

  Solution. (a) The first integral is $\lim_{C\to-\infty} \int_C^0 \frac{dx}{x^2+1}$. Now

$$\int_C^0 \frac{dx}{x^2+1} = \tan^{-1} x]_C^0 = \tan^{-1} 0 - \tan^{-1} C = -\tan^{-1} C \to -(\frac{-\pi}{2}) = \frac{\pi}{2}$$

  as $C \to -\infty$. [Drew picture of the $\tan^{-1}$ graph in class.] So $\int_{-\infty}^0 \frac{dx}{x^2+1} = \frac{\pi}{2}$.

  (b) Also, $\int_{-\infty}^\infty \frac{dx}{x^2+1} = \int_{-\infty}^0 \frac{dx}{x^2+1} + \int_0^\infty \frac{dx}{x^2+1}$. To do $\int_0^\infty \frac{dx}{x^2+1}$ we observe that

$$\int_0^C \frac{dx}{x^2+1} = \tan^{-1} x]_0^C = \tan^{-1} C - \tan^{-1} 0 = \tan^{-1} C \to \frac{\pi}{2}$$

  as $C \to \infty$ [Drew picture of the $\tan^{-1}$ graph in class]. So $\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$, so that

$$\int_{-\infty}^\infty \frac{dx}{x^2+1} = \int_{-\infty}^0 \frac{dx}{x^2+1} + \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

- Example. Determine whether or not the integral $\int_0^1 \frac{dx}{\sqrt{x}}$ is improper. Give a reason for your answer. Also, by taking an appropriate limit, compute the value of the integral.

  Solution. This is an improper integral of the second kind, since $\frac{1}{\sqrt{x}}$ blows up at 0. The integral is $\lim_{C\to0^+} \int_C^1 \frac{dx}{\sqrt{x}}$. Now

$$\int_C^1 \frac{dx}{\sqrt{x}} = 2x^{\frac{1}{2}}]_C^1 = 2 - 2C^{\frac{1}{2}} \to 2$$

  as $C \to 0^+$. So the improper integral converges, and $\int_0^1 \frac{dx}{\sqrt{x}} = 2$.

- Example. Determine whether or not the integral $\int_0^1 (\frac{1}{\sqrt{x}} - \frac{1}{(x-1)^2})\,dx$ is improper. Give a reason for your answer. Also, compute the value of the integral.

  Solution. This is an improper integral of the second kind, since $(\frac{1}{\sqrt{x}} - \frac{1}{(x-1)^2})$ blows up both at 0 and at 1. Let us write

$$\int_0^1 (\frac{1}{\sqrt{x}} - \frac{1}{(x-1)^2})\,dx = \int_0^{\frac{1}{2}} (\frac{1}{\sqrt{x}} - \frac{1}{(x-1)^2})\,dx + \int_{\frac{1}{2}}^1 (\frac{1}{\sqrt{x}} - \frac{1}{(x-1)^2})\,dx.$$

  The first integral is $\lim_{C\to0^+} (2x^{\frac{1}{2}} + \frac{1}{x-1})]_C^{\frac{1}{2}}$, which is a finite number. However the second integral is

$$\lim_{C\to1^-} (2x^{\frac{1}{2}} + \frac{1}{x-1})]_{\frac{1}{2}}^C = \lim_{C\to1^-} (2C^{\frac{1}{2}} + \frac{1}{C-1}) - (\sqrt{2} - 2) = -\infty,$$

  since $\frac{1}{C-1} \to -\infty$ as $C \to 1^-$. So the integral diverges and $\int_0^1 (\frac{1}{\sqrt{x}} - \frac{1}{(x-1)^2})\,dx$ is a finite number plus $-\infty$, which is $-\infty$.

- Example. Determine whether or not the integral $\int_{-2}^1 \frac{1}{x^{\frac{4}{5}}}\,dx$ is improper. Give a reason for your answer. Also, compute the value of the integral.

Solution. This is an improper integral because $\frac{1}{x^{\frac{4}{5}}}$ 'blows up' at 0, and 0 is between -2 and 1. We write $\int_{-2}^{1} \frac{1}{x^{\frac{4}{5}}}\, dx = \int_{-2}^{0} \frac{1}{x^{\frac{4}{5}}}\, dx + \int_{0}^{1} \frac{1}{x^{\frac{4}{5}}}\, dx$, a sum of two of the second kind of improper integral. To do the first integral here,

$$\int_{-2}^{C} \frac{1}{x^{\frac{4}{5}}}\, dx = 5x^{\frac{1}{5}}]_{-2}^{C} = 5C^{\frac{1}{5}} - 5(-2)^{\frac{1}{5}} = 5C^{\frac{1}{5}} + 5 \cdot 2^{\frac{1}{5}} \to 5 \cdot 2^{\frac{1}{5}},$$

as $C \to 0^-$. So $\int_{-2}^{0} \frac{1}{x^{\frac{4}{5}}}\, dx = 5 \cdot 2^{\frac{1}{5}}$. To do the second integral,

$$\int_{C}^{1} \frac{1}{x^{\frac{4}{5}}}\, dx = 5x^{\frac{1}{5}}]_{C}^{1} = 5 - 5C^{\frac{1}{5}} \to 5$$

as $C \to 0^+$. So $\int_{0}^{1} \frac{1}{x^{\frac{4}{5}}}\, dx = 5$. Thus

$$\int_{-2}^{1} \frac{1}{x^{\frac{4}{5}}}\, dx = \int_{-2}^{0} \frac{1}{x^{\frac{4}{5}}}\, dx + \int_{0}^{1} \frac{1}{x^{\frac{4}{5}}}\, dx = 5 \cdot 2^{\frac{1}{5}} + 5.$$

- A "Comparison test": If $0 \leq f(x) \leq g(x)$ on my interval, and if the integral of g on the interval converges then the integral of f converges too. If the integral of f on the interval diverges then the integral of g diverges too.

- Do the integrals converge or diverge: (a) $\int_{1}^{\infty} \frac{dx}{\sqrt{1+x^2}}\, dx$, (b) $\int_{1}^{\infty} \frac{dx}{\sqrt{1+x^3}}$.

  Solution. (b) $\frac{1}{\sqrt{1+x^3}} < \frac{1}{\sqrt{x^3}} = \frac{1}{x^{\frac{3}{2}}}$ if $x \geq 1$. Now since $p = \frac{3}{2} > 1$, we know from the last page that $\int_{1}^{\infty} \frac{dx}{x^{\frac{3}{2}}}$ converges. So by the 'comparison test' above, $\int_{1}^{\infty} \frac{dx}{\sqrt{1+x^3}}$ converges.

  (a) $\frac{1}{\sqrt{1+x^2}} > \frac{1}{2x}$ if $x \geq 1$ (to see this note that this is saying that $2x > \sqrt{1+x^2}$ or $4x^2 > 1 + x^2$ or $3x^2 > 1$, which is true if $x \geq 1$). However $\int_{1}^{\infty} \frac{dx}{2x}$ diverges as we saw on the last page. So by the 'comparison test' above, $\int_{1}^{\infty} \frac{dx}{\sqrt{1+x^2}}$ diverges too.

- *Gabriel's horn* (textbook p. 624). In the bible, Gabriel is an archangel, and the horn is the trumpet blown at the end of time, heralding a cataclysmic event described in the book of Revelation. In this problem, we obtain this infinite trumpet by taking the graph of $f(x) = \frac{1}{x}$ for $1 \leq x < \infty$, and rotating it around the x-axis [Picture drawn in class.] The volume formula for this from Calc I is

$$V = \pi \int_{1}^{\infty} (f(x))^2\, dx = \pi \int_{1}^{\infty} \frac{dx}{x^2} = \pi,$$

  as we saw recently. So you can easily fill up Gabriel's horn with water. However in the homework you will be asked to show that the surface area formula from Calc 1, applied here, gives that Gabriel's horn has infinite surface area. So you can fill it up, but you could never paint it! Interesting... .

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Chapter 11. Infinite Series.

First we recall 'sigma-notation'. For example,

$$\sum_{k=2}^{4} k^2 \;=\; 2^2+3^2+4^2 = 4+9+16 = 29\ .$$

Here '$\sum$' is read as 'sigma' but it should be interpreted as 'sum of'.

Example. Write in sigma-notation: $e+2e^2+3e^3+4e^4+5e^5$.
Solution. The kth term here is ke^k. So the series is $\sum_{k=1}^{5} ke^k$.

Example. Write in sigma-notation: $\frac{4}{3}+\frac{4}{5}+\frac{8}{7}+\frac{8}{9}+\frac{12}{11}$.
Solution. This is almost identical to a sequence example in the previous chapter; the nth term is $a_n = \frac{2n+1+(-1)^{n+1}}{2n+1} = 1+\frac{(-1)^{n+1}}{2n+1}$. So the series is $\sum_{k=1}^{5}(1+\frac{(-1)^{k+1}}{2k+1})$.

In most of the rest of this course we study 'infinite series'. These are expressions of the form

$$\sum_{k=m}^{\infty} a_k \;=\; a_m + a_{m+1} + a_{m+2} + \cdots \qquad (*)$$

Let us call this expression (*).
What does expression (*) mean? In fact we shall see shortly that the expression means two things.
Usually $m=0$ or 1, that is, (*) usually is

$$a_0+a_1+a_2+\cdots$$

or

$$a_1+a_2+a_3+\cdots\ .$$

We call the number a_k the *kth term in the series.* Sometimes we will be sloppy and write $\sum_k a_k$ when we mean (*).
The most important question about an infinite series, just as for an infinite sequence, is 1) does the series converge? and 2) if it converges, what is its sum? We will explain these in a minute.

Example 1. Write in sigma notation:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\ .$$

Solution. Here the first term in the series is 1, the second term is $\frac{1}{2}$, the third term is $\frac{1}{2^2}$, the fourth term is $\frac{1}{2^3}$, and so on. Clearly the kth term in the series is $\frac{1}{2^{k-1}}$. We may rewrite the series as

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$$

or

$$\sum_{k=0}^{\infty} \frac{1}{2^k}\ .$$

Example 2. Write in sigma notation:

$$-1+1-1+1-1+\cdots\ .$$

Solution. Note that the kth term is $(-1)^k$. The series may be rewritten $\sum_{k=1}^{\infty}(-1)^k$.

Example 3. Write in sigma notation:

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots \;=\; \frac{1}{2\cdot 1}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3}+\frac{1}{5\cdot 4}+\cdots$$

Solution. Here the first term in the series is $\frac{1}{2\cdot 1}$, the second term is $\frac{1}{3\cdot 2}$, the third term is $\frac{1}{4\cdot 3}$, and so on. Clearly the kth term in the series is $\frac{1}{(k+1)k}$. We may rewrite the series as

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)k} \;.$$

We still have not said what expressions like (*) or those in Examples 1, 2, and 3 mean. In fact an expression like (*), or those in Examples 1, 2, and 3, has two meanings:

Meaning # 1: A 'formal sum'. That is, it is a way to indicate that we are thinking about adding up all these numbers in the expression (*), in the order given. It does not mean that these numbers do add up.

Before we go to Meaning # 2, let me say how you 'add up all the numbers in an infinite series'. To do this, we define the *nth partial sum* s_n to be the sum of the first n terms in the series. In this way we get a *sequence*

$$s_1, s_2, s_3, \cdots$$

called the *sequence of partial sums*. We say the original series *converges* if the *sequence* $\{s_n\}$ converges. If it does not converge then we say it *diverges*. We call $\lim_{n\to\infty} s_n$ the *sum of the series* if this limit exists.

Meaning # 2: $\sum_k a_k \;=\; \lim_{n\to\infty} s_n$ if this limit exists.

Back to Example 2 above: The first meaning of $-1+1-1+1-1+\cdots$ (or $\sum_{k=1}^{\infty}(-1)^k$) is as a 'formal sum'. That is, we we want to think about adding up all the numbers in the expression in the order given. To get to the second meaning, we must compute the s_n's. Clearly in this example, $s_1=-1, s_2=-1+1=0, s_3=-1+1-1=-1, s_4=-1+1-1+1=0$, and so on. In fact the sequence $s_1, s_2, s_3, \cdots$ is the sequence

$$-1,\ 0,\ -1,\ 0,\ -1,\ 0,\ -1,\ \cdots$$

Does this *sequence* converge? No. So the series *diverges*. It does not have 'Meaning # 2', and it has no sum.

Back to Example 1 above: The first meaning of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots \;=\; \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$, is 'a formal sum'. That is, it is a way to indicate that we would like to add up all these numbers. To actually add them up, we need to look at the sequence s_n of partial sums. In this example, $s_1=1$, and s_2 is the sum of the first two terms, that is $s_2=1\frac{1}{2}$. Also s_3 is the sum of the first three terms:

$$s_3 = 1+\frac{1}{2}+\frac{1}{4} = 1\frac{3}{4}$$

Similarly

$$s_4 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1\frac{4+2+1}{8} = 1\frac{7}{8}\;.$$

Now you can spot the pattern. The sequence $s_1, s_2, s_3, \cdots$ is the sequence

$$1,\ 1\frac{1}{2},\ 1\frac{3}{4},\ 1\frac{7}{8},\ 1\frac{15}{16},\ \cdots$$

which has limit 2. Thus the original *series* $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$ *converges*, and its sum is 2. Thus we have Meaning # 2:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 2$$

or

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} \;=\; 2.$$

Example 4: Consider the series $1+1+1+\cdots$. Here the nth partial sum s_n, namely the sum of the first n terms, is n. Since $\lim_{n\to\infty} s_n = \lim_{n\to\infty} n = \infty$ in this case, the series diverges. But its sum, namely $\lim_{n\to\infty} s_n$, is ∞. So this series does have a Meaning # 2, namely its sum $1+1+1+\cdots = +\infty$.

Back to Example 3 above: The first meaning of $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots \;=\; \sum_{k=1}^{\infty} \frac{1}{(k+1)k}$, is 'a formal sum'. That is, it is a way to indicate that we would like to add up all these numbers. To actually add them up, we need to look at the sequence s_n. In this example, $s_1 = \frac{1}{2}$, and s_2 is the sum of the first two terms, that is

$$s_2 = \frac{1}{2}+\frac{1}{6} = \frac{3+1}{6} = \frac{2}{3}\,.$$

Similarly

$$s_3 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{6+2+1}{12} = \frac{3}{4}\,.$$

Similarly, $s_4 = \frac{4}{5}$, and now you can spot the pattern, it seems that $s_n = \frac{n}{n+1}$. This can be proved using the method of 'telescoping series': to use this method we consider the kth term $\frac{1}{(k+1)k}$ and use partial fractions. The partial fraction technique asks you to write

$$\frac{1}{(x+1)x} \;=\; \frac{A}{x+1}+\frac{B}{x}$$

Multiplying through by $(x+1)x$ gives $1 = Ax + B(x+1)$, so that $B = 1$ and $A = -1$. Thus

$$\frac{1}{(k+1)k} \;=\; \frac{1}{k} \;-\; \frac{1}{k+1}\;\;.$$

Therefore setting $k = 1$, $k = 2$, $k = 3, \cdots, k = n$ in the last equation gives that

$$s_n = \frac{1}{2\cdot 1}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3}+\cdots+\frac{1}{(n+1)n} = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\;.$$

This is called 'telescoping', because most of the last expression cancels, and the expression cancels to yield

$$s_n \;=\; 1-\frac{1}{n+1} \;=\; \frac{n}{n+1}$$

as claimed. Thus the sequence $s_1, s_2, s_3, \cdots$ is the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

which has limit 1. Thus the original *series* $\sum_{k=1}^{\infty} \frac{1}{(k+1)k}$ *converges*, and its sum is 1. Thus we have Meaning # 2:

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots = 1$$

or

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)k} \;=\; 1.$$

- In a sum like $\sum_{k=1}^{\infty} \frac{1}{(k+1)k}$, the k is a 'dummy index'. That is, it is only used internally inside the sum, and we can feel free to change its name, to $\sum_{j=1}^{\infty} \frac{1}{(j+1)j}$, for example,

- In a series $\sum_{k=m}^{\infty} a_k$ let us call m the 'starting index'. Thus for example, the starting index of $\sum_{k=2}^{\infty} \frac{k-1}{k^2}$ is 2. Any series can be 'renumbered' so that its starting index is 0. That is, any infinite series may be rewritten as $\sum_{k=0}^{\infty} a_k$. For example, $\sum_{k=m}^{\infty} a_k$, which is the same as $a_m + a_{m+1} + a_{m+2} + \cdots$, can be relabelled by letting $j = k - m$, or equivalently $k = j + m$. Then $\sum_{k=m}^{\infty} a_k = \sum_{j=0}^{\infty} a_{j+m}$.

Example: Rewrite $\sum_{k=2}^{\infty} \frac{k-1}{k^2}$ as a series $\sum_{k=0}^{\infty} a_k$.
Solution. Letting $j = k - 2$, so that $k = j + 2$, the sum becomes

$$\sum_{j=0}^{\infty} \frac{j+2-1}{(j+2)^2} = \sum_{j=0}^{\infty} \frac{j+1}{(j+2)^2}$$

Of course j is 'dummy' so we can rewrite this as $\sum_{k=0}^{\infty} \frac{k+1}{(k+2)^2}$.

There is no reason of course why we chose 0 for the starting index. One can make all series begin with the starting index 1 if you wanted to, by a similar trick. However it is convenient to fix one starting index, so it may as well be 0. Many of the following results are therfore phrased in terms of series $\sum_{k=0}^{\infty} a_k$.

- **Geometric series:** This is a series of form $c + cx + cx^2 + cx^3 + \cdots$, or $\sum_{k=0}^{\infty} cx^k$, for constants c and x. We call x the 'constant ratio' of the geometric series. Note that if you divide any term in the series by the previous term, you get x. We assume $c \neq 0$, otherwise this is the trivial series with sum 0.

  The MAIN FACT about geometric series, is that such a series converges if and only if $|x| < 1$, and in that case its sum is $\frac{c}{1-x}$.

  [Proof: If $c = 0$ then the series is $0 + 0 + 0 + \cdots$ and the result is obvious. So we can assume that $c \neq 0$. The sum of the first n terms is

  $$s_n = c + cx + cx^2 + \cdots + cx^{n-1} = c(1 + x + x^2 + \cdots + x^{n-1}) \ .$$

  There is a well known result in algebra which says that

  $$(1 + x + x^2 + \cdots + x^{n-1})(1 - x) = 1 - x^n$$

  (to prove it multiply out the parentheses and cancel). Thus if $x \neq 1$ then $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$, so that

  $$s_n = c + cx + cx^2 + \cdots + cx^{n-1} = c\frac{1-x^n}{1-x} \ .$$

  (It is worth memorizing the last formula, the sum of n terms of a geometric series.) The only thing that depends on n on the right hand side here is the x^n, which we saw in 10.4 converges to 0 if $|x| < 1$, and diverges otherwise. If $x = 1$ then $s_n = c + c + \cdots + c$ (n times) which equals nc. Thus $\lim_{n\to\infty} s_n = c\frac{1}{1-x}$ if $|x| < 1$. If $|x| \geq 1$ then $\{s_n\}$ diverges, so that the original series diverges.]

- **Example.** Is the series $6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots$ a geometric series? What is its sum?

Solution: Note that any one term, divided by the previous term, is $\frac{1}{3}$. So it is a geometric series, with constant ratio $x = \frac{1}{3}$ and first term $c = 6$, so that its sum is

$$\frac{6}{1-\frac{1}{3}} \quad = \quad 6 \cdot \frac{3}{2} \quad = \quad 9 \ .$$

- **Example.** Find the sum of the series $\sum_{k=1}^{\infty} \frac{3}{(1.2)^k}$ if it converges.

  Solution: This is a geometric series, with constant ratio $x = \frac{1}{1.2}$ and first term $c = \frac{3}{1.2}$, so that its sum is

$$\frac{\frac{3}{1.2}}{1-\frac{1}{1.2}} \ = \ \frac{3}{1.2-1} \ = \ \frac{3}{0.2} = 15.$$

- **Example: Recurring decimals.** Express the number $5.1233333\cdots$ (recurring) as an improper fraction.

  Solution: $5.1233333\cdots$ equals

$$5+\frac{1}{10}+\frac{2}{100}+\frac{3}{1000}+\frac{3}{10000}+\frac{3}{100000}+\cdots \quad = \quad \frac{512}{100}+\frac{3}{1000}+\frac{3}{10000}+\frac{3}{100000}+\cdots \ .$$

  Since the last part of this is a geometric series with first term $\frac{3}{1000}$ and constant ratio $\frac{1}{10}$, the last sum equals

$$\frac{512}{100}+\frac{3}{1000}\left(\frac{1}{1-\frac{1}{10}}\right) \ = \ \frac{512}{100}+\frac{3}{1000}\frac{10}{9} \ = \ \frac{512}{100}+\frac{1}{300} \ = \ \frac{1537}{300} \ .$$

- FACT: If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ both converge, and if c is a constant, then:
- $\sum_{k=0}^{\infty} (a_k+b_k)$ converges, with sum $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$;
- $\sum_{k=0}^{\infty} (a_k-b_k)$ converges, with sum $\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} b_k$;
- $\sum_{k=0}^{\infty} (ca_k)$ converges, with sum $c\sum_{k=0}^{\infty} a_k$.

  [Proof: We just prove the first one, the others are quite similar. The nth partial sum of $\sum_{k=0}^{\infty} (a_k+b_k)$ is $\sum_{k=0}^{n-1} (a_k+b_k) \ = \ \sum_{k=0}^{n-1} a_k \ + \ \sum_{k=0}^{n-1} b_k$. By a fact about sums of *sequences* from a few weeks ago, this converges, as $n \to \infty$, to $\sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$.]

- **Example:** Does $\sum_{k=0}^{\infty} \left(\frac{2}{3^k}+\frac{(-1)^k}{2^k}\right)$ converge? If so, what is its sum?

  Solution. It converges. To see this notice that by the previous item we can write

$$\sum_{k=0}^{\infty}\left(\frac{2}{3^k}+\frac{(-1)^k}{2^k}\right) = \sum_{k=0}^{\infty} \frac{2}{3^k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} = \sum_{k=0}^{\infty} \frac{2}{3^k} + \sum_{k=0}^{\infty} \left(\frac{-1}{2}\right)^k$$

  which is a sum of two geometric series. By the rule for geometric series on the previous page, the series do converge and our sum is

$$\frac{2}{1-\frac{1}{3}} \ + \ \frac{1}{1-\left(\frac{-1}{2}\right)} \ = \ 3\frac{2}{3}.$$

- For any positive integer m we can write $\sum_{k=0}^{\infty} a_k = (a_0+a_1+\cdots+a_{m-1}) + \sum_{k=m}^{\infty} a_k$.
  Indeed $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=m}^{\infty} a_k$ converges. [This is because the nth partial sum of the $\sum_{k=0}^{\infty} a_k$ series, and the nth partial sum of the $\sum_{k=m}^{\infty} a_k$ series differ by

a fixed constant.] If these series converge, then their sum also obeys the rule:

$$\sum_{k=0}^{\infty} a_k = (a_0 + a_1 + \cdots + a_{m-1}) + \sum_{k=m}^{\infty} a_k \ .$$

- **Example.** You are told that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Find $\sum_{k=3}^{\infty} \frac{1}{k^2}$.

  Solution. $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2} + \sum_{k=3}^{\infty} \frac{1}{k^2} = \frac{pi^2}{6}$. So $\sum_{k=3}^{\infty} \frac{1}{k^2} = \frac{pi^2}{6} - 1 - \frac{1}{2}$.

- From the last fact it follows that the 'first few terms' of a series, do not affect whether the series converges or not. It will affect the sum though.

- **The Divergence Test:** If $\lim_{k\to\infty} a_k \neq 0$ then the series $\sum_k a_k$ diverges.

  A matching statement: If $\sum_k a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

  [Beware: If $\lim_{k\to\infty} a_k = 0$ we cannot conclude that $\sum_k a_k$ converges.

  [Proof: Suppose that $\sum_{k=0}^{\infty} a_k = s$. If s_n is the nth partial sum then $s_n \to s$ as $n \to \infty$. Clearly $s_{n+1} \to s$ too, as $n \to \infty$. Thus $s_{n+1} - s_n = a_n \to s - s = 0$.]

- **Example:** Determine whether the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$ converges or diverges.

  Solution: It diverges by the Divergence Test, since $\lim_{k\to\infty} \frac{k}{k+1} = 1 \neq 0$.

- **Example:** Determine whether the series $\sum_{k=1}^{\infty} (-1)^k \frac{k}{\ln(2k)}$ converges or diverges.

  Solution: It diverges by the Divergence Test, since $\lim_{k\to\infty} \frac{k}{\ln(2k)} = \infty \neq 0$. (One can use LHospital to see that $\lim_{x\to\infty} \frac{x}{\ln(2x)} = \infty$).

- **Example:** Determine whether the series $\sum_k \frac{3^k-2}{3^k}$ converges or diverges.

  Solution: It diverges by the Divergence Test, since $\lim_{k\to\infty} \frac{3^k-2}{3^k} = 1 \neq 0$.

11.2-3. Nonnegative Series, and tests for series convergence.

- A series $\sum_k a_k$ is called a *nonnegative series* if all the terms a_k are ≥ 0.

- For a nonnegative series, the sequence $\{s_n\}$ of the partial sums is a nondecreasing (or increasing) sequence.

  [Proof: If $s_n = a_0 + a_1 + \cdots + a_{n-1}$ say, then $s_{n+1} = a_1 + a_2 + \cdots + a_{n-1} + a_n$, so that $s_{n+1} - s_n = a_n \geq 0$.]

- Therefore, by a fact we saw in Chapter 10 for a nonnegative sequence, the sum of the series equals the least upper bound of the sequence $\{s_n\}$ of partial sums. Thus the sum of the series always exists, but may be ∞. More importantly, the series converges if and only if the $\{s_n\}$ sequence is bounded above. The latter happens if and only if the sum of the series is finite. Thus to indicate that a nonnegative series converges we often simply write $\sum_k a_k < \infty$.

- **Example:** The HARMONIC SERIES is the important series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \;=\; \sum_{k=1}^{\infty} \frac{1}{k}\,.$$

This is a nonnegative series, so to see if it converges we need only check if the sequence $\{s_n\}$ is bounded above, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$. A trick to do this is to look at $\int_1^{n+1} \frac{1}{x}\,dx$, interpreted as the shaded area in the graph below [Picture drawn in class]. This shaded area is less than the area of the n rectangles shown. Hence

$$1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + \cdots 1 \cdot \frac{1}{n} \geq \int_1^{n+1} \frac{1}{x}\,dx.$$

So $s_n \;\geq\; \ln(n+1) - \ln(1) \;\to \infty$ as $n \to \infty$.
Thus the harmonic series diverges; it has sum $+\infty$.

- The trick used in the previous example can be used in the same way to prove:

The Integral Test: If $f(x)$ is a continuous decreasing positive function defined on $[1, \infty)$ [Picture drawn in class], then $\sum_{k=1}^{\infty}\, f(k)$ converges if and only if $\int_1^{\infty}\, f(x)\,dx$ converges (i.e. is finite).

- To redo the previous example using this test, let $f(x) = \frac{1}{x}$, which is certainly continuous, decreasing, and positive on $[1, \infty)$. Also

$$\int_1^{\infty} \frac{1}{x}\,dx \;=\; \lim_{c\to\infty} \int_1^{c} \frac{1}{x}\,dx \;=\; \lim_{c\to\infty} (\ln c - \ln 1) \;=\; +\infty$$

So $\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{k=1}^{\infty}\, f(k)$ does not converge.

- **p-series.** An almost identical argument shows that $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$. These are called 'p-series'.

- **Example.** Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$ and $\sum_{k=1}^{\infty} \frac{1}{k\sqrt[3]{k}}$converge or diverge.

<u>Solution:</u> Use the p-series test with $p = 1/4$ and $4/3$. So the first series diverges, the second converges.

- **Example.** Determine whether the series $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$ converges or diverges.

<u>Solution:</u> First note that it does'nt matter that the series starts with $k = 3$, since the first few terms of a series do not affect whether the series converges or not. Next let $f(x) = \frac{1}{x(\ln x)^2}$, and apply the Integral Test. Note that

$$\int_3^{\infty} \frac{1}{x(\ln x)^2}\,dx \;=\; \lim_{c\to\infty} \int_3^{c} \frac{1}{x(\ln x)^2}\,dx\,.$$

Substituting $u = \ln x$ we have $du = \frac{dx}{x}$ so that

$$\int_3^{c} \frac{1}{x(\ln x)^2}\,dx \;=\; \int_{\ln 3}^{\ln c} \frac{1}{u^2}\,du \;=\; -u^{-1}\Big|_{\ln 3}^{\ln c} \;=\; \frac{1}{\ln 3} - \frac{1}{\ln c} \;\to\; \frac{1}{\ln 3}$$

as $c \to \infty$. So both the integral and the series converge.

- **Example.** Determine whether the series $\sum_{k=3}^{\infty} \frac{1}{k\sqrt{\ln k}}$ converges or diverges.

Solution: Similar to the last, let $f(x) = \frac{1}{x(\ln x)^{\frac{1}{2}}}$, and apply the Integral Test. Note that

$$\int_3^\infty \frac{1}{x(\ln x)^{\frac{1}{2}}}\, dx \;=\; \lim_{c\to\infty} \int_3^c \frac{1}{x(\ln x)^{\frac{1}{2}}}\, dx\ .$$

Substituting $u = \ln x$ we have $du = \frac{dx}{x}$ so that

$$\int_3^c \frac{1}{x(\ln x)^{\frac{1}{2}}}\, dx \;=\; \int_{\ln 3}^{\ln c} \frac{1}{u^{\frac{1}{2}}}\, du \;=\; 2u^{\frac{1}{2}}\Big|_{\ln 3}^{\ln c} \;=\; 2(\sqrt{\ln c} - \sqrt{\ln 3}) \;\to\; \infty$$

as $c \to \infty$. So both the integral and the series diverge.

- **Basic Comparison Test:** Suppose that $0 \le a_k \le b_k$ for all k.
  1) If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.
  2) If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.

- **Limit Comparison Test:** If $0 < \lim\limits_{k\to\infty} \frac{a_k}{b_k} < \infty$, then $\sum_k a_k$ converges if and only if $\sum_k b_k$ converges.

- Asked whether a nonnegative series $\sum_k a_k$ converges, one may first ask: does $a_k \to 0$? If not, the answer is 'no', by the divergence test. If yes, one may ask how *fast* does $a_k \to 0$? If very fast, like faster than the terms in a series that you know converges, then it converges too by something like the Basic Comparison Test.

- **11.3. Root Test:** If $\sum_k a_k$ is a nonnegative series with $\lim\limits_{k\to\infty} (a_k)^{\frac{1}{k}} \;=\; r$. If $0 \le r < 1$ then $\sum_k a_k$ converges. If $1 < r \le \infty$ then $\sum_k a_k$ diverges.

- **11.3. Ratio Test:** If $\sum_k a_k$ is a nonnegative series with $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} \;= \lambda$. If $0 \le \lambda < 1$ then $\sum_k a_k$ converges. If $1 < \lambda \le \infty$ then $\sum_k a_k$ diverges.

- Examples. Determine whether the following series converge or diverge: (a) $\sum_{k=1}^\infty \frac{1}{\sqrt{k+k^3}}$; (b) $\sum_{k=1}^\infty \frac{1}{k+\sqrt{k}}$; (c) $\sum_{k=2}^\infty \frac{1}{\sqrt{k^3-k}}$; (d) $\sum_{k=1}^\infty \frac{k^3}{2^k}$; (e) $\sum_{k=3}^\infty \frac{k!}{(2k)!}$; (f) $\sum_{k=3}^\infty \frac{1}{(\ln k)^k}$.

Solution: (a) The k^3 here is much more important that the k here, so think of the series as being comparable to $\sum_k \frac{1}{\sqrt{k^3}}$. This is a p-series with $p = \frac{3}{2} > 1$, so $\sum_k \frac{1}{\sqrt{k^3}}$ converges by the p-series test above. Now we can use either the basic comparison test or the limit comparison test to see that $\sum_{k=1}^\infty \frac{1}{\sqrt{k+k^3}}$ converges. For example, since $k^3 < k + k^3$ we have $\sqrt{k^3} < \sqrt{k+k^3}$, so that $\frac{1}{\sqrt{k+k^3}} < \frac{1}{\sqrt{k^3}}$. Since $\sum_k \frac{1}{\sqrt{k^3}}$ converges, our other series converges by the basic comparison test.

(b) We may start similarly to (a), compare with $\sum_{k=1}^\infty \frac{1}{k}$ which is divergent. Lets apply the limit comparison test, with $b_k = \frac{1}{k+\sqrt{k}}$ and $a_k = \frac{1}{k}$. We have

$$\frac{a_k}{b_k} \;=\; \frac{k+\sqrt{k}}{k} \;=\; 1 + \frac{1}{\sqrt{k}} \;\to 1$$

as $k \to \infty$. Since this limit is > 0, the limit comparison test tells us that $\sum_{k=1}^\infty \frac{1}{k+\sqrt{k}}$ diverges.

(c) Similar to (b), but compare with the convergent p-series $\sum_k \frac{1}{\sqrt{k^3}}$ (see (a)), using the limit comparison test. We let $a_k = \frac{1}{\sqrt{k^3}}$ and $b_k = \frac{1}{\sqrt{k^3-k}}$, then

$$\frac{a_k}{b_k} = \frac{\sqrt{k^3-k}}{\sqrt{k^3}} = \sqrt{1-\frac{k}{k^3}} \to 1 > 0$$

as $k \to \infty$. Thus the limit comparison test tells us that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3-k}}$ converges.

(d) You could use the ratio or the root test here. If we use the ratio test with $a_k = \frac{k^3}{2^k}$, then $a_{k+1} = \frac{(k+1)^3}{2^{k+1}}$, so that

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^3}{2 \cdot 2^k} \cdot \frac{2^k}{k^3} = \frac{1}{2}\left(\frac{k+1}{k}\right)^3 = \frac{1}{2}\left(1+\frac{1}{k}\right)^3 \to \frac{1}{2}$$

as $k \to \infty$. Since this limit is less than 1, the ratio test tells us that the series converges.

(e) We use the ratio test with $a_k = \frac{k!}{(2k)!}$. Then $a_{k+1} = \frac{(k+1)!}{(2(k+1))!}$. But $(k+1)!$ may be written as $(k+1)k(k-1)\cdots 3{\cdot}2{\cdot}1 = (k+1)k!$, and similarly, $(2k+2)! = (2k+2)(2k+1)(2k)!$. Thus

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)k!}{(2k+2)(2k+1)(2k)!} \cdot \frac{(2k)!}{k!} = \frac{(k+1)}{(2k+2)(2k+1)} = \frac{1}{2(2k+1)} \to 0$$

as $k \to \infty$. Since this limit is less than 1, the ratio test tells us that the series converges.

(f) Converges by the root test with $a_k = \frac{1}{(\ln k)^k}$, since $(a_k)^{\frac{1}{k}} = \frac{1}{(\ln k)} \to 0$ as $k \to \infty$.

- Examples. Determine whether the following series converge or diverge: (a) $\sum_{k=1}^{\infty} \frac{k}{\sqrt{1+k^2}}$; (b) $\sum_{k=10}^{\infty} \frac{1}{\sqrt{k-3}}$; (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+1}}$; (d) $\sum_{k=1}^{\infty} \frac{1}{3^k+2}$; (e) $\sum_{k=1}^{\infty} \frac{k!}{2^k}$; (f) $\sum_{k=1}^{\infty} \frac{\sin(\frac{1}{k})}{\sqrt{k}}$; (g) $\sum_{k=10}^{\infty} \frac{1}{3k^2-4k+5}$ (h) $\sum_{k=10}^{\infty} \frac{k^2+10}{4k^3-k^2+7}$; (i) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k^3+1}}$; (j) $\sum_{k=1}^{\infty} \frac{e^{2k}}{k^k}$; (k) $\sum_{k=1}^{\infty} (\frac{2k+1}{3^k})^k$ (l) $\sum_{k=1}^{\infty} \frac{k^2 2^k}{3^k}$; (m) $\sum_{k=1}^{\infty} \frac{k!}{2^k}$.

[These are for extra practice. Some were done in class or review session; all worked in Pam B's online notes. Items (a)–(d), (g)–(i) can also be answered very quickly by the 'winning term' trick. For example, in (h) the 'winning terms' in numerator and denominator give $\frac{k^2}{4k^3} = \frac{1}{4k}$. So our series behaves like $\sum_k \frac{1}{4k} = \frac{1}{4}\sum_k \frac{1}{k}$, which diverges (basically is the harmonic series). So the series in (h) diverges too. That is the trick. To fully justify this though, if pressed for a proof, one would use the limit comparison test, with $b_n = \frac{1}{4k}$.

In (f), use the limit comparison test, with $a_n = \frac{\sin(\frac{1}{k})}{\sqrt{k}}$ and $b_n = \frac{1}{k\sqrt{k}}$. Then $\frac{a_n}{b_n} = \frac{\sin(\frac{1}{k})}{\frac{1}{k}} \to 1$ as $k \to \infty$. Since $\sum_k b_k$ is a convergent p-series, the series in (f) converges by the limit comparison test.]

Order of tests for nonnegative series: If you dont recognize it as a geometric or p-series, etc, I'd use the following order: divergence, limit comparison, root, comparison, integral, ratio. Use the ratio test if you have factorials, the root test if you have powers, limit comparison test if you have 'winning terms', integral if terms are decreasing.

If you are not asked for working, then many such problems may be done instantaneously if you use the following rules (memorize): $\sum_k \frac{a^k}{k!}$, $\sum_k \frac{a^k}{k^k}$, $\sum_k \frac{k!}{k^k}$ all converge rapidly for any number a. This is because the top grows much much more slowly than the bottom. The same argument applies to series like $\sum_k \frac{\ln k}{k^k}$, etc.

11.4. Absolute and conditional convergence

A series $\sum_k a_k$ is called *absolutely convergent* if $\sum_k |a_k|$ converges.

- Example. $1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}+\cdots$ is absolutely convergent, because $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ converges (by the p-series test with $p=2$).

- **Key Fact in 11.4:** Any absolutely convergent series is convergent.

- We shall see that the converse is false, a series may be convergent, but not absolutely convergent. Such a series is called *conditionally convergent*.

- (Proof of Key Fact: If $\sum_k |a_k|$ converges, then so does $\sum_k 2|a_k|$. However $0 \leq a_k + |a_k| \leq 2|a_k|$. So by the basic comparison test, $\sum_k (a_k + |a_k|)$ converges. By the 'difference rule' (3rd 'bullet' in the FACT towards the end of the notes for Section 11.1 only), $\sum_k a_k = \sum_k (a_k + |a_k|) - \sum_k |a_k|$ converges.)

- Example. Does series $1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\frac{1}{49}+\cdots$ converge or diverge?

  Solution. It converges, since as we saw in the previous example, this series is absolutely convergent. So by the Key Fact above it is convergent.

- Example. Does the series $\sum_k \frac{\sin(k\pi^2)}{k^2}$ converge or diverge?

  Solution. The series $\sum_k |\frac{\sin(k\pi^2)}{k^2}|$ converges by the comparison test, since $|\frac{\sin(k\pi^2)}{k^2}| \leq \frac{1}{k^2}$, and $\sum_k \frac{1}{k^2}$ is a convergent p-series. So the series $\sum_k \frac{\sin(k\pi^2)}{k^2}$ converges absolutely. So by the Key Fact above it is convergent.

- **The Alternating Series Test:** Suppose that $a_0 > a_1 > a_2 > \cdots$, and that $\lim_k a_k = 0$. Then $a_0 - a_1 + a_2 - a_3 + \cdots$ (which in sigma notation is $\sum_k (-1)^k a_k$) converges.

  [Proof: The $2n$th partial sum is

$$s_{2n} = a_0 - a_1 + a_2 - a_3 + \cdots - a_{2n-1} = (a_0 - a_1) + (a_2 - a_3) + \cdots + (a_{2n-2} - a_{2n-1}).$$

  Each bracketed term is nonnegative, so that $s_2, s_4, s_6, \cdots$ is an increasing sequence, so it has a limit s say. Similarly

$$s_{2n+1} = a_0 - (a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{2n-1} - a_{2n})$$

  and each bracketed term is nonnegative, so that $s_1, s_3, s_5, \cdots$ is a decreasing sequence, and so has a limit t say. But $s_{2n+1} - s_{2n} = a_{2n}$ which has limit 0 as $n \to \infty$. So $s = t$ and this is a finite number. Thus $\{s_n\}$ converges.]

- Example. Determine whether the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent, converges absolutely, converges conditionally, or diverges.

  Solution: The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent by the Alternating Series Test. But it is not absolutely convergent, because $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is the divergent harmonic series (which we met close to the start of Section 11.2). So $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is conditionally convergent. We'll see later that its sum is $\ln 2$.

- Example. Determine whether the following series are convergent, converges absolutely, converges conditionally. (a) $\sum_k \frac{(-1)^k}{\sqrt{k}}$, (b) $\sum_k \frac{(-1)^k}{2^k}$, (c) $\sum_k (-1)^k (\sqrt{k+1} - \sqrt{k})$.

  Solution: Both series are convergent by the Alternating Series Test. But (a) is not absolutely convergent, because $\sum_k \frac{1}{\sqrt{k}}$ is divergent by the p-series test of Section 11.2. So (a) is conditionally convergent. Series (b) is a convergent geometric series, and so is $\sum_k \frac{1}{2^k}$, so series (b) is absolutely convergent and hence not conditionally convergent. Series (c) is not absolutely convergent, conditionally convergent, convergent, and not divergent (this example will probably be done in the review).
- A much more difficult fact to prove is that any 'rearrangement' of an absolutely convergent series is convergent and has the same sum.

- Example. $1 - \frac{1}{9} + \frac{1}{4} - \frac{1}{36} + \frac{1}{16} - \frac{1}{81} + \frac{1}{25} - \cdots$ converges, since it is a 'rearrangement' of the absolutely convergent series considered at the beginning of this section.

- Example[1]. For any x show that $\lim_{k\to\infty} \frac{x^k}{k!} = 0.$

  Solution: Consider the series $\sum_k \frac{x^k}{k!}$. This series is absolutely convergent, since $\sum_k \frac{|x|^k}{k!}$ is convergent, as one can check using the ratio test: for if $a_k = \frac{|x|^k}{k!}$ then as $k \to \infty$

$$\frac{a_{k+1}}{a_k} = \frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = \frac{|x|}{(k+1)} \to 0 \,.$$

  Thus $\sum_k \frac{x^k}{k!}$ is convergent, so by the line after the Divergence Test, $\lim_{k\to\infty} \frac{x^k}{k!} = 0.$

- If $\sum_{k=1}^{\infty} (-1)^k a_k$ is a convergent alternating series as in the 'alternating series test' above, with sum s, and if $s_n = \sum_{k=1}^{n} (-1)^k a_k$, then $|s_n - s| \leq |a_{n+1}|$. Note that $|s_n - s|$ is the error in approximating s by the sum of the first n terms in the series.

- Example. Approximate the sum of $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k!}$ by its first six terms, and estimate the error in your approximation.

  Solution. $\sum_{k=1}^{6} (-1)^k \frac{1}{k!} \approx 0.63194$ (calculator). The error in this approximation is less than $|a_7| = \frac{1}{7!} = 0.00019$ (calculator).

- Example. Approximate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4}$ with error less than 0.001.

  Solution: We have by the last bullet that $|s_n - s| \leq |a_{n+1}| = \frac{1}{(n+1)^4}$, and this will be less than 0.001 if $\frac{1}{(n+1)^4} < 0.001$, or equivalently if $(n+1)^4 > 1000$. This will be true if $n = 5$. So an approximation to $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$ with error less than 0.001 will be $s_5 = \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k^4} = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} - \frac{1}{5^4}$, which is 0.94754 (calculator).

11.5. Taylor Series.

- We are now ready to finish the Calculus 2 syllabus with a discussion of Taylor and Power series. Up until now we have pretty much proved everything in these notes for Chapter 11 (although you are not expected to read most of these proofs). However from now on the proofs become too lengthy to include.

[1]We did this example another way towards the end of Chapter 10.

- The *nth Taylor polynomial* of a function $f(x)$ is defined to be

$$P_n(x) \quad = \quad f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n$$

which in sigma notation is $\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\,x^k$.

The *Taylor series* or *MacLaurin series* of a function $f(x)$ is defined to be

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

which in sigma notation is $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$.

- Example. Find the 7th Taylor polynomial of $\sin x$. Also, find the Taylor series of $\sin x$.

Solution: If $f(x) = \sin(x)$ then $f'(x) = \cos x$, $f''(x) = -\sin(x), f'''(x) = -\cos x$, and then it starts repeating, $f^{(4)}(x) = f(x) = \sin(x), f^{(5)}(x) = f'(x) = \cos x$, and so on. Thus we have $f(0) = 0 = f^{(4)}(0)$, $f'(0) = 1 = f^{(5)}(0)$, $f''(0) = 0 = f^{(6)}(0)$, $f'''(0) = -1 = f^{(7)}(0)$, and so on. Thus

$$P_7(x) \quad = \quad 0 + 1 \cdot x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 + \frac{-1}{7!}x^7 \quad = \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

and the Taylor series of $\sin x$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad .$$

- Example. Find the Taylor polynomial $P_4(x)$ for e^x. Also, find the Taylor series of e^x.

Solution: If $f(x) = e^x$ then $e^x = f'(x) = f''(x) = f'''(x) = \cdots$. Thus we have $1 = f(0) = f'(0) = f''(0) = f'''(0) = f^{(4)}(0)$, and so on. Thus

$$P_4(x) \quad = \quad 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \quad = \quad 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

The Taylor series of e^x is

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

- Example. Find the nth Taylor polynomial, and the Taylor series, of $\ln(1 - x)$.

Solution: If $f(x) = \ln(1 - x)$ then $f'(x) = -(1 - x)^{-1}$, and

$$f''(x) = -(-1) \cdot (1 - x)^{-2} \cdot (-1) = -(1 - x)^{-2}$$

Similarly, $f'''(x) = -(-2)(1 - x)^{-3} \cdot (-1) = -2(1 - x)^{-3}$, and

$$f^{(4)}(x) = -(-3) \cdot 2(1 - x)^{-4} \cdot (-1) = -3 \cdot 2(1 - x)^{-4}$$

and so on. In general, the pattern is $f^{(k)}(x) = -\frac{(k-1)!}{(1-x)^k}$. Also $f(0) = \ln(1) = 0, f'(0) = -1, f''(0) = -1, f'''(0) = -2, f^{(4)}(0) = -3 \cdot 2$, and in general $f^{(k)}(0) = -(k - 1)!$. Therefore

$$\frac{f^{(k)}(0)}{k!} = -\frac{(k-1)!}{k!} = -\frac{1}{k}$$

for $k = 1, 2, 3, \cdots$. So the nth Taylor polynomial of $\ln(1 - x)$ is

$$P_n(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n}.$$

The Taylor series of $\ln(1-x)$ is $-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ In sigma notation this is $-\sum_{k=1}^{\infty} \frac{x^k}{k}$.

- We define the *Taylor remainder* to be

$$R_n(x) = f(x) - P_n(x).$$

This is the error in approximating $f(x)$ by $P_n(x)$. Note

$$f(x) = P_n(x) + R_n(x).$$

- Example. Find the Taylor remainder $R_5(x)$ for $f(x) = \sin x$.

Solution: $R_5(x) = \sin x - P_5(x) = \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}$, by the example above.

- **Taylor's Theorem.** If $f^{(n+1)}$ is continuous on an open interval I containing 0, then for every $x \in I$ we have

$$f(x) \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \;+\; R_n(x)$$

where $R_n(x) = \frac{1}{n!}\int_0^x \; f^{(n+1)}(t)(x-t)^n\; dt$. There exists a number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\, x^{n+1}.$$

This is the *Lagrange formula*, or the *Lagrange form of the remainder*. Thus if $|f^{(n+1)}(t)| \leq M$ for all t between 0 and x then

$$|R_n(x)| \;\leq\; \frac{M|x|^{n+1}}{(n+1)!}.$$

We can take M to be the maximum of $|f^{(n+1)}(t)|$ on the interval $[0, x]$.

- Note that the Mean Value Theorem from Calculus I is the case $n = 0$ of Taylor's Theorem.
- Example. Assume that f is a function such that $|f^{(4)}(x)| \leq 2$ for all x. Find the maximal possible error if $P_3(\frac{1}{2})$ is used to approximate $f(\frac{1}{2})$.

Solution: We may take $M = 2$. The error in using $P_3(\frac{1}{2})$ to approximate $f(\frac{1}{2})$ is $R_3(\frac{1}{2})$, and have

$$|R_3(\frac{1}{2})| \leq \frac{M|\frac{1}{2}|^4}{4!} \leq \frac{1}{2^3 \cdot 4!} = \frac{1}{192} = 0.0052.$$

So the maximal possible error is less than 0.0052.

- Example. Assume that f is a function such that $|f^{(4)}(x)| \leq 2$ for all x. Find a small n such that the maximal possible error is smaller than 0.001, if $P_3(\frac{1}{2})$ is used to approximate $f(\frac{1}{2})$.

Solution: Just as in the previous example, $M = 2$. The error in using $P_n(\frac{1}{2})$ to approximate $f(\frac{1}{2})$ is $R_n(\frac{1}{2})$, and have

$$|R_n(\frac{1}{2})| \leq \frac{M|\frac{1}{2}|^{n+1}}{(n+1)!} = \frac{1}{2^n(n+1)!}$$

If we want the error smaller than 0.001 we can solve $\frac{1}{2^n \cdot (n+1)!} < 0.001$. That is, $1000 < 2^n \cdot (n+1)!$. Clearly $n = 4$ will do that job: $1000 < 2^4 5! = 16 \cdot 120$. So if $n = 4$ then the maximal possible error (if $P_3(\frac{1}{2})$ is used to approximate $f(\frac{1}{2})$) is smaller than 0.001.

- Example. Find the Lagrange form of the Taylor remainder R_n for the function $f(x) = e^{2x}$ and $n = 3$.

  Solution: We have $f'(x) = 2e^{2x}, f''(x) = 2^2e^{2x}, f'''(x) = 2^3e^{2x}, f^4(x) = 2^4e^{2x}$. So the Lagrange form is

$$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{2^4e^{2c}}{4!}x^4 = \frac{2e^{2c}x^4}{3}.$$

- KEY POINT: Up until now, we have not considered at all the question of whether the Taylor series of $f(x)$ converges. Clearly it does converge when $x = 0$ to $f(0)$ though. The partial sums of the Taylor series of $f(x)$ are just the $P_n(x)$, so it follows that the Taylor series converges to $f(x)$ if and only if $P_n(x) \to f(x)$, or equivalently, if and only if $\lim_{n\to\infty} R_n(x) = 0$.

  To check if $\lim_{n\to\infty} R_n(x) = 0$, the last fact in Taylor's theorem is very useful, as we shall now see in many examples:

- Example 1. Show that the series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ converges to $\sin x$ for every number x. Thus for every number x,

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

  Solution: The series in this example is the Taylor series of $f(x) = \sin x$, as we saw on the previous page. To show that the series converges to $\sin x$ for all x, by the KEY POINT above, we need to show that $\lim_{n\to\infty} R_n(x) = 0$. Note that $|f^{(n+1)}(x)| = |\sin x|$ or $|\cos x|$, so that $|f^{(n+1)}(x)| \le 1$ for every number x. So we can take $M = 1$ in the last part of Taylors theorem, and

$$|R_n(x)| \;\le\; \frac{|x|^{n+1}}{(n+1)!}$$

  But the right hand side here converges to 0 as we saw close to the top of page 9. So by the squeezing or pinching rule $\lim_{n\to\infty} R_n(x) = 0$.

- Example 2. An exactly similar argument shows that for every x,

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \;\cdots\; = \;\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- Example 3. Using the example a couple of pages back of the Taylor series of $\ln(1-x)$, show
  (a) that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ has sum equal to $\ln 2$.
  (b) Approximate $\ln 2$ with error less than 0.05, using a 'Taylor approximation'.
  (c) Find a value of n so that $P_n(x)$ approximates $\ln 2$ with error less than 0.001.

  Solution: (a): Let $f(x) = \ln(1-x)$. We saw in an example a couple of pages back that the Taylor series of $\ln(1-x)$ is $-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$. Putting $x = -1$ we see that the Taylor series of $\ln(1-x)$ is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ when $x = -1$. And $f(-1) = \ln 2$. But this is not a proof that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$ yet. What we need to show is that the Taylor series converges to $f(x)$ when $x = -1$, but by the KEY POINT above, this is the same as showing that $R_n(-1) \to 0$ as $n \to \infty$. So let us show the latter.

We also saw in that example a couple of pages back that $f^{(n+1)}(t) = -\frac{n!}{(1-t)^{(n+1)}}$, so if $-1 \leq t \leq 0$ then $1 - t \geq 1$, so that $|1-t|^{n+1} \geq 1$, so that we have

$$|f^{(n+1)}(t)| = \frac{n!}{|1-t|^{n+1}} \leq n!.$$

So we can take $M = n!$ and $x = -1$ in the last line of Taylors theorem and get

$$|R_n(-1)| \leq \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$$

as $n \to \infty$. Thus by the KEY POINT above, the Taylor series of $\ln(1-x)$ when $x = -1$, which is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, converges to $f(-1) = \ln(2)$.

To get (b), we want the Taylor remainder $|R_n(-1)|$, which is the error in approximating $f(-1)$ by $P_n(-1)$, to be less that 0.05. We just saw that $|R_n(-1)| \leq \frac{1}{n+1}$. So if $\frac{1}{n+1} < 0.05$ we will be done. But $\frac{1}{n+1} < 0.05$ if $n \geq 20$. Thus $P_{20}(-1)$, which equals $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{20}$, is a Taylor approximation to $\ln 2$ with error less than 0.05.

Item (d) is just like (c), we want $|R_n(-1)| < 0.001$. In (a) we saw that $|R_n(-1)| \leq \frac{1}{n+1}$. So if $\frac{1}{n+1} < 0.001$ we will be done. But $\frac{1}{n+1} < 0.001$ if $n + 1 > 1000$. So choose $n = 1000$.

- Example. Estimate the error if $P_4(\frac{\pi}{10})$ is used to approximate $\sin(\frac{\pi}{10})$. If you do not have a calculator do not simplify your estimate too much, but it should be a number (that is, it should have no variables in it).

  Solution: We want to estimate $R_4(\frac{\pi}{10})$, when $f(x) = \sin x$. Since $|f^{(5)}| = |\cos x| \leq 1$, in the estimate for $R_n(x)$ in the last line of Taylor's Theorem, we can take $M = 1$, and so that estimate becomes:

  $$|R_4(\frac{\pi}{10})| \leq \frac{1}{5!}(\frac{\pi}{10})^5 = \frac{\pi^5}{120 \cdot 10^5}.$$

  Final answer: $\frac{\pi^5}{120 \cdot 10^5}$.

- Example. Let P_n be the nth Taylor Polynomial of the function $f(x)$. Assume that f is a function such that $|f^{(n)}(x)| \leq 1$ for all n and x (the sine and cosine functions have this property.) Find the least integer n for which $P_n(0.5)$ approximates $f(0.5)$ to within 0.001.

  Solution: In the estimate for $R_n(x)$ in the last line of Taylor's Theorem, we can take $M = 1$, and so that estimate becomes:

  $$|R_n(0.5)| \leq \frac{1}{(n+1)!}(0.5)^{n+1} = \frac{1}{2^{n+1}(n+1)!},$$

  and this is < 0.001 if $2^{n+1}(n+1)! > 1000$. This is true if $n = 4$ (since $2^5 5! = 3840$). 'Answer': $n = 4$.

11.6. The Taylor series of a function $f(x)$ *about a number* a, is the series

$$f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^3 + \cdots$$

which in sigma notation is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$.

- The *nth Taylor polynomial* of a function $f(x)$ *about a number* a, is defined to be

$$P_n(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \frac{f'''(a)}{3!}\,(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n$$

and the Taylor remainder $R_n(x) = f(x) - P_n(x)$.

- Example. Suppose that g is a function which has continuous derivatives, and that $g(2) = 3, g'(2) = -4, g''(2) = 7, g'''(2) = -5$. Find the Taylor polynomial of degree 3 for g centered at $x = 2$.

  Solution: $P_3(x) = 3 - 4(x-2) + \frac{7}{2}(x-2)^2 - \frac{5}{6}(x-2)^3$.

- Example. Find the Taylor series of $f(x) = x^2 + 3x - 1$ about $x = 1$.

  Solution: $f(1) = 3, f'(x) = 2x + 3, f'(1) = 5, f''(x) = 2, f'''(x) = 0$, and so on. So the Taylor series about $x = 1$ is

$$f(1) + f'(1)\,(x-1) + \frac{f''(1)}{2!}\,(x-1)^2 + 0 + 0 + \cdots \quad = \quad 3 + 5(x-1) + (x-1)^2.$$

- Example. Expand $f(x) = x^2 + 3x - 1$ in powers of $(x-1)$.

  Solution: This is just another way to ask for the Taylor series about $x = 1$. So the answer is $3 + 5(x-1) + (x-1)^2$.

- Example. Find the Taylor series of $\ln x$ about $x = 1$. Also find the nth Taylor polynomial of $\ln x$ about $x = 1$.

  Solution: Let $f(x) = \ln x$. Then $f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = +2x^{-3}, f^{(4)}(x) = -3{\cdot}2x^{-4}$. In general $f^{(k)}(x) = (-1)^{k-1}{\cdot}(k-1)!x^{-k}$. Thus $f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = 2, f^{(4)}(1) = -3 \cdot 2$, and in general $f^{(k)}(1) = (-1)^{k-1} \cdot (k-1)!$. Thus $\frac{f^{(k)}(1)}{k!} = (-1)^{k-1}\frac{1}{k}$. So the Taylor series about $x = 1$ is

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

  The nth Taylor polynomial about $x = 1$ is

$$P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + (-1)^{n-1}\frac{1}{n}x^n.$$

- Just as in 11.5 we have **Taylor's Theorem:** If $f^{(n+1)}$ is continuous on an open interval I containing a, then for every $x \in I$ we have

$$f(x) \;=\; f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \;+\; R_n(x)$$

  and $R_n(x) = \frac{1}{n!}\int_a^x \; f^{(n+1)}(t)(x-t)^n\,dt$, and indeed

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

  for some number c between a and x. This is the *Lagrange formula*, or the *Lagrange form of the remainder*. Thus if $|f^{(n+1)}(t)| \leq M$ for all t between a and x then

$$|R_n(x)| \;\leq\; \frac{M|x-a|^{n+1}}{(n+1)!}.$$

  Again we can take M to be the maximum of $|f^{(n+1)}(t)|$ for t between a and x.

- Example. Using the last example, approximate $\ln(1.1)$ with error less than 0.001.

  Solution: Let $a = 1, x = 1.1$ in Taylor's Theorem above. In the previous example we saw that $f^{(n+1)}(t) = (-1)^n \cdot n! t^{-n-1}$, so that $|f^{(n+1)}(t)| = n! t^{-n-1} \leq n!$, if $1 \leq t \leq 1.1$. From the last line of Taylor's Theorem,

$$|R_n(1.1)| \leq \frac{n!\,(0.1)^{n+1}}{(n+1)!} = \frac{(0.1)^{n+1}}{n+1}.$$

  Notice if $n = 2$ then $|R_2(1.1)| \leq \frac{0.001}{3} < 0.001$. So the approximation we want is the 2nd Taylor polynomial, which by what we did in the previous example will be $P_2(x) = (x-1) - \frac{(x-1)^2}{2}$. Thus $\ln(1.1)$ is approximately $P_2(1.1) = (0.1) - \frac{(0.1)^2}{2} = 0.1 - 0.005 = 0.095$.

- Example. Find the Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt{x}$ about $x = 4$. Also find the Lagrange form of the remainder $R_2(x)$ for the function $f(x) = \sqrt{x}$ about $x = 4$.

  Solution: We have $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, and $f''(x) = -\frac{1}{2}\frac{1}{2}x^{-\frac{3}{2}}$, and $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$. So $f(4) = 2, f'(4) = \frac{1}{2}4^{-\frac{1}{2}} = \frac{1}{4}, f''(4) = -\frac{1}{4}4^{-\frac{3}{2}} = -\frac{1}{32}, f'''(4) = \frac{3}{8}4^{-\frac{5}{2}} = \frac{3}{8\cdot 32} = \frac{3}{256}$. So $P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$.

  The Lagrange form of the remainder is

$$R_2(x) = \frac{f^{(3)}(c)}{3!}(x-4)^3 = \frac{\frac{3}{8}c^{-\frac{5}{2}}}{6}(x-4)^3 = \frac{c^{\frac{5}{2}}}{16}(x-4)^3.$$

  Here c is a number between 4 and x.

11.7. Power Series.

- A *power series* is a series of the form $\sum_{k=0}^{\infty} c_k\, x^k$, or in longhand,

$$c_0 + c_1\, x + c_2\, x^2 + c_3\, x^3 + \cdots$$

  where $c_0, c_1, c_2, \cdots$ are constants.

- Example 1. $1 + x + x^2 + x^3 + \cdots$.
- Example 2. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.
- Example 3. $\sum_{k=1}^{\infty} \frac{x^k}{k}$.

- KEY RESULT IN 11.7: For any power series $\sum_{k=0}^{\infty} c_k\, x^k$, there is a constant R, called the *radius of convergence* of the power series. We have $0 \leq R \leq \infty$ and
  (a) If $R = 0$ then the power series only converges when $x = 0$.
  (b) If $R = \infty$ then the power series converges for every number x.
  (c) If $0 < R < \infty$ then the power series converges if $-R < x < R$, and it diverges if $|x| > R$.

  Thus the set of numbers x for which the power series converges is an interval centered at the origin on the number line. This interval is called the *interval of convergence*. [Picture drawn in class].

- Examples. Find the radius of convergence and the interval of convergence for each of Examples 1, 2, 3 above.

  Solutions: Example 1 above is a geometric series, and from the geometric series test, we know it converges only for $x \in (-1, 1)$. This is its interval of convergence. Its radius of convergence is therefore 1.

  Example 2: we said in 11.5 that this series converges for all x to $\cos x$. So its interval of convergence is $(-\infty, \infty)$, and its radius of convergence is therefore ∞.

  Put $x = 1$ in Example 3, and one gets the divergent harmonic series (see Section 11.2). Put $x = -1$ in Example 3, and one gets a convergent alternating series (this was a major example in Section 11.4). Thus the interval of convergence must be $[-1, 1)$, and the radius of convergence is 1.

- **A formula for the radius of convergence R:**

$$R = \lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|}$$

  if this limit exists.

- Example 4. Find the radius of convergence and the interval of convergence for the power series $\sum_{k=1}^{\infty} k\, x^k$.

  Solution: The radius of convergence of the power series $\sum_{k=1}^{\infty} k\, x^k$ is 1 since

$$\lim_{k\to\infty} \frac{k}{k+1} \quad = \quad \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} \quad = \quad 1.$$

  Since this series diverges when $x = 1$ or $x = -1$, but the radius of convergence is 1, the interval of convergence must be $(-1, 1)$.

- If the limit in the 'formula for R' above does not exist, try

$$R \;=\; \lim_{k\to\infty} \frac{1}{|c_k|^{\frac{1}{k}}}$$

  if this limit exists.
  [These two 'formulae for R' are easily proved using the 'ratio test' and 'root test'.]

- **The sum function of a power series:** Suppose that $\sum_{k=0}^{\infty} c_k\, x^k$ is a power series, and suppose that I is its interval of convergence. For x in I, define $f(x)$ to be the sum of the series; that is $f(x) = \sum_{k=0}^{\infty} c_k\, x^k$ (in the sense of Meaning # 2 of the sum of a series (see 2nd page of these typed notes on Chapter 11)). Then $f : I \to (-\infty, \infty)$ is a function defined on the interval of convergence. We call f the *sum function*. It is actually a continuous function on I, and differentiable in the interior of I as we shall see in the next section.

- Examples. In Example 1 above, the sum function is, by the geometric series formula, $f(x) = \frac{1}{1-x}$. In Example 2, the sum function is $\cos x$.

In Example 3, the sum function $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$ is defined on the interval of convergence. On the last page we saw that the interval of convergence is $[-1,1)$. So $\sum_{k=1}^{\infty} \frac{x^k}{k}$ is a continuous function on $[-1,1)$, and it is differentiable on $(-1,1)$.

In Example 4, the sum function $f(x) = \sum_{k=1}^{\infty} k\, x^k$ is defined on the interval of convergence. What is this interval? Since this series diverges when $x = 1$ or $x = -1$, but the radius of convergence is 1, the interval of convergence must be $(-1,1)$. So $\sum_{k=1}^{\infty} k\, x^k$ is a continuous function on $(-1,1)$.

- The results in this section above have variants for power series $\sum_{k=0}^{\infty} c_k\, (x-a)^k$ about a number a. One simply replaces 0 by a, and x by $x - a$ in places, etc. The radius of convergence R still has the formula $R = \lim_{k\to\infty} \frac{1}{|c_k|^{\frac{1}{k}}}$ if this limit exists (or $\lim_{k\to\infty} \frac{|c_k|}{|c_{k+1}|}$). The interval of convergence of this power series about $x = a$ will be either $[a-R, a+R]$, $(a-R, a+R]$, $[a-R, a+R)$, or $(a-R, a+R)$ [Picture drawn in class]. Outside of the interval of convergence it diverges, but inside its sum function is continuous.

- Example 4. Find the radius of convergence and the interval of convergence for the power series $\sum_{k=1}^{\infty} k\, (x-1)^k$.

  Solution: The radius of convergence of the power series $\sum_{k=1}^{\infty} k\, (x-1)^k$ is 1 since

$$\lim_{k\to\infty} \frac{k}{k+1} \quad = \quad \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} \quad = \quad 1.$$

  Since this series diverges when $x = 0$ or $x = 2$, but the radius of convergence is 1, the interval of convergence must be $(0,2)$.

11.8. Differentiation and integration of power series.

- **The differentiated power series** of a power series $c_0 + c_1\, x + c_2\, x^2 + c_3\, x^3 + c_4\, x^4 + \cdots$, is the power series

$$c_1 + 2c_2\, x + 3c_3\, x^2 + 4c_4\, x^3 + \cdots$$

- Example. The differentiated power series of the power series $1 + x + x^2 + x^3 + \cdots$ is the power series $1 + 2x + 3x^2 + 4x^3 + \cdots$.

- KEY RESULTS ON DIFFERENTIATING POWER SERIES: Suppose that a power series $\sum_{k=0}^{\infty} c_k\, x^k$ has a radius of convergence $R > 0$. Then:

  A) the differentiated power series has the same radius of convergence R.

  Let $f(x)$ be the sum function of the power series (defined on the previous page), and let $g(x)$ be the sum function of the differentiated power series.

  B) $f(x)$ is differentiable on $(-R, R)$, and $f'(x) = g(x)$ for all x in $(-R, R)$. That is, if $-R < x < R$ then

$$f'(x) \quad = \quad \sum_{k=1}^{\infty} kc_k\, x^{k-1}.$$

C) We can iterate this process, and look at the 'second differentiated power series' (i.e, the differentiated power series of differentiated power series) $\sum_{k=2}^{\infty} k(k-1)c_k\ x^{k-2}$. By A) and B), this has the same radius of convergence R, and its sum function equals $f''(x)$ on $(-R, R)$. Thus if $-R < x < R$ then

$$f''(x) \quad = \quad \sum_{k=2}^{\infty} k(k-1)c_k\ x^{k-2}.$$

Similarly for $f'''(x), f^{(4)}(x)$, and so on.

D) Putting $x = 0$ in $f(x), f'(x), f''(x), f'''(x)$ and so on, we find from B) and C) that $f(0) = c_0, f'(0) = c_1, f''(0) = 2c_2$, and more generally, $f^{(k)}(0) = k!c_k$, or $c_k = \frac{f^{(k)}(0)}{k!}$.

E) Because of D), the Taylor series of $f(x)$ is our original power series $\sum_{k=0}^{\infty} c_k\ x^k$.

- Part E is saying that the sum function $f(x)$ of a power series on its interval of convergence, has Taylor series equal to the original power series.

- Example: Find a simple formula for the sum of the series $x + 2\,x^2 + 3\,x^3 + 4\,x^4 + \cdots$.

  Solution: The geometric series $1 + x + x^2 + \cdots$ converges to $\frac{1}{1-x}$ for $x \in (-1, 1)$. Differentiating this, we have by B) above that $1+2x+3x^2+\cdots$ converges to $\frac{d}{dx}(\frac{1}{1-x}) = (1-x)^{-2}$, for all $x \in (-1, 1)$. Multiplying by x shows that $x+2\,x^2+3\,x^3+4\,x^4+\cdots$ converges to $x\,(1-x)^{-2}$ for $x \in (-1, 1)$. This is the desired sum function.

- **Equality of power series.** Suppose that $\sum_{k=0}^{\infty} a_k\ x^k$ and $\sum_{k=0}^{\infty} b_k\ x^k$ are two power series whose sums are equal (i.e. $\sum_{k=0}^{\infty} a_k\ x^k = \sum_{k=0}^{\infty} b_k\ x^k$) for all x in an open interval containing 0. Then $a_0 = b_0,\ a_1 = b_1,\ \cdots, a_k = b_k$ for every k.

  [Proof: Let $f(x) = \sum_{k=0}^{\infty} a_k\ x^k = \sum_{k=0}^{\infty} b_k\ x^k$ for all x in this interval. By D), we have $a_k = \frac{f^{(k)}(0)}{k!} = b_k$ for every k.]

- **The integrated power series** of of a power series $c_0+c_1\ x+c_2\ x^2+c_3\ x^3+c_4\ x^4+\cdots$, is the power series

$$c_0\ x + c_1\ \frac{x^2}{2} + c_2\ \frac{x^3}{3} + c_3\ \frac{x^4}{4} + \cdots.$$

- RESULT ON INTEGRATING POWER SERIES: Suppose that $\sum_{k=0}^{\infty} c_k\ x^k$ is a power series with radius of convergence $R > 0$. Then the integrated power series has the same radius of convergence R. If $f(x)$ is the sum function of $\sum_{k=0}^{\infty} c_k\ x^k$, and if $F(x)$ is the sum function of the integrated power series, then $\int_0^x\ f(t)\ dt = F(x)$, for $|x| < R$.

- Example 1. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for $-1 < x < 1$.

  Solution: Consider the series $1 - x^2 + x^4 - x^6 + \cdots$. This is a geometric series which converges for $-1 < x < 1$, with sum $\frac{1}{1+x^2}$. By the last bullet,

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad = \quad \int_0^x \frac{dt}{1+t^2} \quad = \quad \tan^{-1} x - \tan^{-1} 0 = \tan^{-1} x$$

- Example 2. Show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for $-1 \le x \le 1$.

  Solution: Note that the power series here converges when $x = 1$ and when $x = -1$, by the Alternating Series Test (from Section 11.4). So the sum function $f(x)$ of this power series is continuous on $[-1, 1]$ by the second last bullet or so in Section 11.7 above. Hence

$$f(1) = \lim_{x \to 1^-} f(x) \;=\; \lim_{x \to 1^-} \tan^{-1} x \;=\; \tan^{-1} 1.$$

  That is, $\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. Similarly $f(-1) = \tan^{-1}(-1)$.

- Example 3. Show that $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$.

  Solution: By the previous example $\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

- Example. Note $\cosh x = \frac{1}{2}(e^x + e^{-x})$. But

$$e^x + e^{-x} = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) + (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots) = 2(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots).$$

  Thus $\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$. Similarly,

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

- Example. Expand $x^2 \cos x^3$ in powers of x.

  Solution: We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

  Thus

$$\cos x^3 = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \cdots = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots .$$

  Thus

$$x^2 \cos x^3 = x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!} + \cdots .$$

- Read Abel's theorem, bottom of page 693 in text.

- Similar results about differentiation and integration hold for power series about a point $x = a$. For example: If $c_0 + c_1\,(x-a) + c_2\,(x-a)^2 + c_3\,(x-a)^3 + c_4\,(x-a)^4 + \cdots$ is a power series with radius of convergence $R > 0$, then the 'differentiated power series' is the power series

$$c_1 + 2c_2\,(x-a) + 3c_3\,(x-a)^2 + 4c_4\,(x-a)^3 + \cdots ,$$

  and this has the same radius of convergence R. If $f(x)$ is the sum function of the power series, and if $g(x)$ be the sum function of the differentiated power series, then

$f(x)$ is differentiable on $(a-R, a+R)$, and $f'(x) = g(x)$ for all x in $(a-R, a+R)$. That is, if $a - R < x < a + R$ then

$$f'(x) \quad = \quad \sum_{k=1}^{\infty} kc_k \, (x-a)^{k-1}.$$

The integrated power series is the power series

$$c_0 \, (x-a) + c_1 \, \frac{(x-a)^2}{2} + c_2 \, \frac{(x-a)^3}{3} + c_3 \, \frac{(x-a)^4}{4} + \cdots .$$

The integrated power series has the same radius of convergence R. If $f(x)$ is the sum function of $\sum_{k=0}^{\infty} c_k \, x^k$, and if $F(x)$ is the sum function of the integrated power series, then $\int_0^x \, f(t) \, dt = F(x)$, for $|x - a| < R$.

END OF COURSE