Lecture 2

Injective and ternary envelopes and multipliers of operator spaces

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I. The injective envelope.

A *ternary ring of operators* or *TRO* is a closed subspace Z of $B(K, H)$ (or of a C^*-algebra), with $ZZ^\star Z \subset Z$.

Example: $pA(1 - p)$, for a C^*-algebra A and a projection p in A (or in $M(A)$).

A *subTRO* of a TRO Z is a closed subspace Y with $YY^\star Y \subset Y$.

A *triple morphism* between TROs is a linear map with $T(xy^*z) = T(x)T(y)^*T(z) \ \forall x, y, z$.

TROs behave exactly like C*-algebras, and triple morphisms behave exactly like $*$-homomorphisms.

An operator space Z is *injective* if for any c. contractive $u\colon X \to Z$ and for any operator space Y containing X as a closed subspace, there exists a c. contractive extension $\widehat{u}\colon Y \to Z$ such that $\widehat{u}_{|X} = u$.

Theorem (Wittstock) If H and K are Hilbert spaces then $B(K, H)$ is an injective operator space.

A map $\Phi : X \to X$ is idempotent if $\Phi \circ \Phi = \Phi$

Exercise: Show that an operator space is injective iff it is linearly c. isometric to the range of a completely contractive idempotent map on $B(H)$, for some Hilbert space H.

An *extension* of an op. space X is an op. space Y, together with a linear completely isometric map $i \colon X \to Y$.

Say Y is a *rigid* extn. of X if I_Y is the only linear completely contractive map $Y \to Y$ which restricts to the identity map on $i(X)$.

Say Y is an *essential* extension of X if whenever $u \colon Y \to Z$ is a c. contractive map into another operator space Z such that $u \circ i$ is a complete isometry, then u is a complete isometry.

Say that (Y, i) is an *injective envelope* of X if Y is injective, and if there is no injective subspace of Y containing $i(X)$.

Lemma Let (Y, i) be an extension of an operator space X such that Y is injective. The following are equivalent:

(i) Y is an injective envelope of X,

(ii) Y is a rigid extension of X,

(iii) Y is an essential extension of X.

Lemma If (Y_1, i_1) and (Y_2, i_2) are two injective envelopes of X, then there exists a surjective c. isometry $u \colon Y_1 \to Y_2$ such that $u \circ i_1 = i_2$.

Theorem If an operator space X is contained in an injective operator space W, then there is an injective envelope Y of X with $X \subset Y \subset W$.

Hence every operator space has an injective envelope. We write it as $(I(X), j)$, or $I(X)$ for short. It is essentially unique by the above.

If X is a unital operator space, then $I(X)$ can be taken to be a unital C^*-algebra, and j is a unital map.

This is because if $I_H \in X \subset B(H)$, then by the above there is an injective envelope R of X with $X \subset R \subset B(H)$, and a c. contractive idempotent map Φ from $B(H)$ onto R. By a fact from Lecture 1, Φ is completely positive, and by the Choi-Effros result, R is a C^*-algebra with a new product.

An important construction:

Fix an operator space $X \subset B(H)$, and consider the Paulsen system $\mathcal{S}(X) \subset M_2(B(H))$

As we said earlier, there is a completely positive idempotent map Φ on $M_2(B(H))$ whose range is an injective envelope $I(\mathcal{S}(X))$ of $\mathcal{S}(X)$. Also, $I(\mathcal{S}(X))$ is a unital C^*-algebra in a new product

Write p and q for the canonical projections $I_H \oplus 0$ and $0 \oplus I_H$. Since $\Phi(p) = p$ and $\Phi(q) = q$, it follows from a fact at the end of Lecture 1 that $\Phi(px) = p\Phi(x)$, $\Phi(xp) = \Phi(x)p$, etc. That is, Φ is 'corner-preserving'.

The '1-2-corner' P of Φ is an idempotent map on $B(H)$.

By definition of the new product, it is clear that p and q are complementary projections in the C^*-algebra $I(\mathcal{S}(X))$. With respect to these projections, $I(\mathcal{S}(X))$ may be viewed as consisting of 2×2 matrices. Let $I_{kl}(X)$, or simply I_{kl}, denote its 'k-l-corner', for $k, l = 1, 2$.

Thus I_{11} is the unital C^*-algebra $pI(\mathcal{S}(X))p$, I_{22} is $(1-p)I(\mathcal{S}(X))(1-p)$, and $I_{12} = pI(\mathcal{S}(X))(1-p) = \mathsf{Ran}(P)$.

Write J for the canonical map from X into $I_{12}(X)$. We have:

$$X \hookrightarrow \mathcal{S}(X) = \begin{bmatrix} \mathbb{C} & X \\ X^\star & \mathbb{C} \end{bmatrix}$$

$$\hookrightarrow I(\mathcal{S}(X)) = \begin{bmatrix} I_{11}(X) & I_{12}(X) \\ I_{21}(X) & I_{22}(X) \end{bmatrix}.$$

Note that the 'corner' $Z = I_{12}(X)$ is a I_{11}-I_{22}-bimodule. It is also a TRO, since

$$ZZ^\star Z \subset Z, \ ZZ^\star \subset I_{11}, \ \text{and} \ Z^\star Z \subset I_{22}.$$

Thus X inherits a natural C^*-algebra valued 'inner product'

$$\langle x, y \rangle = j(x)^* j(y) \in I_{22}, \qquad x, y \in X$$

We call this the 'Shilov inner product'

Theorem (Hamana–Ruan) If X is an operator space, then $Z = I_{12}(X)$ is an injective envelope of X.

Proof We suppose that $X \subset B(H)$, and we use the notation established above. Clearly $Z = pA(1 - p)$ is injective. Let $v \colon Z \to Z$ be a completely contractive map extending the identity map on $J(X)$. We need to show that $v = I_Z$. By Paulsen's lemma v gives rise to a canonically associated map on $\mathcal{S}(Z)$, the latter viewed as a subset of A. Since A is injective, we may extend further to a complete contraction Ψ from A to itself. Note that the restriction of Ψ to $\mathcal{S}(X)$ is the identity map. By the 'rigidity property', both Ψ and v are the identity map. $\square$

Corollary (Hamana–Ruan) An operator space X is injective if and only if $X \cong pA(1-p)$ completely isometrically, for a projection p in an injective C^*-algebra A.

Proof If X is injective, then in notation of the last few slides,

$$I(X) = pI(\mathcal{S}(X))(1-p)$$

But if X is injective then it equals its injective envelope $\square$

Remark. The C^*-algebras $I_{11}, I_{22}, I(\mathcal{S}(X))$ do not depend on a particular embedding $X \subset B(H)$.

Corollary (Hamana–Kirchberg–Ruan)   A surjective complete isometry between TROs is a triple morphism.

Proof A surjective complete isometry $u \colon X \to Y$ gives, by Paulsen's lemma, a c. isometric unital isomorphism between the operator systems $\mathcal{S}(X)$ and $\mathcal{S}(Y)$.

This isomorphism extends, by an earlier result, to a completely isometric unital surjection θ between $I(\mathcal{S}(X))$ and $I(\mathcal{S}(Y))$.

By a fact at the end of Lecture 1, θ is a $*$-isomorphism.

Since $\theta(1 \oplus 0) = 1 \oplus 0$, θ is 'corner-preserving'.

Since the 1-2-corner θ_{12} of θ is the restriction of θ to a subtriple, it is a triple morphism. However, $\theta_{12} = u$ on the copy of X. $\square$

The triple envelope or noncommutative Shilov boundary

$X \rightsquigarrow \mathcal{T}(X)$, a TRO or C*-module

The triple envelope of an op. space X is a pair $(\mathcal{T}(X), j)$ consisting of a TRO $\mathcal{T}(X)$ and a c. isometry $j : X \to \mathcal{T}(X)$ such that for any other TRO Z and c. isometry $i : X \to Z$, there exists a triple morphism $\theta : Z \to \mathcal{T}(X)$ such that $\theta \circ i = j$.

(Here we are only considering c. isometries $i : X \to Z$ s.t. $\nexists$ nontrivial subTRO of Z containing $i(X)$)

So $\mathcal{T}(X)$ is the *smallest* TRO that containing X.

Theorem (Hamana) If $(I(X), j)$ is an injective envelope of X, then the smallest subTRO of $I(X)$ containing $j(X)$, is a triple envelope of X. So can take $\mathcal{T}(X) \subset I(X)$.

II. The left multiplier algebra of an operator space X

$X \rightsquigarrow \mathcal{M}_\ell(X)$, a unital operator algebra

$X \rightsquigarrow \mathcal{A}_\ell(X)$, a unital C*-algebra

These algebras consist of operators on X, and $\mathcal{A}_\ell(X)$ is a C^*-subalg. of $\mathcal{M}_\ell(X)$

By an *operator algebra* we mean a closed subalgebra of $B(H)$. It is *unital* if it has an identity of norm 1 (or equiv. $I_H \in A$).

Theorem (B-Ruan-Sinclair) Up to completely isometric isomorphism, unital operator algebras are precisely the operator spaces A which is a unital algebra whose product satisfies $\|ab\|_n \leq \|a\|_n \|b\|_n$ for all $a, b \in M_n(A), n \in \mathbb{N}$.

Examples

(1) If A is an approx. unital operator algebra, then $\mathcal{M}_l(A)$ is the usual left multiplier algebra $LM(A)$. If A is a C*-algebra, then $\mathcal{A}_l(A) = M(A)$.

(2) If E is a Banach space, then $\mathcal{M}_l(MIN(E))$ coincides with the classical 'function multiplier algebra' $\mathcal{M}(E)$, whereas $\mathcal{A}_l(MIN(E))$ is the classical 'centralizer algebra', studied in the 60s and 70s.

(3) If Z is a right Hilbert C^*-module, then we shall show later that $\mathcal{M}_l(Z)$ and $\mathcal{A}_l(Z)$ are respectively the algebras of bounded right module maps, and adjointable maps, on Z.

- Although $\mathcal{M}_l(X)$ and $\mathcal{A}_l(X)$ are defined purely in terms of the matrix norms and vector space structure on X, they often encode 'operator algebra structure'.

- Operator spaces with trivial multiplier algebras are exactly the spaces lacking 'operator algebraic structure' in a sense which one can make precise.

We define a *left multiplier* of X to be a linear map $u \colon X \to X$ such that there exists a Hilbert space H, an $S \in B(H)$, and a linear complete isometry $\sigma \colon X \to B(H)$ with $\sigma(ux) = S\sigma(x)$ for all $x \in X$.

The *multiplier norm* of u, is the infimum of $\|S\|$ over all possible H, S, σ as above. We define $\mathcal{M}_l(X)$ to be the set of left multipliers of X.

Notice that we may replace the $B(H)$ in the definition of $\mathcal{M}_l(X)$ by an arbitrary C^*-algebra.

Set $C_2(X) = M_{2,1}(X)$

For a linear $u \colon X \to X$, define $\tau_u \colon C_2(X) \to C_2(X)$ to be the map $u \oplus I_X$; that is:

$$\tau_u\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u(x) \\ y \end{bmatrix}, \qquad x, y \in X.$$

'Left multipliers' are best viewed as a sequence of equivalent definitions, as in the following theorem. To explain the notation in this result: in (iii)–(v), we are viewing $X \subset \mathcal{T}(X) \subset I(X) \subset I(\mathcal{S}(X))$ as above. Here $\mathcal{T}(X)$ is the triple envelope.

Theorem (B, B-Effros-Zarikian, B-Paulsen)
Let $u\colon X \to X$ be a linear map. The following are equivalent:

(i) u is a left multiplier of X with multiplier norm ≤ 1.

(ii) τ_u is completely contractive.

(iii) There exists an $a \in I_{11}(X)$ of norm ≤ 1, such that $u(x) = ax$ for all $x \in X$.

(iv) u is the restriction to X of a contr. right module map a on $\mathcal{T}(X)$ with $a(X) \subset X$.

(v) u is the restriction to X of a contr. right module map a on a C^*-module Z containing X, with $a(X) \subset X$.

(vi) $[\langle u(x_i)|u(x_j)\rangle] \leq [\langle x_i|x_j\rangle]$, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$.

(The matrices in (v) are indexed on rows by i, and on columns by j.

The 'inner product' in (v) is the Shilov inner product.

Lets just prove one of the implications here:

(i) $\Rightarrow$ (ii) Let σ, S, H be chosen as in the definition of $\mathcal{M}_l(X)$ above. Then for $x, y \in X$, we have

$$\left\| \tau_u \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) \right\| = \left\| \begin{bmatrix} \sigma(ux) \\ \sigma(y) \end{bmatrix} \right\| = \left\| \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \sigma(x) \\ \sigma(y) \end{bmatrix} \right\|$$

$$\leq \max\{\|S\|, 1\} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|.$$

Thus $\|\tau_u\| \leq \max\{\|S\|, 1\}$. A similar argument shows that $\|\tau_u\|_{\mathsf{cb}} \leq \max\{\|S\|, 1\}$. Then (ii) follows by the definition of the 'multiplier norm'.

We define matrix norms on $\mathcal{M}_\ell(X)$ by

$$M_n(\mathcal{M}_\ell(X)) \cong \mathcal{M}_\ell(M_n(X))$$

Theorem If X is an operator space, then the 'multiplier norms' defined above are norms. With these matrix norms $\mathcal{M}_l(X)$ is an operator algebra.

Idea of proof The last theorem shows that $\mathcal{M}_l(X) \cong \{a \in I_{11}(X) : aX \subset X\}$, a subalgebra of the C*-algebra $I_{11}(X)$. $\square$

The inclusion map from $\mathcal{M}_l(X)$ to $CB(X)$ is a one-to-one c. contractive homomorphism.

We write $\mathcal{A}_l(X)$ for the C*-algebra $\mathcal{M}_l(X) \cap \mathcal{M}_l(X)^*$.

The operators in $\mathcal{A}_l(X)$ are called *left adjointable multipliers*

Theorem Let X be an operator space, and $u \colon X \to X$ a linear map. The following are equivalent:

(i) $u \in \mathcal{A}_l(X)$.

(ii) There exists an $a \in I_{11}(X)$, s.t. $u(x) = ax$ for all $x \in X$, and s.t. $a^* X \subset X$.

(iii) u is the restriction to X of an adjointable module map a on $\mathcal{T}(X)$ with $a(X) \subset X$ and $a^*(X) \subset X$.

(iv) There exist a Hilbert space H, an $S \in B(H)$, and a linear complete isometry $\sigma \colon X \to B(H)$ with $\sigma(ux) = S\sigma(x)$ for all $x \in X$, and such that also $S^*\sigma(X) \subset \sigma(X)$.

(v) There exists a linear complete isometry σ from X into a C^*-algebra, and a map $R \colon X \to X$, such that $\sigma(T(x))^*\sigma(y) = \sigma(x)^*\sigma(R(y))$, for $x, y \in X$.

(vi) There exists a map $R \colon X \to X$, such that

$$\langle\, T(x) \,|\, y \,\rangle \;=\; \langle\, x \,|\, R(y) \,\rangle, \qquad x, y \in X.$$

The 'inner product' in (vi) is the Shilov inner product

The canonical inclusion map from $\mathcal{A}_l(X)$ into $CB(X)$ (or into $B(X)$) is an isometric homomorphism.

If X is an operator space, then a linear idempotent $P \colon X \to X$ is a *left* M-*projection* if the map

$$\sigma_P(x) = \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix}$$

is a complete isometry from $X \to C_2(X)$.

We say that a subspace J of an operator space X is a right M-ideal if $J^{\perp\perp} = P(X^{**})$ for a left M-projection P on X^{**}.

We'll say more about M-ideals later.

Theorem (B-Effros-Zarikian) If P is an idempotent linear map on an operator space X, then the following are equivalent:

(i) P is a left M-projection.

(ii) τ_P is completely contractive.

(iii) P is a (selfadjoint) projection in the C^*-algebra $\mathcal{A}_\ell(X)$.

(iv) There exist a completely isometric embedding $\sigma\colon X \to B(H)$, and a projection $e \in B(H)$, such that $\sigma(Px) = e\sigma(x)$ for all $x \in X$.

(v) There exists a completely isometric embedding $\sigma\colon X \to B(H)$ such that

$$\sigma(x)^*\sigma(y) = 0, \qquad x \in P(X),\ y \in (I{-}P)(X).$$

One major use of left multipliers is to improve on previously known 'characterization theorems' for operator algebras, 'operator modules' etc.

For example, we give a quick proof of the B-Ruan-Sinclair theorem: namely that up to completely isometric isomorphism, unital operator algebras are precisely the operator spaces A which is a unital algebra whose product satisfies $\|ab\|_n \leq \|a\|_n\|b\|_n$ for all $a, b \in M_n(A)$.

The one direction of the proof ($\Rightarrow$) is obvious, since if A is an operator algebra then so is $M_n(A)$.

Let $\lambda : A \to B(A)$ be the homomorphism

$$\lambda(a)(b) = ab, \qquad x, y \in A$$

Let $a \in Ball(A)$, and $b, c \in A$:

$$\left\| \begin{bmatrix} ax \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \right\| \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

$$\leq 1 \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

Thus $\lambda(a) \in Ball(\mathcal{M}_\ell(A))$. So $\lambda : A \to \mathcal{M}_\ell(A)$ and $\|\lambda\| \leq 1$. Conversely,

$$\|\lambda(a)\| \geq \|\lambda(a)(1)\| = \|a\|$$

So λ is an isometry. Similarly completely isometric. But $\mathcal{M}_\ell(X)$ is always an operator algebra! $\square$

There is a similar quick proof for the 'charac-terization of operator modules' that also im-proves that characterization:

Definition An *operator module* over a (unital say) operator algebra A, is a closed subspace $X \subset B(H)$ with $\pi(A)X \subset X$ for a c. contr. hom. $\pi : A \to B(H)$.

(Note: c. contr. homs. $= *$-homs if A C^*-algebra)

Theorem. (Christensen-Effros-Sinclair, B) For an operator space X, X is an operator A-module iff $\exists$ c. contr. hom. $A \to \mathcal{M}_\ell(X)$ iff

$$\|ax\|_n \leq \|a\|_n \|x\|_n \; \forall a \in M_n(A), x \in M_n(X)$$

Multipliers and algebraic structure

As we said, even though our multiplier algebras are only defined in terms of norms and vector space structure, they encode algebra miraculously.

Eg. if you have 'forgotten' the product on an operator algebra A, you can recover it from the norm, via the $\mathcal{M}_\ell(X)$ construction:

Suppose A has an identity element 1, and $X = A$ with forgotten product

Form $\mathcal{M}_\ell(X)$, this is an operator algebra whose elements are maps on X; the product is 'composition'

But A is exactly $\mathcal{M}_\ell(X)$, via the map $\theta : T \mapsto T(1)$ from $\mathcal{M}_\ell(X)$ to X

So $ab = \theta(\theta^{-1}(a) \circ \theta^{-1}(b))$

III. Multipliers and duality (Mostly B-Magajna)

We begin with a generalization of a result of Tomiyama

The most interesting modules X over a C*-algebra have a norm satisfying the following condition:

$$\|a_1 x_1 + a_2 x_2\| \leq \sqrt{\|a_1 a_1^* + a_2 a_2^*\|} \sqrt{\|x_1\|^2 + \|x_2\|^2}.$$

Here $a_1, a_2 \in A, x_1, x_2 \in X$.

Call this a 'representable module'.

A module version of Tomiyama's result on conditional expectations:

Theorem $\Phi : X \to X$ linear from a representable module X over C^*-alg A onto a submodule, $\|\Phi\| \leq 1, \Phi \circ \Phi = \Phi$. Then

$$\Phi(ax) = a\Phi(x) \, , a \in A, x \in X.$$

Idea of proof Not hard to prove that the representable modules are precisely the ones isometrically A-isomorphic to an operator A-module (Magajna)

So we can assume X is an operator A-module

WLOG A is a W*-algebra (by going to 2nd dual)

By density of the span of projections p in a W*-algebra it suffices to show

$$p^{\perp}\Phi(px) \;=\; 0, \qquad x \in X$$

Then use trick in well known proof of Tomiyama's result. $\square$

Theorem Suppose that X is a representable module over a C^*-algebra A, and suppose that X is also a dual Banach space. Then the map $x \mapsto ax$ on X is weak* continuous for all $a \in A$.

Proof Let u be the map $x \mapsto ax$. The adjoint of the canonical map $X_* \to X^*$ is a w^*-continuous contr. projection $q \colon X^{**} \to X$

This induces an isometric map $v \colon X^{**}/\mathrm{Ker}(q) \to X$. By basic duality principles, v is w^*-continuous.

By Krein-Smulian, v is a w^*-homeomorphism. We claim that:

$$q(u^{**}(\eta)) \;=\; u\,q(\eta), \qquad \eta \in X^{**}.$$

If so, then u^{**} induces a map $\dot{u}$ in $B(X^{**}/\mathrm{Ker}(q))$. Since u^{**} is w^*-continuous, so is $\dot{u}$, by Banach space principles. It is easy to see that $\dot{u} = v^{-1}uv$. Since $\dot{u}, v$, and v^{-1} are w^*-continuous, so is u, and thus the result is proved.

In fact the Claim follows from the 'generalized Tomiyama' result.

Indeed, X^{**} may be regarded as an operator A^{**}-module. Therefore it is an A-module, with module action $a\eta = u^{**}(\eta)$ in the notation above, and X is an A-submodule. $\square$

Corollary Suppose X is a subspace of a C^*-algebra A. If $a \in A$, with $aX \subset X$ and $a^*X \subset X$, then the map $x \mapsto ax$ on X is w^*-continuous for any Banach space predual of X.

Proof X is a left operator module over the C^*-algebra generated by 1 and a. By the previous result, left multiplication by a is continuous in the w^*-topology of X. $\square$

As an example of the usefulness of this result, it gives a new proof of an important and substantial theorem about 'W^*-modules/WTROs':

Theorem (Zettl, Effros-Ozawa-Ruan) A TRO with a Banach space predual $= pM(1 - p)$ for a von Neumann algebra M and projn. $p \in M$.

Equivalently, a C*-module Z over a vNA, with a predual, is 'selfdual' (that is, every bounded module map from Z into the vNA is of form $f(\cdot) = \langle z | \cdot \rangle$).

Idea of new proof: The hard part is to show 1) the module is self dual, and 2) the 'inner product' is separately weak* continuous. To do this one uses the last Corollary three times!!

Corollary Banach module characterization of the σ-weakly closed spaces of operators which are invariant under the action of a von Neumann algebra M:

They are exactly the 'representable modules' over M such that for all $x \in X$ the canonical map $M \to X$ given by $m \mapsto mx$ is w^*-continuous.

Theorem Every left multiplier of a dual operator space is w^*-continuous.

Proof If $u \in \mathcal{M}_l(X)$, then $u^{**} \in B(X^{**})$. As in the last theorem, let $q \colon X^{**} \to X$ be the canonical projection, which is c. contractive.

As in that result, it suffices to show that

$$q(u^{**}(\eta)) \ = \ u\,q(\eta), \qquad \eta \in X^{**}. \qquad (1)$$

In order to prove (1), we let Z be an injective envelope of X, viewed as a TRO $pA(1-p)$, as in Part I of Lecture 2.

If $E = Z^{**}$ then $E = pA^{**}(1-p)$ is also a TRO. Clearly $X^{**} \subset E$

By injectivity of Z, extend q to a completely contractive map $\theta : E \to Z$. Since $\theta_{|X} = I_X$, by the rigidity property of the injective envelope we must have $\theta_{|Z} = I_Z$.

Thus θ is a completely contractive projection from E onto Z.

By the 'generalized Tomiyama' result, θ is a left pAp-module map. Let $a \in pAp$ be such that $ax = ux$ for all $x \in X$ (using the characterization of left multipliers).

Since θ is a left pDp-module map,

$$\theta(a\eta) = a\theta(\eta) = aq(\eta), \qquad \eta \in X^{**}. \qquad (2)$$

On the other hand, we claim that

$$a\,\eta \;=\; u^{**}(\eta), \qquad \eta \in X^{**}. \qquad (3)$$

To see this, view both sides as functions from X^{**} into E. Then both functions are w^*-continuous. But (3) holds if $\eta \in X$, and by density it must hold for $\eta \in X^{**}$.

By (3), we have that $\theta(a\eta) = \theta(u^{**}(\eta)) = q(u^{**}(\eta))$. This and (2) proves (1). $\square$

This theorem has many applications.

This is because functional analytic questions about spaces of operators often boil down to considerations involving weak*, topologies, and the key point is to prove that certain linear functions are weak* continuous.

For example, we can characterize weak* closed operator algebras:

Theorem (Le Merdy-B-Magajna) An operator algebra which is a *dual operator space*, is completely isometrically homomorphic, via a weak*-homeomorphism, to a weak* closed subalgebra of some $B(H)$.

Proof. By last theorem, the multiplication is separately weak* continuous. Then ... . □

Corollary Given a subspace X of a C^*-algebra A (or of $B(H)$), and $a \in A$ with $aX \subset X$. Then the function $x \mapsto ax$ is weak* continuous on X with respect to any op. space predual of X

Proof. Left multiplication by A is a left multiplier! $\square$

Theorem (B-E-Z) If X is a dual operator space then $\mathcal{A}_\ell(X)$ is a von Neumann algebra.

Subtlety of preduals:

Theorem (B-Magajna) Last several results are not true if you assume Banach space predual instead of operator space predual.

IV. Noncommutative M-ideals

(Joint with Effros and Zarikian)

We want to generalize the classical M-ideal notion to operator spaces, in a way that the classical M-ideals are the left M-ideals in $MIN(X)$, and such that the left M-ideals in C^*-algebras are the left ideals

Recall: Classical M-ideals

M-**projection:** idempotent $P : X \to X$ s.t.

$$X \to X \oplus_\infty X : x \mapsto (P(x), (I - P)(x))$$

is an isometry (This is saying ...)

$J \subset X$ is an M-**ideal** if $J^{\perp\perp}$ is the range of an M-projection on X^{**}

Ex. The M-ideals in a C^*-algebra are the two-sided ideals

Extensive and useful theory - see Harmand, Werner, Werner

Say $J \subset X$ is a **left M-ideal** if $J^{\perp\perp}$ is the range of a left M-projection on X^{**}

Note: our definitions are only in terms of operator space structure, yet often encodes important algebraic information.

Examples:

- Classical M-ideals

- left ideals in C^*-algebras, submodules of Hilbert C^*-modules

- left M-ideals in an operator algebras exactly the left ideals with a right contr. approx. identity

The game then is to generalize theorems from classical M-ideal theory, and apply these theorems in 'noncommutative functional analysis.

Main tool is that $\mathcal{A}_\ell(X^{**})$ is a von Neumann algebra, and so we know exactly how the projections in there behave!

For example: if J_1 and J_2 are right M-ideals then $\overline{J_1 + J_2}$ is a right M-ideal.

In B-Zarikian Memoirs of AMS (2006), we generalize the basic facts about M-ideals to operator spaces

It is really a generalization, to operator spaces, of the theory of submodules of C^*-modules.