Exam 3 Review. If you find any errors, please email me. pamb@math.uh.edu Thanks!

1. a. $\int \frac{x+2}{x+1}dx = \int \frac{x+1+1}{x+1}dx = \int\left(1+\frac{1}{x+1}\right)dx$

$= x + \ln|x+1| + C$

b. $\int \frac{3x^2+3x+3}{x^2+1}dx = \int \frac{3(x^2+1)+3x}{x^2+1}dx$

$= \int\left(3+\frac{3x}{x^2+1}\right)dx$

$= 3x + \frac{3}{2}\ln|x^2+1| + C$

c. $\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}dx = \int_0^{\pi/3} \frac{\cos\theta \, d\theta}{\cos\theta} = \int_0^{\pi/3} d\theta$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta}$

$= \cos\theta$

$\frac{\sqrt{3}}{2} = \sin\theta \quad \theta = \pi/3$

$0 = \sin\theta \quad \theta = 0$

$= \theta \Big|_0^{\pi/3}$

$= \pi/3$

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d. $\int \frac{x^2}{(x+1)(x-1)^2}\,dx$ $= \int \left(\frac{1/4}{x+1} + \frac{3/4}{x-1} + \frac{1/2}{(x-1)^2} \right) dx$

$= \frac{1}{4}\ln|x+1| + \frac{3}{4}\ln|x-1| - \frac{1}{2(x-1)} + C$

$$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$x^2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

$x=1$ $\quad 1 = 2C \Rightarrow C = 1/2$

$x=-1$ $\quad 1 = 4A \Rightarrow A = 1/4$

$x=0$ $\quad 0 = A - B + C$

$0 = 1/4 - B + 1/2 \Rightarrow B = 3/4$

$\frac{1}{2}\int (x-1)^{-2}\,dx\,c$

$\frac{(x-1)^{-1}}{-1} + C$

e. $\int \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx$ $= \int \left(\frac{-1}{x+1} + \frac{x+3}{x^2+1} \right) dx$

$= \int \left(\frac{-1}{x+1} + \frac{2x}{x^2+1} + \frac{3}{x^2+1} \right) dx$

$$\frac{x^2+5x+2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$= -\ln|x+1| + \ln|x^2+1| + 3\arctan x + C$

$$x^2+5x+2 = A(x^2+1) + (Bx+C)(x+1)$$

$x=-1$ $\quad -2 = 2A \Rightarrow A = -1$

x^2 terms $\quad 1 = A + B \Rightarrow B = 2$

$x=0$ $\quad 2 = A + C \Rightarrow C = 3$

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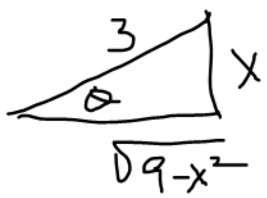
f. $\int \frac{2x^2}{\sqrt{9-x^2}}\,dx = \int \frac{2\cdot 9\sin^2\theta}{\cancel{3\cos\theta}} \cdot \cancel{3\cos\theta}\,d\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta}$

$= 3\cos\theta$



$= 18\int \sin^2\theta\,d\theta$

$= \frac{18}{2}\int (1-\cos 2\theta)\,d\theta$

$= 9\left(\theta - \frac{\sin 2\theta}{2}\right) + C$

$= 9\left(\arcsin\frac{x}{3} - \frac{1}{2}\cdot 2\cdot \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$

$= 9\arcsin\frac{x}{3} - x\sqrt{9-x^2} + C$

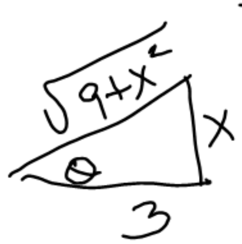
g. $\int \frac{2}{x\sqrt{9+x^2}}\,dx = \int \frac{2}{3\tan\theta\cdot \cancel{3}\cancel{\sec\theta}} \cdot \cancel{3}\sec^{\cancel{2}}\theta\,d\theta$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\,d\theta$

$\sqrt{9+x^2} = \sqrt{9+9\tan^2\theta}$

$= 3\sec\theta$



$= \frac{2}{3}\int \frac{\cos\theta}{\sin\theta\cos\theta}\,d\theta$

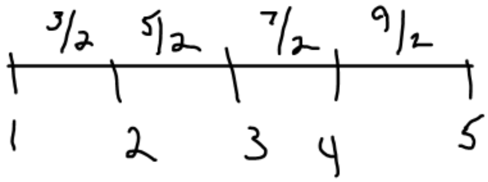
$= \frac{2}{3}\int \csc\theta\,d\theta$

$= \frac{2}{3}\ln\left|\csc\theta - \cot\theta\right| + C$

$= \frac{2}{3}\ln\left|\frac{\sqrt{9+x^2}}{x} - \frac{3}{x}\right| + C$

2. Given $\int_1^5 (2x+1)\,dx$

Use the trapezoid method with $n = 4$ to approximate the integral.
Use the midpoint method with $n = 4$ to approximate the integral.
Use Simpson's rule with $n = 4$ to approximate the integral.



$$T_4 = \frac{1}{2}(1)\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right]$$

$$= \frac{1}{2}\left[3 + 2(5) + 2(7) + 2(9) + 11\right]$$

$2x+1$

$$= \frac{1}{2}(56) = 28$$

$$M_4 = 1\left[f(3/2) + f(5/2) + f(7/2) + f(9/2)\right]$$

$$= 4 + 6 + 8 + 10$$

$$= 28$$

$$S_4 = \frac{1}{3}(28) + \frac{2}{3}(28)$$

$$= 28$$

3. Determine the values of *n* which guarantee a theoretical error less than ε if the integral is estimated by the trapezoidal rule if ε = 0.01.

$$\int_1^3 \left(\frac{1}{4}x^2 + 3x - 2\right) dx$$

$$|E| \leq \frac{(b-a)^3}{12n^2} f''(c) < \frac{1}{100}$$

$$f(x) = \frac{1}{4}x^2 + 3x - 2$$

$$f'(x) = \frac{1}{2}x + 3$$

$$f''(x) = \frac{1}{2} = m$$

$$\frac{(3-1)^3}{12n^2} \cdot \left(\frac{1}{2}\right) < \frac{1}{100}$$

$$\frac{8}{24n^2} < \frac{1}{100}$$

$$\frac{800}{24} < n^2$$

$$\frac{100}{3} < n^2$$

$$33\tfrac{1}{3} < n^2$$

$$5.\text{---} < n \quad \therefore \quad n \geq 6 \text{ works}$$

5. Write the equation in polar coordinates:

a. $x^2 + y^2 = 4$

b. $x^2 + y^2 = 4x$

c. $(x^2 + y^2)^2 = 4xy$

d. $x = 4y$

a) $x^2+y^2=4$

$r^2=4$

$r=2$

b) $x^2+y^2=4x$

$r^2=4r\cos\theta$

$r=4\cos\theta$

c) $(x^2+y^2)^2=4xy$

$(r^2)^2=4r\cos\theta\, r\sin\theta$

$r^2=4\cos\theta\sin\theta$

d) $x=4y$

$r\cos\theta=4r\sin\theta$

$\cos\theta=4\sin\theta$

$\frac{1}{4}=\tan\theta$

OR $4=\cot\theta$

6. Write the given equations in rectangular coordinates:

a. $r = -2\sin\theta$

b. $r\cos\theta = 5$

a) $r=-2\sin\theta$

$r^2=-2r\sin\theta$

$x^2+y^2=-2y$

b) $r\cos\theta=5$

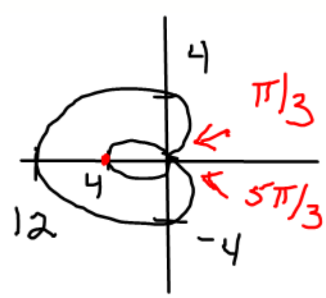
$x=5$

7. Given $r = 4 - 8\cos\theta$, give the formula (only) for the area inside the inner loop.

$4-8\cos\theta=0$

$\cos\theta=\frac{1}{2}$

$\theta=\pi/3,\ 5\pi/3$



$A=2\cdot\frac{1}{2}\int_0^{\pi/3}(4-8\cos\theta)^2\,d\theta$

(there are other integrals that give the same answer)

8. Given $r = 2\sin(3\theta)$, give the formula (only) for the area of one petal.

3 petals; traced in π

$$A = \frac{1}{3} \cdot \frac{1}{2} \int_0^{\pi} (2\sin 3\theta)^2 \, d\theta$$

OR

$3\theta = 0, \pi, 2\pi, \ldots$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$

$$A = \frac{1}{2} \int_0^{\pi/3} (2\sin 3\theta)^2 \, d\theta$$

9. Find the arc length for the following:

a. $f(x) = \frac{2}{3}(x-1)^{3/2} \quad x \in [1,2]$

b. $x(t) = \sin(2t),\ y(t) = \cos(2t), \quad t \in \left[0, \frac{\pi}{2}\right]$

c. $r = 2\sec(\theta), \quad t \in \left[0, \frac{\pi}{4}\right]$

a) $f'(x) = (x-1)^{1/2}$ $\qquad L = \int_a^b \sqrt{1 + [f'(x)]^2}$

$$L = \int_1^2 \sqrt{1 + ((x-1)^{1/2})^2} \, dx$$

$$= \int_1^2 \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \Big|_1^2 = \frac{2}{3}(2\sqrt{2} - 1)$$

b) $L = \int_0^{\pi/2} \sqrt{(2\cos 2t)^2 + (-2\sin 2t)^2} \, dt$ $\qquad L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$

$$= \int_0^{\pi/2} \sqrt{4\cos^2 2t + 4\sin^2 2t} \, dt = \int_0^{\pi/2} 2 \, dt = 2t \Big|_0^{\pi/2} = \pi$$

c) $L = \int_0^{\pi/4} \sqrt{(2\sec\theta)^2 + (2\sec\theta\tan\theta)^2} \, d\theta$ $\qquad L = \int_a^b \sqrt{r^2 + (r')^2} \, d\theta$

$$= \int_0^{\pi/4} \sqrt{4\sec^2\theta(1 + \tan^2\theta)} \, d\theta = \int_0^{\pi/4} 2\sec^2\theta \, d\theta = 2\tan\theta \Big|_0^{\pi/4}$$

$$= 2$$

10. Find the equation of the tangent and the normal lines to the parametric curves at the given points:

a. $x(t) = -2\cos 2t,\ y(t) = 4 + 2t,\ (-2,4)$

b. $x(t) = 3\cos(3t) + 2t,\ y(t) = 1 + 5t,\ (3,1)$

a) $\frac{dy}{dx} = \frac{2}{4\sin 2t}\Big|_{t=0} = \frac{2}{0} \Rightarrow \text{Und.}$ $\quad -2\cos 2t = -2$, $t = 0$

TL vertical $\quad 4 + 2t = 4$, $t = 0$

TL: $x = -2$

NL: $y = 4$ (horizontal)

b) $\frac{dy}{dx} = \frac{5}{-9\sin 3t + 2}\Big|_{t=0} = \frac{5}{2}$ $\quad 3\cos 3t + 2t = 3$, $t = 0$

$1 + 5t = 1$, $t = 0$

TL: $y - 1 = \frac{5}{2}(x - 3)$

NL $y - 1 = \frac{-2}{5}(x - 3)$

8. Find the points (x, y) at which the curve $x(t) = 3 - 4\sin(t)$, $y(t) = 4 + 3\cos(t)$ has:
(a) a horizontal tangent; (b) a vertical tangent.

$\frac{dy}{dx} = \frac{-3\sin t}{-4\cos t}$

a) $-3\sin t = 0$

$t = 0, \pi, 2\pi, \ldots$

$x(0) = 3 \quad x(\pi) = 3$
$y(0) = 7 \quad y(\pi) = 1$

$(3,7) \quad (3,1)$

b) $-4\cos t = 0$

$t = \pi/2, 3\pi/2, 5\pi/2, \ldots$

$x(\pi/2) = -1 \quad x(3\pi/2) = 7$
$y(\pi/2) = 4 \quad y(3\pi/2)\ 4$

$(-1,4) \quad (7,4)$

9. Give an equation relating x and y for the curve given parametrically by

a. $x(t) = -1 + 3\cos t \quad y(t) = 1 + 2\sin t$

b. $x(t) = -1 + 3\cosh t \quad y(t) = 1 + 2\sinh t$

c. $x(t) = -1 + 4e^{t} \quad y(t) = 2 + 3e^{-t}$

a) $\frac{x+1}{3} = \cos t \qquad \frac{y-1}{2} = \sin t \qquad \cos^2 t + \sin^2 t = 1$

$$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$

b) $\frac{x+1}{3} = \cosh t \qquad \frac{y-1}{2} = \sinh t$

$$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-1}{2}\right)^2 = 1$$

c) $\frac{x+1}{4} = e^{t}$

$\frac{4}{x+1} = e^{-t} \qquad y = 2 + 3\left(\frac{4}{x+1}\right)$

10. Find a parameterization for:

a. Line segment from (-1, 3) to (5, 4)

b. Circle with radius 2 and center (2, -1)

a) $x(t) = -1 + t(5 - -1) = -1 + 6t \qquad 0 \le t \le 1$

$y(t) = 3 + t(4 - 3) = 3 + t$

b) $(x-2)^2 + (y+1)^2 = 4$

$\frac{x-2}{2} = \cos t \qquad \frac{y+1}{2} = \sin t$

$x(t) = 2 + 2\cos t \qquad y(t) = -1 + 2\sin t$

11. Write an expression for the nth term of the sequence:

a. 1, 4, 7, 10, ...

b. 2, -1, $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

1 2 3 4

a) 1, 4, 7, 10, ...

3 3 3 $\Rightarrow$ linear

$a_n = 1 + 3(n-1)$
$= 3n - 2$

b) $2, -1, \frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \ldots$

1 2 3 4 5

$2^1 \quad 2^0 \quad 2^{-1} \quad 2^{-2}$

powers of two

$a_n = (-1)^{n+1} \, 2^{2-n}$

alternating; starts with positive

12. Determine if the following sequences are monotonic. Also indicate if the sequence is bounded and if it is give the least upper bound and/or greatest lower bound.

a. $a_n = \dfrac{2n}{1+n} \to 2$

$\therefore$ bounded
monotonic

$\frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5}, \ldots$

$< \; < \; <$

increasing

LUB = 2 GLB = 1

b. $a_n = \dfrac{\cos n}{n} \to 0$

$\frac{\cos 1}{1} > \frac{\cos 2}{2} > \frac{\cos 3}{3}, \ldots$

almost π

$-1 \le \cos n \le 1$

not monotonic
bounded

LUB = $\cos 1$
GLB = $\frac{\cos 3}{3}$

13. Determine if the following sequences converge or diverge. If they converge, give the limit.

a. $\left\{(-1)^n\left(\frac{n}{n+1}\right)\right\}$ no limit, alternates diverges $\frac{n}{n+1} \to 1$

b. $\left\{\frac{6n^2-2n+1}{4n^2-1}\right\} \to \frac{6}{4} = \frac{3}{2}$ converges to $\frac{3}{2}$

c. $\left\{\frac{(n+2)!}{n!}\right\}$ $\frac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}} \to \infty$ diverges

d. $\left\{\frac{3}{e^n}\right\} \to 0$ converges to 0

e. $\left\{\frac{4n+1}{n^2-3n}\right\} \to 0$ converges to 0

f. $\left\{\frac{e^n}{n^3}\right\} \to \infty$ diverges e^n "trumps" n^3