Lecture 4: Von Neumann algebraic Hardy spaces

David Blecher
University of Houston

December 2016

Abstract for Lecture 4

We discuss Arvesons noncommutative H^∞, and associated von Neumann algebraic H^p spaces. First we review the theory for finite von Neumann algebras, then discuss the general von Neumann algebra case (joint work with Louis Labuschagne).

From the Preface, "Banach spaces of analytic functions", by Kenneth Hoffman (1962)

"many of the techniques of functional analysis have a 'real variable' character and are not directly applicable to... analytic function theory... But there are parts ... which blend beautifully ... . These are fascinating areas of study for the general analyst, for three principal reasons: (a) the point of view of the algebraic analyst leads to the formulation of many interesting problems concerned with analytic functions; (b) when such problems are solved by a combination of the tools from the two disciplines, the depth of each discipline is increased; (c) the techniques of functional analysis often lend clarity and elegance to the proofs of the classical theorems, and thereby make the results available in more general situations. "

Section 1. Introduction and the classical case

Beginning in his 1964 UCLA PhD thesis, Arveson found a beautiful way to combine the theory of von Neumann algebras and Hardy spaces via his subdiagonal subalgebra of a von Neumann algebra M

He was inspired by prediction theory (Helson-Lowdenslager, Wiener-Masana), ...

Arveson was also inspired by some emerging ‘abstract analytic function theory’

which they sometimes called ‘analytic function theory without analytic functions’

Arveson was also inspired by some emerging 'abstract analytic function theory'

which they sometimes called 'analytic function theory without analytic functions'

– it was becoming clear then that many famous theorems about analytic functions, and about the Hardy spaces $H^p(\mathbb{D})$, were essentially algebraic in nature

(i.e. they could be generalized to an abstract uniform algebraic setting where they followed from general algebraic, functional analytic (particularly Hilbert space) principles)

Isolating carefully what makes some of the most important theorems about $H^\infty(\mathbb{D})$, such as Beurling, F & M Riesz, etc Beurling's theorem, work, researchers in the 60s (Helson-Lowdenslager, Hoffman, Srinivasan and Wang, and many others) arrived at the following very simple setting:

• X a probability space, A a closed subalgebra of $L^\infty(X)$ containing constants, such that:

$$\int fg = \int f \int g, \qquad f, g \in A$$

(Note: this clearly applies to $H^\infty(\mathbb{D}) \subset L^\infty(\mathbb{T})$)

We suppose A is weak* closed (otherwise replace...)

Define H^p to be the closure of A in the p-norm.

Theorem. (Hoffman, Srinivasan-Wang, ...) For such A, the following eight are equivalent

(i) the weak* closure of $A + \bar{A}$ is all of $L^\infty(X)$.

(ii) A has 'factorization': (i.e. $b \in L^\infty$, $b \geq \epsilon 1 > 0$ iff $b = |a|^2$ for an invertible $a \in A$)

(iii) A is logmodular (similar to (iii) but $b = \lim_n |a_n|^2$...)

(iv) A satisfies Szegö's theorem (i.e. $\exp \int \log g = \inf\{\int |1 - f|^p g : f \in A, \int f = 0\}$, for any $g \in L^1(X)_+$).

(v) $g \in L^1(X)_+$, $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e..

(vi) Beurling type invariant subspace property: every 'simply A-invariant subspace' of L^p is of the form uH^p for a function u with $|u| \equiv 1$.

(vii) Beurling-Nevanlinna factorization property: Every $f \in L^p$ such that $\int \log |f| > -\infty$ has an (essentially unique) 'inner-outer factorization' $f = uh$, u unimodular and $h \in H^p$ 'outer' (i.e. $1 \in [hA]_p$).

(viii) Every normal (i.e. weak* continuous) functional on A has a unique Hahn-Banach extension to L^∞, and this is normal ('Gleason-Whitney property').

(v) $g \in L^1(X)_+$, $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e..

(vi) Beurling type invariant subspace property: every 'simply A-invariant subspace' of L^p is of the form uH^p for a function u with $|u| \equiv 1$.

(vii) Beurling-Nevanlinna factorization property: Every $f \in L^p$ such that $\int \log |f| > -\infty$ has an (essentially unique) 'inner-outer factorization' $f = uh$, u unimodular and $h \in H^p$ 'outer' (i.e. $1 \in [hA]_p$).

(viii) Every normal (i.e. weak* continuous) functional on A has a unique Hahn-Banach extension to L^∞, and this is normal ('Gleason-Whitney property').

- These are the *weak* Dirichlet algebras*.

- Not only does this result visit interesting topics, e.g. Beurling's invariant subspace theorem, outers, etc, but it shows that these topics are tightly connected (and characterize the basic object)

- Almost all of the implications are quite pretty and nontrivial, and this persists when we go to the noncommutative case. Some of these or the properties that follow, were open 30 or 40 years in Arveson's NC case

Other properties of such algebras $A \subset L^\infty$:

- Jensen's inequality: $\log |\int f| \leq \int \log |f|, \; f \in A$
- Jensen's formula: $\log |\int f| = \int \log |f|, \; f \in A^{-1}$
- F & M Riesz theorem

  (if a measure annihilates A then its absolutely continuous and singular parts separately annihilate A; the classical statement of the F & M Riesz theorem does not generalize, but this is an equivalent statement that does generalize).
- Riesz factorization: $h \in H^p$ factors as $h = h_1 h_2$ with $h_1 \in H^q, h_2 \in H^r$, any $1/p = 1/q + 1/r$.
- (Characterization of outers:) $h \in H^p$ satisfies $1 \in [hA]_p$ iff $\log \int |h| = \int \log |h| > 0$.
- Eg. $f \in L^1_+, \& \int \log f > -\infty \;\; \Rightarrow \;\; f = |h|^p$, h outer in H^p
- Szegö's theorem (e.g. $\exp \int \log g = \inf\{\int |1 - f|^p g : f \in A, \int f = 0\}$, for any $g \in L^1(X)_+$).

• Thus these authors from the 1960s replaced ‘analyticity’ by a very *algebraic* situation concerning function algebras in which general functional analysis tools, and in particular Hilbert space tools, yield these fundamental theorems

- Thus these authors from the 1960s replaced ‘analyticity’ by a very *algebraic* situation concerning function algebras in which general functional analysis tools, and in particular Hilbert space tools, yield these fundamental theorems

- Arveson wanted to replace function algebras by operator algebras

- Thus these authors from the 1960s replaced ‘analyticity’ by a very *algebraic* situation concerning function algebras in which general functional analysis tools, and in particular Hilbert space tools, yield these fundamental theorems

- Arveson wanted to replace function algebras by operator algebras

- In particular, one has to take the classical arguments, which feature numerous tricks with functions which fail for operators, and replace them with noncommutative tools coming from the theory of von Neumann algebras and unbounded operators.

For example, in many classical papers on H^p spaces one finds arguments involving expressions of the form $e^{f(x)} \cdots$ but such exponentials behave badly in the noncommutative case, if the exponent is not a normal operator.

Section 2. Arveson's von Neumann algebraic Hardy spaces

In this section M is a von Neumann algebra with a faithful normal tracial state τ. This was mostly the context Arveson worked in

Let $\mathcal{D} \subset M$ be a von Neumann subalgebra

Let A be a weak* closed subalgebra of M with $A \cap A^* = \mathcal{D}$, such that:

the (unique) trace preserving conditional expectation $\Phi : M \to \mathcal{D}$ satisfies:

$$\Phi(a_1 a_2) = \Phi(a_1)\,\Phi(a_2), \quad a_1, a_2 \in A.$$

(In classical case, $\mathcal{D} = \mathbb{C}1$ and $\Phi = \tau(\cdot)1$.)

Need one more condition, and then we will have Arveson's (maximal) subdiagonal algebras

One may define H^p to be the closure of A in the noncommutative L^p space $L^p(M)$, which in turn may be defined to be the closure of M in the norm $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$.

One may define H^p to be the closure of A in the noncommutative L^p space $L^p(M)$, which in turn may be defined to be the closure of M in the norm $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$.

- In the case that $A = M$, the H^p space collapses to $L^p(M)$

- At the other extreme, if A contains no selfadjoint elements except scalar multiples of the identity, and M is commutative, the theory collapses to the classical theory from the 1960s of generalized H^p spaces associated to 'weak* Dirichlet algebras', a class of abstract function algebras

One may define H^p to be the closure of A in the noncommutative L^p space $L^p(M)$, which in turn may be defined to be the closure of M in the norm $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$.

• In the case that $A = M$, the H^p space collapses to $L^p(M)$

• At the other extreme, if A contains no selfadjoint elements except scalar multiples of the identity, and M is commutative, the theory collapses to the classical theory from the 1960s of generalized H^p spaces associated to 'weak* Dirichlet algebras', a class of abstract function algebras

• Thus Arveson's setting formally merges noncomm L^p spaces, with a classical abstract function algebra generalization of H^p spaces

- Arveson also included many interesting examples, showing that his framework synthesized several theories that were emerging in the 1960s (eg. Kadison-Singer nonselfadjoint op algs, ...)

• Arveson also included many interesting examples, showing that his framework synthesized several theories that were emerging in the 1960s (eg. Kadison-Singer nonselfadjoint op algs, ...)

• Contributors over the next decades include: Zsidó, Loebl, Muhly, McAsey, Saito (and his school in Japan), Exel, Marsalli, West, Pisier, Quanhua Xu, Ji, Nakazi, Watatani, Randrianantoanina

B $+$ Labuschagne - long series of papers

More recently: Ueda, Bekjan, Q Xu, Prunaru, Labuschagne, Ji, and many others $\cdots$

Some examples of subdiagonal algebras:

(i) If M is commutative, $\mathcal{D} = \mathbb{C}1$, subdiagonal algebras coincide with the classical weak* Dirichlet algebras

(ii) Upper triangular matrices in M_n, here Φ is the projection onto the diagonal matrices

(iii) (Arveson) Let G be a countable discrete group with a linear ordering which is invariant under left multiplication, say (for example, any free group). The subalgebra generated by G_+ in the group von Neumann algebra of G, gives a subdiagonal algebra

(iv) Mimic the construction of the 'hyperfinite II_1 factor', but with upper triangular matrices

(v) (Zsidó, Loebl-Muhly, Kawamura-Tomiyama) Let α be any one-parameter group of $*$-automorphisms of a von Neumann algebra M satisfying a certain ergodicity property (in particular, any of those arising in the Tomita-Takesaki theory)

In the last example, one gets a subdiagonal algebra $A \subset M$ from the elements of M with ‘spectrum with respect to α’ in $\mathbb{R}_+$

More on noncommutative L^p: If M acts on a Hilbert space H, let $\widetilde{M}$ be the set of unbounded, but closed and densely defined, operators on H which are affiliated to M (that is, $Tu = uT$ for all unitaries $u \in M'$).

• This is a $*$-algebra with respect to the 'strong' sum and product

• We can also use the functional calculus for selfadjoint unbounded operators, for example defining $|T|^p$, $\log |T|$, etc.

• The trace τ extends naturally to the positive operators in $\widetilde{M}$.

• If $1 \leq p < \infty$, then $L^p(M) = \{a \in \widetilde{M} : \tau(|a|^p) < \infty\}$, with norm $\| \cdot \|_p = \tau(| \cdot |^p)^{1/p}$

$L^\infty(M) = M$

$L^1(M)^* \cong M$ isometrically, via the map taking $T \in L^1(M)$ to the normal functional $\tau(T\,\cdot\,)$ on M.

- This isomorphism is an M-module map which respects the natural positive cones on these spaces

$L^\infty(M) = M$

$L^1(M)^* \cong M$ isometrically, via the map taking $T \in L^1(M)$ to the normal functional $\tau(T\,\cdot\,)$ on M.

• This isomorphism is an M-module map which respects the natural positive cones on these spaces

• A special role is played by $L^2(M)$ as you might expect.

This is a Hilbert space, which is just (the completion of) M with inner product $\tau(xy^*)$.

$L^\infty(M) = M$

$L^1(M)^* \cong M$ isometrically, via the map taking $T \in L^1(M)$ to the normal functional $\tau(T\,\cdot\,)$ on M.

- This isomorphism is an M-module map which respects the natural positive cones on these spaces

- A special role is played by $L^2(M)$ as you might expect.

This is a Hilbert space, which is just (the completion of) M with inner product $\tau(xy^*)$.

Note that M has two natural representations on $L^2(M)$, the left and the right regular representation

This is called the standard form—a major noncommutative tool

The Fuglede-Kadison determinant

Remarkable positive scalar valued function Δ defined on M or $L^p(M)$ (or larger domains):

$\Delta(a) = \exp \tau(\log |a|)$ if $|a| > 0$. Otherwise, $\Delta(a) = \inf \Delta(|a| + \epsilon 1)$, the infimum taken over all scalars $\epsilon > 0$.

- Alternatively,

$$\Delta(a) = \exp\Big(\int_0^\infty \log t \, d\nu_{|a|}(t)\Big),$$

where $d\nu_{|a|}$ is the probability measure on $\mathbb{R}_+$ which is the spectral measure of $|a|$ composed with the trace τ

$\Delta(\cdot)$ has the following properties:

(1) $\Delta(h) = \Delta(h^*) = \Delta(|h|)$.

(2) If $h \geq g$ in $L^p(M)_+$ then $\Delta(h) \geq \Delta(g)$.

(3) If $h \geq 0$ then $\Delta(h^q) = \Delta(h)^q$ for any $q > 0$.

(4) $\Delta(hk) = \Delta(h)\Delta(k) = \Delta(kh)$

- Proofs in L. G. Brown (1986), Haagerup-Schultz (2006)

$\Delta(\cdot)$ has the following properties:

(1) $\Delta(h) = \Delta(h^*) = \Delta(|h|)$.

(2) If $h \geq g$ in $L^p(M)_+$ then $\Delta(h) \geq \Delta(g)$.

(3) If $h \geq 0$ then $\Delta(h^q) = \Delta(h)^q$ for any $q > 0$.

(4) $\Delta(hk) = \Delta(h)\Delta(k) = \Delta(kh)$

- Proofs in L. G. Brown (1986), Haagerup-Schultz (2006)

We will also use singular functionals (that is, every nonzero projection dominates a nonzero projection in the kernel of the functional), and the nc Lebesgue decomposition: Functionals on a von Neumann algebra have a unique normal plus singular decomposition $\varphi = \varphi_n + \varphi_s$, and $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

A result that maps out some of the geography of the area:

Theorem (B-L) For such A, the following eight conditions are equivalent:

(i) A is subdiagonal (i.e. $\overline{A + A^*}^{w*} = M$)

(ii) A has factorization (i.e. $b \in M_+$, b invertible iff $b = a^*a$ for an invertible $a \in A$)

(iii) A is logmodular (similar to (ii) but $b = \lim_n |a_n|^2 ...$)

(iv) A satisfies the NC Szegö's formula above

(v) a) L^2-density of $A + A^*$ in $L^2(M)$; and b) if $g \in L^1(M)_+$, $\tau(fg) = \tau(f)$ for all $f \in A$, then $g = 1$

(vi) A Beurling-type invariant subspace condn: every A-invariant subspace of $L^p(M)$ has a decomposition of the form ... (related to Junge-Sherman 05, Nakazi-Watatani, and earlier work).

(vii) (Beurling-Nevanlinna factorization) Every $f \in L^p(M)$ such that $\Delta(f) > 0$ has an 'inner-outer factorization' $f = uh$, u unitary and h outer.

(viii) (Gleason-Whitney property) there exists at most one normal Hahn-Banach extension of any normal functional on A

A key step along our way, was the breakthrough in which my coauthor settled one of Arveson's 35 year old open problems:

Theorem (Labuschagne) Subdiagonal algebras satisfy Szegö's theorem and Jensen's inequality

(viii) (Gleason-Whitney property) there exists at most one normal Hahn-Banach extension of any normal functional on A

A key step along our way, was the breakthrough in which my coauthor settled one of Arveson's 35 year old open problems:

Theorem (Labuschagne) Subdiagonal algebras satisfy Szegö's theorem and Jensen's inequality

• The Fuglede-Kadison determinant and its properties is the most important ingredient of many of the proofs. Also use 'finite von Neumann algebra' techniques, unbounded operators and their functional calculus, noncommutative L^p-space tools, such as the standard form, etc.

Theorem (Generalized Jensen inequality)

$$\Delta(h) \geq \Delta(\Phi(h)), \qquad h \in H^1$$

- We also obtained generalizations of the classical results on outers, inner-outer factorization, etc

Eg. We have characterizations of outers in terms of $\Delta(h) = \Delta(\Phi(h)) > 0$, and e.g.

Theorem If $f \in L^p(M)$ then $\Delta(f) > 0$ iff $f = uh$ for a unitary u and a strongly outer $h \in H^p$. Moreover, this factorization is unique up to a unitary in $\mathcal{D}$.

($u \in A$ iff $f \in H^p$)

Theorem (F & M Riesz theorem) If a functional $\varphi \in M^*$ annihilates A then its absolutely continuous (normal) and singular parts separately annihilate A).

(The classical statement of the F & M Riesz theorem does not generalize, but the above is an equivalent statement that does generalize)

Theorem (F & M Riesz theorem) If a functional $\varphi \in M^*$ annihilates A then its absolutely continuous (normal) and singular parts separately annihilate A).

(The classical statement of the F & M Riesz theorem does not generalize, but the above is an equivalent statement that does generalize)

In our paper we required the hypothesis $\mathcal{D}$ finite dimensional in our F & M Riesz theorem. This hypothesis was later removed by Ueda, and similarly in the next result, as we shall discuss later

Theorem (F & M Riesz theorem) If a functional $\varphi \in M^*$ annihilates A then its absolutely continuous (normal) and singular parts separately annihilate A).

(The classical statement of the F & M Riesz theorem does not generalize, but the above is an equivalent statement that does generalize)

In our paper we required the hypothesis $\mathcal{D}$ finite dimensional in our F & M Riesz theorem. This hypothesis was later removed by Ueda, and similarly in the next result, as we shall discuss later

Gleason-Whitney property: We said that for a subdiagonal algebra A, there exists at most one normal Hahn-Banach extension to M of any normal functional on A. We also proved that if $\mathcal{D}$ finite dimensional, every Hahn-Banach extension to M of any normal functional on A, is normal.

Summary so far: The analytic principles in the classical theory are actually far more algebraic in nature than was even anticipated in the 1960s. That is, the results in the Hoffman et al algebraic approach to analyticity, all extend in an unusually literal manner to Arveson's class of noncommutative algebras in von Neumann algebras with a faithful normal tracial state. And in a way that merges the abstract H^p theory and vNAs/NC L^p

All the results in Srinivasan and Wang's 1966 survey, and all the relevant ones in Hoffman's seminal Acta paper have been transferred to von Neumann algebras with a faithful normal tracial state

Summary so far: The analytic principles in the classical theory are actually far more algebraic in nature than was even anticipated in the 1960s. That is, the results in the Hoffman et al algebraic approach to analyticity, all extend in an unusually literal manner to Arveson's class of noncommutative algebras in von Neumann algebras with a faithful normal tracial state. And in a way that merges the abstract H^p theory and vNAs/NC L^p

All the results in Srinivasan and Wang's 1966 survey, and all the relevant ones in Hoffman's seminal Acta paper have been transferred to von Neumann algebras with a faithful normal tracial state

What about if the von Neumann algebra M has no faithful normal tracial state?

Section 3. Noncommutative Hardy spaces for general von Neumann algebras

From early days it seemed like some of the above theory should be valid for more general von Neumann algebras than those having a faithful normal tracial state. For technical reasons though we should have some condition on M; and the relevant ones seem to be that M is σ-finite (i.e. countably decomposable), or semifinite.

Section 3. Noncommutative Hardy spaces for general von Neumann algebras

From early days it seemed like some of the above theory should be valid for more general von Neumann algebras than those having a faithful normal tracial state. For technical reasons though we should have some condition on M; and the relevant ones seem to be that M is σ-finite (i.e. countably decomposable), or semifinite.

σ-finite: Every collection of mutually orthogonal projections is at most countable. Equivalently, M has a faithful normal state (or even just a faithful state); or has a faithful normal representation possessing a cyclic separating vector. Any von Neumann algebra which is separably acting, or equivalently has separable predual M_*, is σ-finite.

semifinite: 1 is a sum of mutually orthogonal finite projections, or equivalently that every nonzero projection has a nonzero finite subprojection

- Ji, Ohwada, Saito, Xu, Labuschagne, Ueda, Bekjan, Sager, and others have done quite a bit of work on subdiagonal algebras in σ-finite and semifinite von Neumann algebras

- Ji, Ohwada, Saito, Xu, Labuschagne, Ueda, Bekjan, Sager, and others have done quite a bit of work on subdiagonal algebras in σ-finite and semifinite von Neumann algebras

- Perhaps the first thing to be said is that now there is no longer a Fuglede-Kadison determinant. Thus we do not know how to make sense of aspects like the characterization of outers above, the inner-outer factorization theorem, Szegö's theorem, etc

Nonetheless these authors have versions of much of the earlier theory in the faithful normal tracial state case

- E.g. very recently Labuschagne make several significant advances in the σ-finite case, most notably using Haagerup's reduction theory to do the Beurling invariant subspace theory (the easier Beurling theory in the semifinite case had been done slightly earlier). The joint work I am describing flowed out of this.

- E.g. very recently Labuschagne make several significant advances in the σ-finite case, most notably using Haagerup's reduction theory to do the Beurling invariant subspace theory (the easier Beurling theory in the semifinite case had been done slightly earlier). The joint work I am describing flowed out of this.

- Some main aspects that remained undone were the circle of results around the F & M. Riesz type theorem, Gleason-Whitney, etc, and Ueda's beautiful duality and peak set results

- E.g. very recently Labuschagne make several significant advances in the σ-finite case, most notably using Haagerup's reduction theory to do the Beurling invariant subspace theory (the easier Beurling theory in the semifinite case had been done slightly earlier). The joint work I am describing flowed out of this.

- Some main aspects that remained undone were the circle of results around the F & M. Riesz type theorem, Gleason-Whitney, etc, and Ueda's beautiful duality and peak set results

- Indeed these all rely on Uedas peak set result. We are able to show (later lecture) that Uedas peak set result is not provable in ZFC for all von Neumann algebras, not even for $M = l^\infty(\mathbb{R})$. So one cannot hope to do much more than σ-finite von Neumann algebras, by the heuristic argument that there may be nothing strictly between 'countable' and $\mathbb{R}$

So let us assume that M is σ-finite.

So let us assume that M is σ-finite. We can all breathe a sigh of relief, because we are now shielded from the worst terrors of von Neumann algebras, but still are extremely general. (Indeed...).

So let us assume that M is σ-finite. We can all breathe a sigh of relief, because we are now shielded from the worst terrors of von Neumann algebras, but still are extremely general. (Indeed...).

- What is $L^p(M)$ now? Not so simple: Haagerup L^p-spaces, have to deal with modular theory and the things Stefaan did in his Lecture 2 (modular automorphism group, Radon-Nikodym derivative of weights, etc)

In our case the latter is $h = \frac{d\nu}{d\tau_N} \in L^1(M)$ that appears everwhere, for a fixed fns weight ν, and canonical trace τ_N on a crossed product N of M

So let us assume that M is σ-finite. We can all breathe a sigh of relief, because we are now shielded from the worst terrors of von Neumann algebras, but still are extremely general. (Indeed...).

- What is $L^p(M)$ now? Not so simple: Haagerup L^p-spaces, have to deal with modular theory and the things Stefaan did in his Lecture 2 (modular automorphism group, Radon-Nikodym derivative of weights, etc)

In our case the latter is $h = \frac{d\nu}{d\tau_N} \in L^1(M)$ that appears everwhere, for a fixed fns weight ν, and canonical trace τ_N on a crossed product N of M

- What is H^p now? even worse: cant just take the closure of A in the $L^p(M)$ norm, have to take the closure of $h^{\frac{1}{2p}} A h^{\frac{1}{2p}}$. This makes things extremely annoying and surprisingly technical.

- Another technical point arises in the definition of maximal subdiagonal algebras, but this was sorted out by Quanhua Xu (2005), who also found the analogue of Haagerup's reduction theorem for the subalgebra

- Haagerup's reduction theory: From a σ-finite von Neumann algebra M one constructs a larger semifinite algebra R and a faithful normal conditional expectation $\Phi : R \to M$. Inside R one may then construct an increasing weak* dense sequence R_n of finite von Neumann algebras and expectations $\Phi_n : R \to R_n$ with $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

- Another technical point arises in the definition of maximal subdiagonal algebras, but this was sorted out by Quanhua Xu (2005), who also found the analogue of Haagerup's reduction theorem for the subalgebra

- Haagerup's reduction theory: From a σ-finite von Neumann algebra M one constructs a larger semifinite algebra R and a faithful normal conditional expectation $\Phi : R \to M$. Inside R one may then construct an increasing weak* dense sequence R_n of finite von Neumann algebras and expectations $\Phi_n : R \to R_n$ with $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

- In the same manner R is constructed from M, one constructs a subalgebra $\hat{A}$ of R from A, and $\hat{A}$ is a subdiagonal algebra in R. The subalgebras $\hat{A}_n = \hat{A} \cap R_n$ are an increasing sequence of subdiagonal subalgebras of the finite von Neumann algebras R_n, with the restriction of $\hat{\mathcal{E}}$ to R_n acting multiplicatively on $\hat{A}_n$, and mapping R_n onto $\hat{D} \cap R_n$. The union of the $\hat{A}_n$ are weak* dense in $\hat{A}$.

The following helps with Kaplansky density type results in unital operator spaces or operator systems.

Lemma Let M be a unital operator space. Let σ be any linear topology on M weaker than the norm topology, e.g. the weak or weak* topology (the latter if M is a dual space too). Let X be a subspace of M for which $\mathrm{Ball}(X)$ is dense in $\mathrm{Ball}(M)$ in the topology σ. Then $\{x \in X : x+x^* \geq 0\}$ is dense in $\{x \in M : x + x^* \geq 0\}$ in the topology σ.

Proof Suppose that $x \in M$ with $x + x^* \geq 0$. Then $z = x + \frac{1}{n}$ satisfies $z + z^* \geq 0$ and

$$z + z^* \geq \frac{2}{n} \geq C z^* z$$

for some constant $C > 0$. This implies that $C^2 z^* z - C(z + z^*) + 1 = (1 - Cz)^*(1 - Cz) \leq 1$. We may then approximate $1 - Cz$ in the topology σ by a net $x_t \in \mathrm{Ball}(X)$, and so $\frac{1}{C}(1 - x_t) \to z$ with respect to σ. Since $2 - x_t - x_t^* \geq 0$ we have shown that z is in the closure of $\{x \in X : x + x^* \geq 0\}$ in the topology σ. Hence so is x. $\square$

Theorem (Kaplansky density type) If A is a subdiagonal algebra in a von Neumann algebra M with a faithful state, then $\mathrm{Ball}(A + A^*)$ is weak* dense in $\mathrm{Ball}(M)$. Also, $(A + A^*)_+$ is weak* dense in M_+.

- Uses Haagerup's reduction theory as explained a few slides back. It is true for finite von Neumann algebras, and Haagerup's reduction theory gives a modus operandi to reduce to such.

Hilbert transform (in this generality due to Ji): roughly speaking for $f \in L^p(M)$ gives $f + iH(f) \in H^p(A)$

Hilbert transform (in this generality due to Ji): roughly speaking for $f \in L^p(M)$ gives $f + iH(f) \in H^p(A)$

- A σ-finite von Neumann algebra M has a convenient standard form. Indeed as we said, a characterization of σ-finite algebras is the existence of a normal faithful Hilbert space representation $\mathfrak{H}$ possessing a fixed cyclic and separating vector Ω. Then $\mathfrak{H}$ together with Ω is a 'standard form' for M. By the universality of the standard form we may identify $(\mathfrak{H}, \Omega)$ with $(L^2(M), h^{\frac{1}{2}})$, and work with the copy of M living inside $B(L^2(M))$ as multiplication operators, viewing $h^{\frac{1}{2}}$ as the fixed cyclic and separating vector.

Hilbert transform (in this generality due to Ji): roughly speaking for $f \in L^p(M)$ gives $f + iH(f) \in H^p(A)$

- A σ-finite von Neumann algebra M has a convenient standard form. Indeed as we said, a characterization of σ-finite algebras is the existence of a normal faithful Hilbert space representation $\mathfrak{H}$ possessing a fixed cyclic and separating vector Ω. Then $\mathfrak{H}$ together with Ω is a 'standard form' for M. By the universality of the standard form we may identify $(\mathfrak{H}, \Omega)$ with $(L^2(M), h^{\frac{1}{2}})$, and work with the copy of M living inside $B(L^2(M))$ as multiplication operators, viewing $h^{\frac{1}{2}}$ as the fixed cyclic and separating vector.

The above last many pages are some of the tools needed for our generalization of Ueda's peak set theorem

One formulation (without mentioning peak sets) of our generalization of Ueda's peak set theorem to subalgebras of σ-finite von Neumann algebras:

Generalized Ueda's peak set theorem Suppose A is a subdiagonal subalgebra of a σ-finite von Neumann algebra M. For any singular state φ on M, there is a sequence (p_n) of projections in Ker (φ) with sup in M^{**} in $A^{\perp\perp}$, and sup in M being 1.

One formulation (without mentioning peak sets) of our generalization of Ueda's peak set theorem to subalgebras of σ-finite von Neumann algebras:

Generalized Ueda's peak set theorem Suppose A is a subdiagonal subalgebra of a σ-finite von Neumann algebra M. For any singular state φ on M, there is a sequence (p_n) of projections in Ker (φ) with sup in M^{**} in $A^{\perp\perp}$, and sup in M being 1.

The proof is far too technical to describe here. It uses the tools described above, together with the same basic strategy as Ueda' proof of his case, but becomes enormously more complicated technically. We will describe Ueda' original proof tomorrow, emphasizing where real positivity comes in. I will also explain why this is a theorem about noncommutative peak sets and reformulate it as such, explaining what noncommutative peak sets are

- All the other consequences found by Ueda of his peak set theorem, now go through in our more general case. It is convenient to phrase this as follows:

If A is a weak* closed subalgebra of a von Neumann algebra M then we say that A is an Ueda algebra if Ueda's peak set theorem 'holds' for A.

Ueda's ideas then immediately give the following three generalizations of his beautiful results:

If A is a weak* closed subalgebra of a von Neumann algebra M then we say that A is an Ueda algebra if Ueda's peak set theorem 'holds' for A.

Ueda's ideas then immediately give the following three generalizations of his beautiful results:

Theorem Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. Write A_s^* and A_n^* for the set of restrictions to A of singular and normal functionals on M. Each $\varphi \in A^*$ has a unique Lebesgue decomposition relative to M: $\varphi = \varphi_n + \varphi_s$ with $\varphi_n \in A_n^*$ and $\varphi_s \in A_s^*$. Moreover, $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.

Corollary Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. Then the predual A_* of A is unique

Theorem (F. & M. Riesz type theorem) Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. If $\varphi \in M^*$ annihilates A (that is, $\varphi \in A^\perp$) then the normal and singular parts, φ_n and φ_s, also annihilate A.

One may define an F & M Riesz algebra to be a weak* closed subalgebra A of a von Neumann algebra M, such that if $\varphi \in A^\perp$ then the normal and singular parts, φ_n and φ_s, also annihilate A. The F & M type theorem above says that any Ueda algebra is an F & M Riesz algebra.

Theorem (F. $\&$ M. Riesz type theorem) Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. If $\varphi \in M^*$ annihilates A (that is, $\varphi \in A^\perp$) then the normal and singular parts, φ_n and φ_s, also annihilate A.

One may define an F $\&$ M Riesz algebra to be a weak* closed subalgebra A of a von Neumann algebra M, such that if $\varphi \in A^\perp$ then the normal and singular parts, φ_n and φ_s, also annihilate A. The F $\&$ M type theorem above says that any Ueda algebra is an F $\&$ M Riesz algebra.

By proofs in [B-Labuschagne] (but using the F $\&$ M type theorem above instead of our original one) we have:

Corollary Suppose that A is an F $\&$ M Riesz or Ueda algebra in a von Neumann algebra M such that $A + A^*$ is weak* dense in M. Any normal functional on M is the unique Hahn-Banach extension of its restriction to $A+A^*$, and in particular is normed by $A+A^*$. In addition, any Hahn-Banach extension to M of a weak* continuous functional on A, is normal.

The last assertion of the Corollary is related to the Gleason-Whitney theorem:

Lemma Suppose A is a weak* closed subalgebra of a von Neumann algebra M. Then $A+A^*$ is weak* dense in M iff there is at most one normal Hahn-Banach extension to M of any normal weak* continuous functional on A.

Corollary (Gleason-Whitney type theorem) Suppose that A is an F $\&$ M Riesz or Ueda algebra in a von Neumann algebra M. Then $A + A^*$ is weak* dense in M if and only if every normal functional on A has a unique Hahn-Banach extension to M. This extension is normal.

Of course by our main theorem all of these hold when A is a maximal subdiagonal subalgebra of a σ-finite von Neumann algebra M. Conversely these properties characterize maximal subdiagonal subalgebras (in terms of if and only if every normal functional on A has a unique normal Hahn-Banach extension to M).

Summary of the last 3 pages: Any algebra satisfying the conclusions of Ueda's peak set theorem also has the F $\&$ M Riesz, Gleason-Whitney, unique predual, Lebesgue decomposition, etc. It also satisfies the earlier Kaplansky density theorem

The Lebesgue decomposition result generalizes the Lebesgue decomposition theorem for functionals on a von Neumann algebra, the unique predual result simultaneously generalizes the Ando-Wojtaszczyk result that $H^\infty(\mathbb{D})$ has a unique predual, and the Dixmier-Sakai result that von Neumann algebras have unique predual