Department of Mathematics, University of Houston–Math 1432–David Blecher, Some review questions

Ignore the group numberings below.

1. Group A. Let $g(x) = x^3 + x - 2$. Find an equation for the tangent line to the graph of g^{-1} (the inverse function, not the reciprocal) at the point $(0, 1)$ on the graph of g^{-1}.

2. Group B. Find $\frac{dy}{dx}$ if $y = \frac{(x+3)^4}{(x^4+12)\sqrt{x^2+5}}$ by 'logarithmic differentiation'.

3. Group C. Suppose a culture of bacteria is growing in such a way that the change in the number of bacteria is proportional to the number present. The number of bacteria double every 200 minutes and there are currently 5000 bacteria in the culture. How many bacteria were present 2 hours ago?

4. Group D. Find $\int e^{2x} \cos x \, dx$.

5. Group E.

   (i) Find $h'(x)$ if $h(x) = x^x$, for $x > 0$.

   (ii) Find $\lim_{x\to\infty} x^x$ and $\lim_{x\to 0^+} x^x$.

(iii) Show that $\lim_{x\to\infty} (1 + \frac{2}{x})^x = e^2$.

6. Groups A, B, C: Find the following integrals:

(a) $\int_0^{\frac{\pi}{2}} \sin^2 x \, \cos^3 x \, dx$. (b) $\int_0^1 \sin^{-1} x \, dx$. (c) $\int_1^5 \frac{x}{\sqrt{x-1}} \, dx$.

7. Group D: Find the limit $\lim_{n\to\infty} n^{\frac{1}{n}}$, giving your reasoning (Hint: this can be deduced from Q 4 (ii)). What is $\lim_{n\to\infty} 2^{\frac{1}{n}}$?

8. Groups A–E: In each of the following write down if the series converges or diverges, giving reasons (letters correspond to groups).

(a) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$. (b) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ (c) $\sum_{k=1}^{\infty} \frac{\sin k}{e^k}$,

(d) $\sum_{k=1}^{\infty} \frac{(-1)^k \ln k}{k^2}$, (e) $\sum_{k=1}^{\infty} (\sqrt{k^4+1} - \sqrt{k^4-1})$.

Is the series in (d) absolutely or conditionally convergent?

9. Group B *Gabriel's horn.* In the bible, Gabriel is an archangel, and the horn is the trumpet blown at the end of time, heralding a cataclysmic event described in the book of Revelation. In this problem, we obtain this infinite trumpet by taking the graph of $f(x) = \frac{1}{x}$ for $1 \le x < \infty$, and rotating it around the x-axis [Picture drawn in class.] The volume formula for this from Calc I is

$$V = \pi \int_1^\infty (f(x))^2 \, dx = \pi \int_1^\infty \frac{dx}{x^2}.$$

Show that you can fill up Gabriel's horn with water, and estimate how much water it will take. Also show that the surface area formula from Calc 1 ($S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx$), applied here, gives that Gabriel's horn has infinite surface area. So you can fill it up with paint, but you could never paint it!

10. Group E

(a) Using a fact about geometric and power series (not Taylor series) show that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for $-1 < x < 1$.

(b) Show using (a) together with a Theorem about convergence of a power series at the endpoints of its interval of convergence (you may have to google to find the right theorem), that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, for $-1 \le x \le 1$.

(c) (Leibnitz formula) Deduce that $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$.

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11. Group A and C. Find the radii of convergence and intervals of convergence of the power series $\sum_{k=1}^{\infty} x^k/k$ and $\sum_{k=1}^{\infty} kx^k$.

12. Group B: Find a Taylor series for $\ln(2-x)$. Write down the first four terms out in full.

13. Group C: Find a Taylor approximation for $\sin(1)$ with error < 0.001. What is the name of the formula for the error you are using?

14. Group A: Give an equation of the tangent line to the curve given parametrically as $x(t) = \sin(\pi t), y(t) = 2\cos(\pi t)$ at the point where $t = 1/4$.

15. Group B: Find the length of the polar curve $r = 1 - \cos\theta$, for $0 \leq \theta \leq 2\pi$.

16. Group D: Find a formula in Cartesian coordinates for the curve in question 15.

17. Group E: Sketch the polar curves $r = 4$ and $r = 8\sin\theta$. Find the θ values where these curves intersect. Then compute the area of the region between the graphs of these polar curves.