Chapter 6. Systems of First Order Linear Differential Equations

- We will only discuss first order systems. However higher order systems may be made into first order systems by a trick shown below.

- We will have a slight change in our notation for DE's. Before, in Chapters 1–4, we used the letter x for the independent variable, and y for the dependent variable. For example, $y = \sin x$, or $x^2 \frac{dy}{dx} + 2xy = \sin x$. Now we will use t for the independent variable, and x, y, z, or x_1, x_2, x_3, x_4, and so on, for the dependent variables. For example:

$$\begin{cases} x_1 = \sin t \\ x_2 = t \cos t \end{cases} .$$

  And when we write x_1', for example, we will henceforth mean $\frac{dx_1}{dt}$.

• The *first order systems* (of ODE's) that we shall be looking at are systems of equations of the form

$$\begin{cases} x_1' = & \text{expression in } x_1, x_2, \cdots x_n, t \\ x_2' = & \text{expression in } x_1, x_2, \cdots x_n, t \\ \vdots & \vdots \qquad\qquad\qquad \vdots \\ x_n' = & \text{expression in } x_1, x_2, \cdots x_n, t, \end{cases}$$

valid for t in an interval I. These expressions on the right sides contain no derivatives. A *first order IVP system* would be the same, but now we also have initial conditions $x_1(a) = c_1, x_2(a) = c_2, \cdots, x_n(a) = c_n$. Here a is a fixed number in I, and $c_1, c_2, \cdots, c_n$ are fixed constants.

Example 1. $\begin{cases} x' = y \\ y' = -x + 1. \end{cases}$ (Where is t?)

Example 2. $\begin{cases} x_1' = x_1 x_2 x_3 - \sin(t)\, x_3^2 \\ x_2' = 3x_1 x_3 t + 1 \\ x_3' = e^{x_1 - t}. \end{cases}$

Example 3. $\begin{cases} x' = y \\ y' = \frac{3}{t^2}x + \frac{1}{t}y - 9 \end{cases}, \quad x(1) = 3, y(1) = 6.$

• These are called *first order* systems, because the highest derivative is a first derivative. Example 3 is a *first order IVP system*, the *initial conditions* are $x(1) = 3, y(1) = 6$.

• A *solution* to such a system, is several functions $x_1 = f_1(t), x_2 = f_2(t), \cdots, x_n = f_n(t)$ which satisfy all the equations in the system simultaneously. A solution to a first order IVP system also has to satisfy the initial conditions.

For example, a solution to Ex. 1 above is $x = 1 + \sin t, y = \cos t$. To check this, notice that if $x = 1 + \sin t$ and $y = \cos t$, then clearly $x' = (1 + \sin t)' = \cos t = y$, and $y' = -\sin t = -(1 + \sin t) + 1 = -x + 1$. So both equations are satisfied simultaneously.

Similarly, a solution to the first order IVP system in Ex. 3 above is $x = 3t^2, y = 6t$. (Check it.)

- Just as in Chapter 2, under a 'mild condition' there always exist solutions to a first order IVP system, and the solution will be unique, but 'local' (that is, it may only exist in a small interval surrounding a). The proof is almost identical to the one in Chapter 2.

- Trick to change higher order ODE's (or systems) into first order systems:

For example consider the ODE $y''' - \sin(x)y'' + 2y' - xy = \cos x$. Let $t = x, x_1 = y, x_2 = y', x_3 = y''$. Then $y''' = x_3'$. We do not introduce a variable for the highest derivative. We then obtain the following first order system:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = \cos t + t\, x_1 - 2x_2 + \sin(t)\, x_3 \end{cases}$$

Strategy: solve the latter system; and if $x_1 = f(t)$ then the solution to the original ODE is $y = f(x)$. So for example if $x_1 = 3\cos(2t)$ then $y = 3\cos(2x)$.

- Similarly a higher order IVP like $y''' - \sin(x)y'' + 2y' - xy = \cos x,\ y(0) = 1, y'(0) = -2, y''(0) = 3$, is changed into a 1st order IVP system (the one in the last paragraph), with initial conditions $x_1(0) = 1, x_2(0) = -2, x_3(0) = 3$.

- Using the same trick, any nth order system may be changed into a first order system.

- Combining the existence and uniqueness result a few bullets above, with the 'trick' just discussed, we see that every nth order IVP has a unique local solution under a 'mild condition'.

- **Linear systems.** A *first order linear system* is a first order system of form

$$\begin{cases} x_1' = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots & \vdots \qquad\qquad\qquad\qquad \vdots \\ x_n' = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t). \end{cases}$$

Examples like

$$\begin{cases} x' = y \\ y' = \frac{3}{t^2}x + \frac{1}{t}xy - 9 \end{cases},$$

or

$$\begin{cases} x' = y \\ y' = \frac{3}{t^2}x^2 + \frac{1}{t}y - 9 \end{cases},$$

are not linear (on the right sides the dependent variables, in this case x and y are only allowed to be multiplied by constants or functions of t. We will see some more examples momentarily.

- **Matrix formulation of linear systems.** The *coefficient matrix* of the last system is $A(t) = [a_{ij}(t)]$. That is

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \vec{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

then the system may be rewritten as a single matrix equation

$$\vec{x}' = A(t)\,\vec{x} + \vec{b}(t).$$

Example 1. Consider the system $\begin{cases} x_1' = t^2 x_1 - e^t x_2 + 1 - t \\ x_2' = x_1 + \cos(t)\, x_2. \end{cases}$. The *coefficient matrix* of the last system is

$$A(t) = \begin{bmatrix} t^2 & -e^t \\ 1 & \cos t \end{bmatrix}. \quad \text{And } \vec{b}(t) = \begin{bmatrix} 1-t \\ 0 \end{bmatrix}.$$

If we write $\vec{x}$ for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{x}'$ for $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$, then the system may be rewritten as a single matrix equation

$$\vec{x}' = \begin{bmatrix} t^2 & -e^t \\ 1 & \cos t \end{bmatrix} \vec{x} + \begin{bmatrix} 1-t \\ 0 \end{bmatrix}.$$

Example 2. The IVP system in Example 3 above may be rewritten as a single matrix equation

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ \frac{3}{t^2} & \frac{1}{t} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ -9 \end{bmatrix}, \quad \vec{x}(1) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

- Thus a first order linear system is one that can be written in the form

$$\vec{x}' = A(t)\,\vec{x} + \vec{b}(t).$$

Here $A(t)$ is a matrix whose entries depend only on t, and $\vec{b}(t)$ is a column vector whose entries depend only on t. Linear first order IVP systems always have (unique) solutions if $A(t)$ and $\vec{b}(t)$ are continuous; in fact we will give formulae later for the solution in the 'constant coefficient case' (that is when $A(t)$ is constant, does not depend on t).

- **Vector functions:** The vector $\vec{x}$ above depends on t. Thus it is a 'vector function'. Similarly, $\vec{b}(t)$ above is a 'vector function'. We call it an n-component vector function if it has n entries, that is if it lives in $\mathbb{R}^n$.

You should think of a solution to the 'matrix DE' in the Definition above as a 'vector function'. For example, you can check that $\begin{cases} x = 3t^2 \\ y = 6t. \end{cases}$ is a solution to Example 2 above (which was Example 3 before). We write this solution as the 'vector function' $\vec{u}(t) = \begin{bmatrix} 3t^2 \\ 6t \end{bmatrix}$. One can check that indeed

$$\vec{u}' = \begin{bmatrix} 0 & 1 \\ \frac{3}{t^2} & \frac{1}{t} \end{bmatrix} \vec{u} + \begin{bmatrix} 0 \\ -9 \end{bmatrix}.$$

Do it! (We did it in class.)

- The system above is called *homogeneous* if $\vec{b}(t) = \vec{0}$.

- If

$$\vec{x}' = A(t)\,\vec{x} + \vec{b}(t), \qquad \text{(N)}$$

is not homogeneous then the *associated homogeneous equation* or *reduced equation* is the equation $\vec{x}' = A(t)\,\vec{x}$.

- We can rewrite (N) as $\vec{x}' - A(t)\,\vec{x} = \vec{b}(t)$, or

$$(D - A(t))\vec{x} = \vec{b}(t) \qquad (N)$$

where $D\vec{x} = \vec{x}'$, or simply as

$$L\vec{x} = \vec{b}(t) \qquad (N)$$

where $L = D - A(t)$. It is easy to see as before that $L = D - A(t)$ is *linear*, that is:

$$L(c_1\vec{u}_1 + c_2\vec{u}_2) = c_1 L\vec{u}_1 + c_2 L\vec{u}_2.$$

Thus the main results in Chapters 3 and 5 carry over to give variants valid for first order linear systems, with essentially the same proofs. We state some of these results below. First we discuss homogeneous first order linear systems.

6.2. Homogeneous first order systems

Here we are looking at

$$\vec{x}' = A(t)\,\vec{x}, \qquad \text{(H)}$$

for t in an interval I. Thus is just $L\vec{x} = \vec{0}$ where $L = D - A(t)$ as above. We will fix a number n throughout this section and the next, and assume that we have n variables $x_1, \cdots, x_n$, each a function of t. So $A(t)$ is an $n \times n$ matrix. We will then refer to (H) sometimes as $(\text{H})_n$, reminding us of this fixed number n, so that e.g. $A(t)$ is $n \times n$, etc. The proofs of the next several results are similar (usually almost identical) to the matching proofs in Chapter 3 (and Chapters 5 and 2).

- Of course the zero vector $\vec{0}$ is a solution of (H). As before, this solution is called the *trivial solution*.

• **Theorem** If $\vec{u}_1$ and $\vec{u}_2$ are solutions to (H) on I then so is $\vec{u}_1 + \vec{u}_2$ and $c\vec{u}_1(x)$ solutions to (H), for any constant c.

• So the sum of any two solutions of (H) is also a solution of (H). Also, any constant multiple of a solution of (H) is also a solution of (H).

• Again, a *linear combination* of $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ is an expression

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n,$$

for constants $c_1, \cdots, c_n$.

• The *trivial linear combination* is the one where all the constants c_k are zero. This of course is zero.

• **Theorem** Any linear combination of solutions to (H) is also a solution of (H).

• Two vector functions $\vec{u}$ and $\vec{v}$ whose domain includes the interval I, are said to be *linearly dependent* on I if $\vec{u}$ is a constant times $\vec{v}$, or $\vec{v}$ is a constant times $\vec{u}$. If they are not linearly dependent they are called *linearly independent*.

• Another way to say it: $\vec{u}$ and $\vec{v}$ are linearly independent if the only linear combination of $\vec{u}$ and $\vec{v}$ which equals zero, is the trivial one.

• More generally, $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k$ are linearly independent if no one of $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k$ is a linear combination of the others, not including itself. Equivalently: $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k$ are linearly independent if the only way

$$c_1\vec{u}_1(t) + c_2\vec{u}_2(t) + \cdots + c_k\vec{u}_k(t) = 0$$

for all t in I, for constants $c_1, \cdots, c_n$, is when all of these constants $c_1, \cdots, c_n$ are zero.

• The *Wronskian* of n n-component vector functions $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$, written $W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)(t)$ or $W(t)$ or $W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)$, is the determinant of the matrix $[\vec{u}_1 : \vec{u}_2 : \cdots : \vec{u}_n]$. This last matrix is the matrix whose jth column is $\vec{u}_j(t)$.

• **Proposition** If $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are linearly dependent on an interval I, then

$$W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)(t) = 0$$

for all t in I.

Proof. This follows from the equivalence of (2) and (8) in the 12 part theorem proved in Homework 11. □

• **Corollary** If $W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)(t_0) \neq 0$ at some point t_0 in I then $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are linearly independent.

• **Theorem** There exist n solutions $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ to $(H)_n$ which are linearly independent.

• *Proof.* Similar to the matching proof in Chapter 3, or we will give a formula for the solution later. □

• n solutions $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ to $(\mathrm{H})_n$ which are linearly independent, are called a *fundamental set of solutions* to (H). Then the matrix $X(t) = [\vec{u}_1 : \vec{u}_2 : \cdots : \vec{u}_n]$ met above is called the *fundamental matrix.*

• **Theorem** If $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are solutions to $(\mathrm{H})_n$ on an open interval I then either $W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)(t) = 0$ for all t in I; or $W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)(t) \neq 0$ for all t in I.

• This means that an $n \times n$ matrix $X(t)$ whose columns are solutions to $(\mathrm{H})_n$ is a fundamental matrix if and only if $X(t)$ is invertible for all t in I, and if and only if $X(t)$ is invertible for some t in I. By the 12 part theorem proved in Homework 11 this can be phrased in many equivalent ways.

• If $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are solutions to $(\mathrm{H})_n$ on I, and if every solution to $(\mathrm{H})_n$ on I is of the form $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$, for constants $c_1, \cdots, c_n$, then we say that $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ is the *general solution* to $(\mathrm{H})_n$.

• **Theorem** Suppose that $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are solutions to $(\mathrm{H})_n$ on an open interval I. The following are equivalent:

(i) $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n$ are a fundamental set of solutions to (H) on I,
(ii) $W(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n)(t) \neq 0$ for some (or all) t in I,
(iii) $c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n$ is the general solution to (H).

Example. Show that $\vec{u} = \begin{bmatrix} t^3 \\ 3t^2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} \frac{1}{t} \\ -\frac{1}{t^2} \end{bmatrix}$ are a fundamental set for the linear system

$$\begin{cases} x' = y \\ y' = \frac{3}{t^2}x + \frac{1}{t}y \end{cases}$$

on the interval $(0, \infty)$. Also, find the general solution to this system.

Solution. This is the system $\vec{x}' = A(t)\vec{x}$ where

$$A(t) = \begin{bmatrix} 0 & 1 \\ \frac{3}{t^2} & \frac{1}{t} \end{bmatrix}.$$

Check that $\vec{u}' = A(t)\vec{u}$ (we checked this in class), and $\vec{v}' = A(t)\vec{v}$. Then note that

$$W(\vec{u}, \vec{v}) = \det(\begin{bmatrix} t^3 & \frac{1}{t} \\ 3t^2 & -\frac{1}{t^2} \end{bmatrix}) = t^3(-\frac{1}{t^2}) - 3t^2\frac{1}{t} = -t - 3t = -4t \neq 0.$$

So by the last theorem $\vec{u}, \vec{v}$ is a fundamental set of solutions, and the general solution to this system is $C\vec{u} + D\vec{v}$. That is, the general solution is $x = Ct^3 + \frac{D}{t}$ and $y = 3Ct^2 - \frac{D}{t^2}$. (Explained in more detail in class.)

6.3/6.4. Homogeneous first order systems with constant coefficients I-II

Here we are looking at

$$\vec{x}' = A\vec{x}, \qquad \text{(H)}$$

for t in an interval I. Here A is an $n \times n$ matrix with constants (numbers) as entries. Again we are fixing a number n throughout, and assume that we have n variables $x_1, \cdots, x_n$, each a function of t. We will again refer to (H) sometimes as $(\text{H})_n$, reminding us of this fixed number n, so that e.g. A is $n \times n$, etc.

• If λ is an eigenvalue of A with eigenvector $\vec{v}$, set $\vec{x} = e^{\lambda t}\vec{v}$. Note that $\vec{x}' = e^{\lambda t}\lambda\vec{v}$, whereas

$$A\vec{x} = e^{\lambda t} A\vec{v} = e^{\lambda t}\lambda\vec{v}.$$

So $\vec{x}' = A\vec{x}$; that is $e^{\lambda t}\vec{v}$ is a solution to (H).

• First suppose that the $n \times n$ matrix A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$, and that λ_k is the eigenvalue associated with the eigenvector $\vec{v}_k$. Then the general solution is

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots c_n e^{\lambda_n t}\vec{v}_n,$$

where $c_1, c_2, \cdots$, are arbitrary constants.

That is, a fundamental set is: $e^{\lambda_1 t}\vec{v}_1, e^{\lambda_2 t}\vec{v}_2, \cdots, e^{\lambda_n t}\vec{v}_n$. The fundamental matrix $X(t)$ is the matrix with these as columns.

• [Proof: The Wronskian $W(0) = \det(X(0)) \neq 0$ by (2) $\Leftrightarrow$ (8) in the 12 part theorem in Homework 11, since $X(0)$ is the matrix with columns $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$, which are linearly independent.]

• Of course, if the $n \times n$ matrix A has n distinct eigenvalues, then if we find one eigenvector for each eigenvalue, these will be linearly independent by Theorem 1 in Section 5.8. Also, from more advanced linear algebra it is known that if $A = A^T$, that is if A is *symmetric*, then there will exist n linearly independent eigenvectors. So the method above will work.

Example 1. Solve $\begin{cases} x' = x - 3y \\ y' = -2x + 2y \end{cases}$.

Solution. This is just $\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$. We first find the eigenvalues and corresponding eigenvectors of A. We have

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

Thus the eigenvalues are $\lambda = -1$ and 4. To find an eigenvector corresponding to $\lambda = 4$ we need to solve $A\vec{x} = 4\vec{x}$. We solve

$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ ? \end{bmatrix} = 4\begin{bmatrix} 1 \\ ? \end{bmatrix}.$$

From the first row, we see that $1-3? = 4$, so that $? = -1$. Thus an eigenvector corresponding to $\lambda = 4$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Similarly, to find an eigenvector corresponding to $\lambda = -1$ we solve

$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ ? \end{bmatrix} = - \begin{bmatrix} 1 \\ ? \end{bmatrix}.$$

Thus $1 - 3? = -1$ so that $? = \frac{2}{3}$. Thus an eigenvector corresponding to $\lambda = -1$ is: $\begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$, or, multiplying by 3, we get the eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

A general solution to the system is then $\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-t}$, where c_1, c_2 are arbitrary constants. We can rewrite this as $\vec{x} = \begin{bmatrix} c_1 e^{4t} + 3c_2 e^{-t} \\ -c_1 e^{4t} + 2c_2 e^{-t} \end{bmatrix}$. This corresponds to the solution $\begin{cases} x = c_1 e^{4t} + 3c_2 e^{-t} \\ y = -c_1 e^{4t} + 2c_2 e^{-t} \end{cases}$.

Example 2. Find the general solution to $\begin{cases} x_1' = x_1 + x_2 - x_3 \\ x_2' = -x_1 + 3x_2 - x_3 \\ x_3' = -x_1 + x_2 + x_3 \end{cases}$. Also solve the IVP consisting of this system with initial conditions $x_1(0) = 1, x_2(0) = 2, x_3(0) = 2$.

Solution. This is just $\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. One can show that this matrix A has eigenvalues $1, 2, 2$ (I will skip the work, it is just as in the previous chapter). For the eigenvalue 1 we only want one eigenvector, and you can check that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector (using the usual method as in the previous chapter, similar to what follows:). To find evectors for the eigenvalue 2 we must solve $(A - 2I)\vec{x} = \vec{0}$, which is

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \vec{x} = \vec{0}.$$

This has solution (after Gauss elimination):

$$\begin{cases} x_1 = s - t \\ x_2 = s \\ x_3 = t \end{cases}.$$

Setting $s = 0, t = 1$, then $t = 0, s = 1$, we get linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the general solution to the system is $\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$.

Reading this row by row, we can rewrite this as $\begin{cases} x_1 = c_1 e^t + c_2 e^{2t} - c_3 e^{2t} \\ x_2 = c_1 e^t + c_2 e^{2t} \\ x_3 = c_1 e^t + c_3 e^{2t} \end{cases}$.

To solve the IVP we must solve

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2 \cdot 0} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2 \cdot 0} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

That is,

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Using the solution method of your choice (row reduction, inverse, Cramers rule), the solution is: $c_1 = 1, c_2 = 1, c_3 = 1$. The solution of the initial-value problem is $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$. Reading this row by row, we can rewrite this as $\begin{cases} x_1 = e^t \\ x_2 = e^t + e^{2t} \\ x_3 = e^t + e^{2t} \end{cases}$.

- **What if some of the eigenvalues are complex?** If A is a real matrix, then its complex eigenvalues occur in pairs $\alpha \pm i\beta$, with α, β real. Suppose that $\vec{v}$ is an eigenvector corresponding to an eigenvalue $\lambda = \alpha + i\beta$. Write $\vec{v} = \vec{r} + i\vec{s}$ where $\vec{r}$ and $\vec{s}$ have only real entries. Then in the general solution to the linear system include terms:

$$Ce^{\alpha t}(\vec{r}\cos(\beta t) - \vec{s}\sin(\beta t)) \ + \ De^{\alpha t}(\vec{r}\sin(\beta t) + \vec{s}\cos(\beta t))$$

Here C, D are constants. (This also takes care of the $\alpha - i\beta$ case, so ignore that case).

Example 3. Solve the linear system $\vec{x}' = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \vec{x}$.

Solution. We have $det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2 = \lambda^2 + 4\lambda + 5$. Using the quadratic equation $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ you can easily show that $\lambda = -2 \pm i$ are the two eigenvalues. To find an eigenvector corresponding to $\lambda = -2 + i$ we need to solve $A\vec{x} = (-2+i)\vec{x}$. So we solve

$$\begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ ? \end{bmatrix} = (-2+i) \begin{bmatrix} 1 \\ ? \end{bmatrix}.$$

From the first row, we see that $-3-? = -2+i$, so that $? = -1-i$. Thus an eigenvector is $\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$. We write this as $\vec{r} + i\vec{s}$ as above:

$$\begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Thus, by the discussion above the Example, the general solution is

$$\vec{x} = Ae^{-2t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin t\right) + Be^{-2t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos t\right),$$

where A, B are arbitrary constants. This may be rewritten. Reading row by row we get:

$$\begin{cases} x_1 = Ae^{-2t}\cos t + Be^{-2t}\sin t \\ x_2 = Ae^{-2t}(-\cos t + \sin t) + Be^{-2t}(-\sin t - \cos t) \end{cases}.$$

• **What if A is $n \times n$, but you cannot find n linearly independent eigenvectors?** In this case, if λ is an eigenvalue of A of multiplicity k, but there are fewer than k linearly independent eigenvectors for A, then use *generalized eigenvectors* for λ. For example, if λ is an eigenvalue of multiplicity 2, but it has at most one linearly independent eigenvector $\vec{v}$, solve the equation $(A - \lambda I)\vec{w} = \vec{v}$. Then a linearly independent pair of solution vectors corresponding to λ are $e^{\lambda t}\vec{v}$ (as before) and $e^{\lambda t}\vec{w} + te^{\lambda t}\vec{v}$. So as part of the general solution to (H) we will have

$$Ce^{\lambda t}\vec{v} + De^{\lambda t}(\vec{w} + t\vec{v}).$$

If λ is an eigenvalue of multiplicity 3 and one can only find two linearly independent eigenvectors $\vec{v}$ and $\vec{z}$, then a third solution vector corresponding to λ is

$$e^{\lambda t}\vec{w} + te^{\lambda t}\vec{v}$$

where $\vec{w}$ is a solution to the equation $(A - \lambda I)\vec{w} = \vec{v}$. So as part of the general solution to (H) we will have

$$Ce^{\lambda t}\vec{v} + De^{\lambda t}\vec{z} + Ee^{\lambda t}(\vec{z} + t\vec{v}).$$

If λ is an eigenvalue of multiplicity 3, but it has at most *one* linearly independent eigenvector $\vec{v}$, solve the equations

$$(A - \lambda I)\vec{w} = \vec{v}\ , \qquad (A - \lambda I)\vec{z} = \vec{w}.$$

Then three linearly independent solution vectors corresponding to λ are $e^{\lambda t}\vec{v}$ (as before) and $e^{\lambda t}\vec{w} + te^{\lambda t}\vec{v}$, and

$$e^{\lambda t}\vec{z} + te^{\lambda t}\vec{w} + t^2e^{\lambda t}\vec{v}.$$

So as part of the general solution to (H) we will have

$$Ce^{\lambda t}\vec{v} + De^{\lambda t}(\vec{w} + t\vec{v}) + Ee^{\lambda t}(\vec{z} + t\vec{w} + t^2\vec{v}).$$

Examples worked in class. (p. 298–300 in Text). (a) Find the general solution to the system $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\vec{x}$.

(b) Find the general solution to the system $\vec{x}' = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \vec{x}$.

Solution. (a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$. We first find the eigenvalues and corresponding eigenvectors of A. We have

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Thus the eigenvalues are $\lambda = 2$ (multiplicity 2). To find an eigenvector corresponding to $\lambda = 2$ we need to solve $A\vec{x} = 2\vec{x}$. We solve $(A - 2I)\vec{x} = \vec{0}$. The usual Gauss elimination (done in class) yields

$$\begin{cases} x_1 = -t \\ x_2 = t \end{cases},$$

so we are only able to find at most one linearly independent eigenvector

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The recipe above now tells us to solve $(A - 2I)\vec{x} = \vec{v}$, that is,

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The usual Gauss elimination (done in class) yields a solution to this of

$$\begin{cases} x_1 = 1 - s \\ x_2 = s \end{cases}.$$

We can take s to be anything, say $s = 0$, giving a solution vector

$$\vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

A general solution to the system is then $Ce^{\lambda t}\vec{v} + De^{\lambda t}(\vec{w} + t\vec{v})$, or

$$\vec{x} = Ce^{2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + De^{2t}(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}),$$

, where c_1, c_2 are arbitrary constants. We can rewrite this as

$$\begin{cases} x_1 = -Ce^{2t} + De^{2t}(1-t) \\ x_2 = Ce^{2t} + Dte^{2t} \end{cases}.$$

(b) Is very similar to (a) and is solved on page 299 of the online textbook. Or at least one can find there, in the second last line of that solution, a fundamental set of three solutions $\vec{x}_1, \vec{x}_2, \vec{x}_3$. The general solution is $\vec{x} = c_1\vec{x}_1 + c_2\,\vec{x}_2 + c_3\vec{x}_3$.

• There is another way to solve any linear system

$$\vec{x}' = A\vec{x}, \qquad (H)$$

without using eigenvalues and eigenvectors at all! But we will need...

• Matrix exponentials: if A is an $n \times n$ matrix consider the sum

$$I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots .$$

This is the power series formula for e^A from Calculus II. It is not hard to show that this converges in the sense of Calculus 2 to an $n \times n$ matrix which we write as $\exp(A)$. Or, simply get a computer to add the first 50 terms of this sum to approximate it with great precision.

• **Theorem** Let $\Psi(t) = \exp(tA)$. Then $\Psi(t)$ is a fundamental matrix for (H). The n columns of this matrix are a fundamental set for (H). (Note that the jth column is $\Psi(t)\vec{e}_j$.) So the general solution to (H) is

$$\vec{x} = \Psi(t) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Indeed, for any vector $\vec{a}$, a solution to the IVP system which is (H) together with initial condition $\vec{x}(0) = \vec{a}$, is $\Psi(t)\,\vec{a}$.

• *Proof.* One can show easily that $\frac{d}{dt}(\exp(tA)) = A\exp(tA) = \exp(tA)\,A$. Thus if $\vec{a}$ is any vector, and if $\vec{v} = \Psi(t)\,\vec{a}$ then

$$\frac{d}{dt}\vec{v} = \frac{d}{dt}(\exp(tA))\vec{a} = A\exp(tA)\,\vec{a} = A\,\vec{v}.$$

Thus $\vec{v} = \Psi(t)\,\vec{a}$ is a solution to the linear system $\vec{x}' = A\vec{x}$. Setting $\vec{v} = \vec{e}_j$, we see that the columns of $\Psi(t)$ are solutions to (H). As we said earlier, to check $\Psi(t)$ is a fundamental matrix for (H) we need only check that $\Psi(t)$ is invertible. However $\Psi(t)^{-1} = \Psi(-t)$ since

$$\exp(tA)\,\exp(-tA) = \exp(tA - tA) = \exp(0) = I.$$

Since $\Psi(0) = I$, if $\vec{v} = \Psi(t)\,\vec{a}$ then $\vec{v}(0) = I\vec{a} = \vec{a}$. Thus $\vec{v}$ is a solution to the linear IVP system $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{a}$. $\square$

• Similarly, a solution to the linear IVP system $\vec{x}' = A\vec{x}$, $\vec{x}(t_0) = \vec{a}$ is $\exp((t - t_0)A)\,\vec{a} = \exp(tA)\exp(-t_0A)\vec{a}$.

6.5. Solving nonhomogeneous systems.

Here we are looking at

$$\vec{x}' = A\,\vec{x} + \vec{b}(t), \qquad \text{(N)}$$

for t in an interval I. Here A is an $n \times n$ matrix with constants (numbers) as entries. Again we are fixing a number n throughout, and assume that we have n variables $x_1, \cdots, x_n$, each a function of t.

- The reduced equation or associated homogeneous equation for (N) is

$$\vec{x}\,' = A\vec{x}, \qquad \text{(H)}$$

- **Key fact:** As in Chapters 3 (and 5 and 2), finding the general solution to $\vec{x}\,' = A(t)\vec{x} + \vec{b}(t)$, breaks into two steps: Step 1: find the general solution to the associated homogeneous equation (H). Step 2: Find one solution (called a *particular solution*) to (N). Then add what you get in Steps 1 and 2. (The proof of this is the same as before.)

- The proof is the same as we saw in Chapters 3 (and 5 and 2): let $L\vec{x} = \vec{x}' - A\vec{x}$. This is linear. So if $\vec{x}_p$ is a particular solution to (N) and $\vec{x}_H$ is a solution to (H), then $L(\vec{x}_p + \vec{x}_H) = L\vec{x}_p + L\vec{x}_H = \vec{b} + \vec{0} = \vec{b}$. If $\vec{z}$ is any solution to (N) then $\vec{z} - \vec{x}_p$ is a solution to (H), since $L(\vec{z} - \vec{x}_p) = \vec{0} - \vec{0} = \vec{0}$. So if $\vec{x}_H$ is this solution to (H), $\vec{z} = \vec{x}_p + \vec{x}_H$. $\square$

- **The basic formula in this section** is that a particular solution to the linear system $\vec{x}\,' = A(t)\vec{x} + \vec{b}(t)$ is given by the formula

$$\vec{x} = X(t) \int_0^t X(s)^{-1}\vec{b}(s)\,ds,$$

where $X(t)$ is the fundamental matrix. This is 'variation of parameters', and a proof can be found on page 305 of the text book.

- Lets see what this means, and how to use it in examples:

Example. Find the general solution to the linear system $\vec{x}\,' = A\vec{x} + \vec{b}(t)$, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ and $\vec{b}(t) = \begin{bmatrix} t \\ e^t \end{bmatrix}$.

Note that this is the same as asking you to find the general solution to the linear system $\begin{cases} x_1' = 4x_1 - 3x_2 + t \\ x_2' = 2x_1 - x_2 + e^t. \end{cases}$ You need to be able to jump between both ways of writing the system.

Solution: First we solve the associated homogeneous system $\vec{x}\,' = A\vec{x}$, using the methods of 6.3. I will skip the working (see 6.3), and just write down the answer:

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Thus the fundamental matrix is $X(t) = \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}$ (see the discussion in Section 6.2 above). To find a particular solution to the original system, by the basic formula above, we need to compute $X(t) \int_0^t X(s)^{-1}\vec{b}(s)\,ds$.

We have $X(s) = \begin{bmatrix} e^s & 3e^{2s} \\ e^s & 2e^{2s} \end{bmatrix}$. The determinant of this is $2e^{3s} - 3e^{3s} = -e^{3s}$. Thus

$$X(s)^{-1} = \frac{1}{-e^{3s}} \begin{bmatrix} 2e^{2s} & -3e^{2s} \\ -e^s & e^s \end{bmatrix} = \begin{bmatrix} -2e^{-s} & 3e^{-s} \\ e^{-2s} & -e^{-2s} \end{bmatrix}.$$

Hence

$$X(s)^{-1}\vec{b}(s) = \begin{bmatrix} -2e^{-s} & 3e^{-s} \\ e^{-2s} & -e^{-2s} \end{bmatrix} \begin{bmatrix} s \\ e^s \end{bmatrix} = \begin{bmatrix} -2se^{-s} + 3 \\ se^{-2s} - e^{-s} \end{bmatrix}.$$

Now integrate this:

$$\int_0^t X(s)^{-1}\vec{b}(s)\, ds = \begin{bmatrix} \int_0^t (-2se^{-s} + 3)\, ds \\ \int_0^t (se^{-2s} - e^{-s})\, ds \end{bmatrix}.$$

Now $\int_0^t (-2se^{-s}+3)\, ds = 2e^{-t}(t+1)+3t-2$; and $\int_0^t (se^{-2s} - e^{-s})\, ds = e^{-2t}(\frac{-t}{2} - \frac{1}{4}) + e^{-t} - \frac{3}{4}$. (Check - this just uses Calculus 2 integrals).

Thus $\int_0^t X(s)^{-1}\vec{b}(s)\, ds = \begin{bmatrix} 2e^{-t}(t+1) + 3t - 2 \\ e^{-2t}(\frac{-t}{2} - \frac{1}{4}) + e^{-t} - \frac{3}{4} \end{bmatrix}$. Multiplying this by $X(t)$ we get the particular solution

$$\begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} 2e^{-t}(t+1) + 3t - 2 \\ e^{-2t}(\frac{-t}{2} - \frac{1}{4}) + e^{-t} - \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{t}{2} + \frac{5}{4} + (3t+1)e^t - \frac{9}{4}e^{2t} \\ t + \frac{3}{2} + 3te^t - \frac{3}{2}e^{2t} \end{bmatrix}.$$

The last step here looks complicated, but is just algebra (Check it!) On the test, the algebra won't work out quite as ugly.

Thus by the Key Fact (mentioned at the start of the section above), the general solution to our original system is

$$\vec{x} = \begin{bmatrix} \frac{t}{2} + \frac{5}{4} + (3t+1)e^t - \frac{9}{4}e^{2t} \\ t + \frac{3}{2} + 3te^t - \frac{3}{2}e^{2t} \end{bmatrix} + c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

- The general solution to equation (N) is

$$\vec{x} = X(t) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + X(t) \int_a^t X(s)^{-1}\vec{b}(s)\, ds,$$

since the general solution to the reduced equation (H) is

$$X(t) \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

- It follows that the solution to the IVP which is equation (N) with initial condition $\vec{x}(a) = \vec{x}_0$ is

$$\vec{x} = X(t) \int_a^t X(s)^{-1}\vec{b}(s)\, ds + X(t)X(a)^{-1}\vec{x}_0,$$

where $X(t)$ is the fundamental matrix.

- [Proof: setting $t = a$ in the last formula we get
$$\vec{x}(a) = X(a) \int_a^a X(s)^{-1} \vec{b}(s)\, ds + X(a) X(a)^{-1} \vec{x}_0 = \vec{0} + I \vec{x}_0 = \vec{x}_0.$$

So the initial condition is satisfied. Also, that last formula equals
$$X(t) \int_0^t X(s)^{-1} \vec{b}(s)\, ds + X(t)\vec{w} + X(t)\vec{y} = X(t) \int_0^t X(s)^{-1} \vec{b}(s)\, ds + X(t)\vec{c},$$
where
$$\vec{w} = \int_a^0 X(s)^{-1} \vec{b}(s)\, ds, \quad \vec{y} = X(a)^{-1} \vec{x}_0, \quad \vec{c} = \vec{w} + \vec{y}.$$
But $X(t) \int_0^t X(s)^{-1} \vec{b}(s)\, ds + X(t)\vec{c}$ is of the form of the general solution to (N) in the second last bullet. $\square$]