Corollary: If \( f: [a,b] \to \mathbb{R} \) is continuous, then its range \( f([a,b]) \) is an interval of form \([m_1, m_2]\).

*Proof:* By max-min theorem, if minimum \( m_1 \), maximum \( m_2 \) in \( f([a,b]) \). If \( m_1 < z < m_2 \) then by IVT, \( \exists c \in [a,b] \) s.t. \( f(c) = z \) So \( z \in f([a,b]) \). Thus \( f([a,b]) = [m_1, m_2] \)

Chapter 6 Differentiation

Section 25 The Derivative

**Def:** (Cal.1) If \( f: (a,b) \to \mathbb{R} \), \( c \in (a,b) \) and

\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \quad \text{then we say} \quad f \text{ is differentiable at } c,
\]

and \( f'(c) = L \). If \( f \) is differentiable at \( c \), \( \forall c \in (a,b) \), we say \( f \) is differentiable on \((a,b)\)

\( f: (a,b) \to \mathbb{R} \) in this chapter \( a < b \), \( c \in (a,b) \)

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{if this exists}
\]

**Theorem:**

If \( f \) is differentiable at \( c \) then \( f \) is continuous at \( c \).
Proof: \[ \lim_{x \to c} f(x) = \lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \right) (x - c) + f(c) \]

Rules for limits:
\[ \lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \right) = f'(c) \]
\[ \lim_{x \to c} (x - c) = 0 \]
\[ \lim_{x \to c} f(c) = f(c) \]

Thus \( f \) is continuous at \( c \).

Ex: Let \( f(x) = |x| \). Where is \( f \) differentiable?

Soh: Let's show \( f \) is not differentiable at \( 0 \). We will show
\[ \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist.} \]

To this, note \[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \to 0 \], but if \( g(x) = \frac{|x|}{x} \), then \( g(\frac{1}{n}) = 1 \to 1 \)

Also \[ -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots \to 0 \], but if \( g(-\frac{1}{n}) = \frac{-1}{n} \), then \( g(-\frac{1}{n}) = -1 \to -1 \)

So, by "Another consequence" on 10/27/04,
\[ \lim_{x \to 0} \frac{|x|}{x} \text{ does not exist.} \]

If \( c \neq 0 \), \( f(x) \) is differentiable at \( c \).
For example, on \((0, \infty)\), \( f(x) = x \) which has derivative 1.

* Theorem: (Cal. I)

Suppose \( f, g \) are differentiable at \( c \), then

(sum rule) \( f + g \) is differentiable at \( c \), and \( (f + g)'(c) = f'(c) + g'(c) \)

(constant multiple) \( kf \) is differentiable at \( c \), and \( (kf)'(c) = kf'(c) \) \( k \) a constant

(product rule) \( fg \) is differentiable at \( c \), and \( (fg)'(c) = f'(c)g(c) + f(c)g'(c) \)

(quotient rule) \( f/g \) is differentiable at \( c \), if \( g(c) \neq 0 \) and \( (f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2} \)
**Proof:** \[ \lim_{x \to c} \frac{f(x) + g(x) - (f(c) + g(c))}{x - c} = \lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c} \right) \]

\[ = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = f'(c) + g'(c) \]

The proof for the product rule is similar, and is left as easy exercise.

**Product Rule:** \[ \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \]

\[ = \lim_{x \to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} \]

\[ = \lim_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \cdot g(x) + \frac{g(x) - g(c)}{x - c} \cdot f(c) \right] = f'(c)g(c) + f(c)g'(c) \]

**Quotient Rule:** \[ \lim_{x \to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} = \lim_{x \to c} \frac{f(x)g(c) - f(c)g(x)}{x - c} \]

\[ = \lim_{x \to c} \frac{f(x)g(c) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{g(x)g(c)(x - c)} \]

\[ = \lim_{x \to c} \left( \frac{f(x) - f(c)}{x - c} \frac{g(x)}{g(c)} - \frac{g(x) - g(c)}{x - c} \frac{f(c)}{g(c)} \right) = \frac{f(c)g(c) - g'(c)f(c)}{(g(x))^2} \]

---

*Exercise:*
Theorem: (The Chain Rule)

Let \( f : (a, b) \to \mathbb{R} \), and \( g : (m, n) \to \mathbb{R} \), where \( f((a, b)) \subseteq (m, n) \).

If \( c \in (a, b) \), and \( f \) is differentiable at \( c \), and \( g \) is differentiable at \( f(c) \), then \( g \circ f \) is differentiable at \( c \), and

\[
(g \circ f)'(c) = g'(f(c)) f'(c)
\]

*Proof:* We need to show:

\[
\lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} = g'(f(c)) f'(c)
\]

By Theorem 20.8, it is enough to show that if \( (S_n) \) is a sequence, \( S_n \to c \), \( S_n \neq c \), then

\[
\frac{g(f(S_n)) - g(f(c))}{S_n - c}
\]

converges to \( g'(f(c)) f'(c) \)

We consider two cases:

**Case 1:**

\( f(S_n) = f(c) \) for infinite many numbers \( n \in \mathbb{N} \).

Then \( f'(c) = \lim_{n \to \infty} \frac{f(S_n) - f(c)}{S_n - c} = 0 \), since \( f(S_n) - f(c) = 0 \) for infinitely many values of \( n \), \( n \in \mathbb{N} \).

(Suppose \( f(S_{n_k}) = f(c) \) for all \( k \in \mathbb{N} \), \( n_1 < n_2 < \ldots \))

Then the subsequence \( \frac{f(S_{n_k}) - f(c)}{S_{n_k} - c} = 0 \to 0 \), so by 19.4, text

\[
\lim_{n \to \infty} \frac{f(S_n) - f(c)}{S_n - c} = 0
\]

On the other hand,

\[
\frac{g(f(S_{n_k})) - g(f(c))}{S_{n_k} - c} = \frac{g(f(c)) - g(f(c))}{S_{n_k} - c} = 0 \to 0
\]
If \( S_n \) is such that \( f(S_n) \neq f(c) \), then

\[
\frac{g(f(S_n)) - g(f(c))}{S_n - c} = \frac{g(f(S_n)) - g(f(c))}{f(S_n) - f(c)} \cdot \frac{f(S_n) - f(c)}{S_n - c} \rightarrow g'(f(c)) \cdot f'(c),
\]

by Theorem 20.8, applied to

\[
\lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} \quad \text{and to} \quad \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.
\]

1/10/04 Recall last time

Case 1:

\[ \exists \text{ infinitely many } n \in \mathbb{N} \text{ s.t. } f(S_n) = f(c) \]

Let \( \{n \in \mathbb{N} : f(S_n) = f(c)\} = \{n_1, n_2, n_3, \ldots\} \), \( n_1 < n_2 < n_3 \ldots \)

Subsequence \( S_{n_k} \rightarrow c \) (by 19.4 text). Thus

\[
\frac{f(S_{n_k}) - f(c)}{S_{n_k} - c} \rightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c), \quad \text{by 20.8} \quad \star
\]

Since \( f(S_{n_k}) = f(c) \), \( f'(c) = 0 \).

So we need to show \( a_n = \frac{g(f(S_n)) - g(f(c))}{S_n - c} \rightarrow 0 \)

Simple principle for sequences:

If \((a_n)\) is any sequence, and if \( J \) and \( K \) are two subsets of \( \mathbb{N} \) which are disjoint (\( J \cap K = \emptyset \)) and \( J \cup K = \mathbb{N} \), and if \( J \) is infinite, and if \( a_n = 0 \ \forall n \in K \), and if \( J = \{m_1, m_2, m_3, \ldots\}, m_1 < m_2 < m_3 < \ldots \), and if \( \lim_{i \to \infty} a_{m_i} = 0 \), then

\[ \lim_{n \to \infty} a_n = 0 \]
Picture: \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots \)

J. \( \frac{a_1}{2} \) \( \frac{a_2}{2} \) \( \frac{a_3}{2} \) \( \frac{a_4}{2} \) \( \frac{a_5}{2} \) \( \frac{a_6}{2} \) \( \frac{a_7}{2} \) \( \frac{a_8}{2} \) \( \ldots \) \( \rightarrow 0 \)

2 parts \( K \) break into \( a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots \)

Conclusion: \( a_n \rightarrow 0 \) as \( n \rightarrow \infty \)

In our case, \( K = \{n \in \mathbb{N} : f(S_n) = f(c)\} = \{n_1, n_2, n_3, \ldots \} \) above

Clearly \( a_n = \frac{g(f(S_n)) - g(f(c))}{S_n - c} = 0 \quad \forall n \in K \)

If \( J = \mathbb{N} \setminus K = \{m_1, m_2, m_3, \ldots \}, m_1 < m_2 < m_3 < \ldots \), then

\[ S_{m_i} \rightarrow c \text{ as } i \rightarrow \infty \quad (\text{by 19.4}) \]

and

\[ a_{m_i} = \frac{g(f(S_{m_i})) - g(f(c))}{S_{m_i} - c} = \frac{g(f(S_{m_i}) - g(f(c) \cdot f(S_{m_i}) - f(c))}{S_{m_i} - c} \quad \frac{S_{m_i} - c}{f'(c)} = \frac{S_{m_i} - c}{f'(c)} \]

To see (**), let \( t_i = f(S_{m_i}) \), then \( t_i \rightarrow f(c) \), since \( f \) is continuous at \( c \), and using 20.8

Thus \( \frac{g(t_i) - g(f(c))}{t_i - c} \rightarrow \lim_{y \rightarrow f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)) \), by 20.8 (very similar to *)

By "simple principle" above, \( a_n \rightarrow 0 \) as required. This ends proof of Case 1.
Case 2: If only finitely many \( n \in \mathbb{N} \) s.t. \( f(S_n) = f(c) \)

We need to show that

\[
q_n = \frac{g(f(S_n)) - g(f(c))}{S_n - c} \to g'(f(c))f'(c), \text{ just as before}
\]

We don't care about first few terms of a sequence (fact 7).
Thus we can assume \( f(S_n) \neq f(c) \) \( \forall n \in \mathbb{N} \)

The rest of the proof is almost identical to last part of
Step 1. Since \( S_n \to c \) as \( n \to \infty \) (by 19.4)

and \( q_n = \frac{g(f(S_n)) - g(f(c))}{S_n - c} = \frac{g(f(S_n)) - g(f(c))}{f(S_n) - f(c)} \cdot \frac{f(S_n) - f(c)}{S_n - c} \to g'(f(c))f'(c) \)

\( \Rightarrow g'(f(c))f'(c) \)

by same argument as:*
Section 26

THE Mean Value THEOREM

**Theorem** (1st derivative Cal. I test)
If \( f: (a, b) \to \mathbb{R} \) is differentiable, and if \( c \in (a, b) \) and if \( f(x) \) has a local maximum or local minimum at \( c \), then \( f'(c) = 0 \).

**Proof:** By shrinking \( (a, b) \), we can remove the word "local". Suppose \( f(c) \) is a maximum (leave minimum case as exercise). Then

\[
\lim_{{x \to c^+}} \frac{f(x)-f(c)}{x-c} \leq 0 \quad \text{(look at 20.12, with } a = 0) \]

Similarly

\[
\lim_{{x \to c^-}} \frac{f(x)-f(c)}{x-c} \geq 0 \quad \text{(both numerator & denominator are } \leq 0) \]

But

\[
\lim_{{x \to c^-}} = \lim_{{x \to c^+}} = \lim_{{x \to c}} \quad \text{if the limit exists, (Ex. 20.18)}
\]

so

\[
\lim_{{x \to c}} \frac{f(x)-f(c)}{x-c} = 0 = f'(c)
\]

11/12/04

(Rolle's Theorem):

If \( f: [a, b] \to \mathbb{R} \) is continuous, differentiable on \((a, b)\) and \( f(a) = f(b) = 0 \) then \( \exists \ c \in (a, b) \) s.t. \( f'(c) = 0 \).

**Proof:**

Case 1: \( f \) is constant. This case is obvious. (The derivative of a constant is 0)
\[ h'(c) = f'(c) \frac{b - a}{b - a}, \text{ so } h'(c) = 0. \]
Example: Prove Bernoulli's inequality

\[(1+x)^n \geq 1+nx, \forall n \in \mathbb{N}, \forall x \geq 0\]

Solution: Let \(f(t) = (1+t)^n, \forall t \in [0,x]\)

By MVT applied to \(f\) on \([0,x]\), \(\exists c \in (0,x)\) s.t.

\[f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{(1-x)^n}{x}\]

But \(f'(c) \text{ by chain rule} = n(1+c)^{n-1}\), so \(\frac{(1-x)^n}{x} \geq n(1+c)^{n-1} \Rightarrow n^{n-1}\)

Multiplying by \(x\) and adding 1 gives \((1+x)^n \geq nx + 1\)

**Fact 1: (Calculus)**

If \(f'(x) = 0\), \(\forall x \in (a,b)\) then \(f(x)\) is a constant on \((a,b)\).

*Proof*

If \(a < x < y < b\), then by MVT applied to \(f\) on \([x,y]\), \(\exists c \in (x,y)\) s.t.

\[f(y) - f(x) = f'(c)(y - x) = 0\]

Thus \(f(x) = f(y), \forall x < y \text{ in } (a,b)\) So \(f\) constant.

**Fact (Calculus)**

If \(f'(x) = g'(x), \forall x \in (a,b)\), then \(\exists c \text{ s.t.}\)

\[f(x) = g(x) + C, \forall x \in (a,b)\]

*Proof*

Let \(h(x) = f(x) - g(x)\). Then \(h'(x) = f'(x) - g'(x), \forall x \in (a,b)\), so by last fact, \(\exists c \text{ s.t.} h(x) = C\). Thus \(f(x) = g(x) + C\)
Fact (cal.1)

If $f'(x) > 0 \ \forall x \in (a, b)$ then $f$ is strictly increasing on $(a, b)$

$x < y \Rightarrow f(x) < f(y)$

If $f'(x) > 0 \ \forall x \in (a, b)$ then $f$ is increasing on $(a, b)$

$x \leq y \Rightarrow f(x) \leq f(y)$

If $f'(x) < 0 \ \forall x \in (a, b)$ then $f$ is strictly decreasing on $(a, b)$

$x < y \Rightarrow f(x) > f(y)$

If $f'(x) < 0 \ \forall x \in (a, b)$ then $f$ is decreasing on $(a, b)$

$x \leq y \Rightarrow f(x) \geq f(y)$

Proof:

I'll just do first, you can do others.

If $f'(x) > 0$ on $(a, b)$, and if $a < x < y < b$, then by MVT

on $[x, y]$, $f(y) - f(x) = f'(c)(y-x) > 0$ for some $c \in (x, y)$

Thus $f(y) > f(x)$

Final result in this section:

The inverse function theorem (cal.2)

If $f: (a, b) \to \mathbb{R}$, and if $f'(x) \neq 0 \ \forall x \in (a, b)$ then

$f$ is one-to-one on $(a, b)$, $f((a, b)) = (c, d)$ for numbers $c, d$

and

$(f^{-1})'(y) = \frac{1}{f'(x)} \ \forall y \in (c, d), \ y = f(x)$

$f((a, b)) = \{f(x): x \in (a, b)\}$

Read handout!
Instructions. Show all working and reasoning, the points are almost all for logical, complete reasoning. [Approximate point values are given, total = 100 points plus 15 bonus points].

1. Prove that a decreasing bounded sequence \((a_n)\) converges to \(\inf_n a_n\).

Let \(\alpha = \inf_n a_n\), and suppose \(\varepsilon > 0\) is given and \(a_n < \alpha + \varepsilon\).

Then \(\alpha - \varepsilon < a_n \leq \alpha < a_n < \alpha + \varepsilon + \varepsilon\).

So \(\alpha - \varepsilon < a_n < \alpha + \varepsilon + \varepsilon\).

2. (a) What is the definition of a Cauchy sequence?

(b) Suppose that \((s_n)\) is a sequence with \(|s_{n+1} - s_n| \leq \frac{1}{2^n}\) for all \(n \in \mathbb{N}\). Show that \((s_n)\) is a Cauchy sequence.

(c) Is the sequence in (b) convergent? Why?

\[(a)\] \(\forall \varepsilon > 0 \exists N \text{ s.t. } m, n \geq N \Rightarrow |a_m - a_n| < \varepsilon.\)

\[(b)\] \(|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \ldots + s_{n+1} - s_n| \\
\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \ldots + |s_{n+1} - s_n| \\
\leq \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \ldots + \frac{1}{2^n} \quad \text{(Geometric series)} \\
< \frac{1}{2^{m-1}} \cdot 2 = \frac{1}{2^{m-1}} \to 0 \text{ as } n \to \infty.\]

\[(c)\] Yes, it is Cauchy.

3. Here \(f : D \to \mathbb{R}\), and \(c\) is an accumulation point of \(D\). Mark each statement True or False. If it is true, give a simple reason. If it is false, give a counterexample (you don’t need to show that it is a counterexample).

(a) Every sequence of real numbers has a convergent subsequence.

(b) If \(\lim_{x \to c} f(x) = L\) then there is a sequence \((s_n)\) in \(D\) which converges to \(c\), but \((f(s_n))\) does not converge to \(L\).

(c) If \(f : D \to \mathbb{R}\) is continuous and bounded on \(D\), then \(f(x)\) has a maximum and a minimum value on \(D\).

\[(a)\] False, consider the sequence \(1, 2, 3, 4, \ldots.\)

\[(b)\] True, this is the contrapositive of \(20.8.\)

\[(c)\] False, consider \(f(x) = x\) on \((0, 1)\).
4. Suppose that \( f : (a, b) \to \mathbb{R} \), \( g : (a, b) \to \mathbb{R} \), \( L \in \mathbb{R} \), and \( a < c < b \).

(a) Prove that if \( \lim_{x \to c} f(x) = L \) then whenever \( (s_n) \) is a sequence in \( (a, b) \setminus \{c\} \) with \( \lim_n s_n = c \), then \( \lim_n f(s_n) = f(c) \).

(b) Prove that if \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M \), then \( \lim_{x \to c} f(x)g(x) = LM \).

(c) Prove that if \( \lim_{x \to c} f(x) = 0 \), and if there is a constant \( M \) such that \( |g(x)| \leq M \) for all \( x \in (a, b) \), then \( \lim_{x \to c} f(x)g(x) = 0 \).

5. (a) Give the \( \varepsilon - \delta \) definition for a function \( f : (a, b) \to \mathbb{R} \) to be continuous at a point \( c \in (a, b) \).

(b) List as many other conditions as you know that are equivalent to \( f : (a, b) \to \mathbb{R} \) being continuous at \( c \in (a, b) \).

(c) Using the \( \varepsilon - \delta \) definition, show that the function \( x^2 + 1 \) is continuous at \( x = -1 \).
Chapter 7
Integration
Section 29 The Riemann Integral

We define and study the Riemann Integral
\[ \int_a^b f(x) \, dx \] (or \[ \int_a^b f \, dx \]) of a function \( f: [a, b] \to \mathbb{R} \) which is bounded (i.e., there exist constants \( m, M \) s.t. \( m \leq f(x) \leq M \) \( \forall x \in [a, b] \)).

Recall from Calculus I, that a partition \( P \) of \([a, b]\) is a set \( P = \{x_0, x_1, x_2, \ldots, x_n\}\) where \( a = x_0 < x_1 < x_2 < \ldots < x_n = b \).

Write \( \mathcal{P} \) for the set of all partitions of \([a, b]\).

The upper sum \( U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k \)

lower sum \( L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k \), where

\[ \Delta x_k = x_k - x_{k-1} \]

\[ m_k = \inf \{ f(x) : x \in [x_{k-1}, x_k] \} \]

\[ M_k = \sup \{ f(x) : x \in [x_{k-1}, x_k] \} \]

Note:

\[ m(b-a) = \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k = L(f, P) \leq U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k \leq M(b-a) \]

From \( \mathcal{P} \), \( \{ L(f, P) : P \in \mathcal{P} \} \) and \( \{ U(f, P) : P \in \mathcal{P} \} \) are bounded, so have inf and sup's.
We that is Riemann case write

\[ L(f, P) \leq L(f, p), \quad p \in P \]

Since \( \neq \) is not

A

\[ \leq a \]

then \( b < a \)

\[ M \forall x \in Q \]

\[ Q \subseteq P \]

Ex 0 1 0 1

\[ Q \quad P \text{then} \quad Q \]
Proof: Note \( Q \) is simply \( P \) with a few points added.

Case 1: \( Q \) is \( P \) with one point added.

So \( P = \{x_0, x_1, x_2, \ldots, x_n\} \), \( Q = \{x_0, x_1, x_2, \ldots, x_{k-1}, x^*, x_k, \ldots, x_n\} \)

\((x^* \text{ is point added})\)

\[ L(f, P) = \frac{1}{n} \sum_{i=1}^{n} m_i \Delta x_i = \left( \frac{1}{k-1} \sum_{i=1}^{k-1} m_i \Delta x_i \right) + m_k \Delta x_k + \left( \frac{1}{n} \sum_{i=k+1}^{n} m_i \Delta x_i \right) \]

Now \( m_k = \inf \{ f(x) : x \in [x_{k-1}, x_k] \} \)

so \( m_k \Delta x_k = \inf \{ f(x) : x \in [x_{k-1}, x_k] \} \left( (x^* - x_{k-1}) + (x_k - x^*) \right) \)

\[ = \inf \{ f(x) : x \in [x_{k-1}, x_k] \} (x^* - x_{k-1}) + \inf \{ f(x) : x \in [x_{k-1}, x_k] \} (x_k - x^*) \]

\[ \leq \inf \{ f(x) : x \in [x_{k-1}, x^*] \} (x^* - x_{k-1}) + \inf \{ f(x) : x \in [x^*, x_k] \} (x_k - x^*) \]

(using this fact \( S_1 \subseteq S_2 \subseteq \mathbb{R} \Rightarrow \inf(S_2) \leq \inf(S_1) \))

Hence \( L(f, P) \leq \left( \frac{1}{k-1} \sum_{i=1}^{k-1} m_i \Delta x_i \right) + \inf \{ f(x) : x \in [x_{k-1}, x^*] \} (x^* - x_{k-1}) \)

\[ + \inf \{ f(x) : x \in [x^*, x_k] \} (x_k - x^*) + \left( \frac{1}{n} \sum_{i=k+1}^{n} m_i \Delta x_i \right) \]

\[ = L(f, Q) \]

By a similar argument, \( U(f, Q) \leq U(f, P) \).

By \( \circ \), \( L(f, Q) \leq U(f, Q) \).

Case 2: Use case 1 repeatedly, for example, if \( Q \) is \( P \) with 2 points \( x^* \) and \( y^* \) added.

Let \( Q' = PU \{x^*\} \), then by case 1 applied twice,

\[ L(f, P) \leq L(f, Q') \leq L(f, Q) \leq U(f, Q) \]

By case 1 applied twice more, \( U(f, Q) \leq U(f, Q') \leq U(f, Q) \)
Corollary: \( P, Q \in \mathcal{P} \Rightarrow L(f, P) \leq U(f, Q) \)

*proof:* Let \( R = P \cup Q \), then \( R \) refines both \( P \) and \( Q \), and so by Lemma, \( L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q) \).

Corollary 2: For any bounded function \( f: [a, b] \rightarrow \mathbb{R} \), \( U(f) \geq L(f) \).

*proof:* By last corollary we have
\[
L(f) = \sup \{ L(f, P) : P \in \mathcal{P} \} \leq U(f, Q), \text{ if } Q \in \mathcal{P}
\]

\( L(f) \) is a lower bound for \( \{ U(f, Q) : Q \in \mathcal{P} \} \).

Taking infimum over \( Q \), \( L(f) \leq U(f) \).  

Smallest lower bound of the \( U(f, Q) \).