### OPERATOR MODULES AND THEIR TENSOR PRODUCTS

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## Plan:

Part 0. Why operator spaces?

Part I. Operator space tensor products

Part II. Operator algebras and their modules

Part III. Tensor products of operator modules

Part IV. Back to  $C^{\ast}\text{-modules}$ 

Part V. Nonselfadjoint algebras and their modules again

Part 0. Why operator spaces?

Aperitif/Appetithäppchen: Let A be your favourite unital subalgebra of a  $C^*$ -algebra, let  $C_n(A)$  be the first column of  $M_n(A)$ , and consider the basic result from ring theory

$$M_n(A) \cong \operatorname{Hom}_A(C_n(A))$$

This relation breaks down when norms are placed on the spaces. That is, there is no sensible norm to put on  $C_n(A)$  so that

 $M_n(A) \cong B_A(C_n(A))$  isometrically

(and even bicontinuous isomorphism breaks down when  $n = \infty$ ).

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(and even bicontinuous isomorphism breaks down when  $n = \infty$ ). However, operator spaces save the day:

 $M_n(A) \cong CB_A(C_n(A))$  completely isometrically

• For  $C^*$ -algebras themselves this is OK, and the usefulness of operator spaces is less obvious, but will see later (must use 'completely bounded morphisms' in many later results, etc).

## Part I. Operator space tensor products

• Tensor products come in from the beginning in operator space theory:  $M_n(X)$  is the spatial or minimal tensor product of  $M_n$  and X.

Minimal or spatial tensor product:  $X \otimes_{\min} Y$  may be defined by the relation  $X \otimes_{\min} Y \hookrightarrow CB(Y^*, X)$  completely isometrically

• We will not say much about this tensor product here

• If X, Y, and W are operator spaces, and  $u: X \times Y \to W$  is bilinear define a bilinear map  $u_n: M_n(X) \times M_n(Y) \to M_n(W)$  by

$$(x,y) \longmapsto \left[\sum_{k=1}^{n} u(x_{ik}, y_{kj})\right]_{i,j},$$

where  $x = [x_{ij}] \in M_n(X)$  and  $y = [y_{ij}] \in M_n(Y)$ . If  $\sup_n ||u_n|| < \infty$  we say that u is completely bounded (in the sense of Christensen and Sinclair), and write this supremum as  $||u||_{cb}$ .

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• Completely bounded multilinear maps of more than 2 variables have a similar definition

• If  $v: X \to B(L, H)$  and  $w: Y \to B(H, K)$  are completely bounded linear maps, then it is easy to see that the bilinear map  $(x, y) \mapsto v(x)w(y)$ is completely bounded in the sense above, and has completely bounded norm dominated by  $\|v\|_{cb} \|w\|_{cb}$ .

• Remarkably the converse is true too: Christensen and Sinclair (the  $C^*$ -algebra case), and Paulsen and Smith (the general case)

Theorem  $u: X \times Y \to B(H, K)$  is completely contractive (as a bilinear map) if and only if there is a Hilbert space L, and there are completely contractive linear maps  $v: X \to B(L, H)$  and  $w: Y \to B(K, L)$ , with u(x, y) = v(x)w(y) for all  $x \in X$  and  $y \in Y$ .

Remark. One has further 'Stinespring' representations for v and w

• For  $n \in \mathbb{N}$  and  $z \in M_n(X \otimes Y)$  we define the Haagerup tensor norm  $\|z\|_h = \inf\{\|x\|\|y\|\},\$ 

where the infimum is taken over all  $p \in \mathbb{N}$ , and all ways to write  $z = x \odot y$ , where  $x \in M_{n,p}(X), y \in M_{p,n}(Y)$ . Here  $x \odot y$  denotes the formal matrix product of x and y using the  $\otimes$  sign as multiplication: namely  $x \odot y = [\sum_{k=1}^{p} x_{ik} \otimes y_{kj}]$ .

Proposition The completion  $X \otimes_h Y$  of  $X \otimes Y$  with respect to  $\|\cdot\|_h$  is an operator space (called the Haagerup tensor product)

• The Haagerup tensor product linearizes completely bounded (in the sense of Christensen and Sinclair) bilinear maps

- The Haagerup tensor product is functorial for completely bounded maps
- It is injective (last bullet but everything completely isometric)
- It is projective (last bullet but everything 'complete quotient')
- It is associative:  $(X_1 \otimes_h X_2) \otimes_h X_3 = X_1 \otimes_h (X_2 \otimes_h X_3).$

• Convenient norm formulae: If  $z \in X \otimes_h Y$  then  $||z||_h < 1$  iff z is a norm convergent sum  $\sum_{k=1}^{\infty} a_k \otimes b_k$  in  $X \otimes_h Y$ , with  $||\sum_{k=1}^{\infty} a_k a_k^*|| < 1$  and  $||\sum_{k=1}^{\infty} b_k^* b_k|| < 1$ , and where the last two sums converge in *norm*.

• That is,  $\|z\|_n = \inf\{\|x\| \|y\| : z = x \odot y\}$ , where  $x \odot y$  is defined to be  $[\sum_k x_{ik} \otimes_A y_{kj}]$ 

• It is self-dual:  $X^* \otimes_h Y^* \subset (X \otimes_h Y)^*$  completely isometrically.

But we will not use the Banach duality theory of tensor norms in this talk.

• There is also a notion of *jointly completely bounded* bilinear maps that I will not discuss here. The operator space projective tensor product  $X \otimes Y$  is defined so as linearize jointly completely bounded bilinear maps:

 $JCB(X,Y;W) \cong CB(X,CB(Y,W)) \cong CB(X \otimes Y,W)$ 

• (Comparison of tensor norms) The 'identity' is a complete contraction  $X \stackrel{\frown}{\otimes} Y \to X \otimes_h Y \to X \otimes_{\min} Y$ 

• There is a calculus of tensor products. If H, K are Hilbert spaces, and if  $m, n \in \mathbb{N}$ , then we have the following complete isometries:

$$\begin{array}{l} (1) \ H^r \otimes_h X = H^r \stackrel{\frown}{\otimes} X, \mbox{ and } X \otimes_h H^c = X \stackrel{\frown}{\otimes} H^c. \\ (2) \ H^c \otimes_h X = H^c \otimes_{\min} X, \mbox{ and } X \otimes_h H^r = X \otimes_{\min} H^r. \\ (3) \ C_n(X) \cong C_n \otimes_h X = C_n \otimes_{\min} X, \mbox{ where } C_n = M_{n,1}(\mathbb{C}), \\ \mbox{ and } R_n(X) \cong X \otimes_h R_n = X \otimes_{\min} R_n. \\ (4) \ (\bar{H}^r \stackrel{\frown}{\otimes} X \stackrel{\frown}{\otimes} K^c)^* = (\bar{H}^r \otimes_h X \otimes_h K^c)^* \cong CB(X, B(K, H)). \\ (5) \ \mathbb{K}(K, H) \cong H^c \otimes_{\min} \bar{K}^r \mbox{ and } \mathbb{K}(K, H) \otimes_{\min} X \cong H^c \otimes_h X \otimes_h \bar{K}^r. \\ (6) \ M_{m,n}(X) \cong C_m \otimes_h X \otimes_h R_n. \\ (7) \ M_{m,n}(X \otimes_h Y) \cong C_m(X) \otimes_h R_n(Y). \\ (8) \ H^c \stackrel{\frown}{\otimes} K^c = H^c \otimes_h K^c = H^c \otimes_{\min} K^c = (H \otimes^2 K)^c, \mbox{ and similarly for row Hilbert spaces.} \\ (9) \ S^1(K, H) \cong \bar{K}^r \stackrel{\frown}{\otimes} H^c. \\ (10) \ CB(S^1(\ell_I^2, \ell_J^2), X) \cong \mathbb{M}_{I,J}(X), \ \mbox{if } I, J \ \mbox{are sets.} \end{array}$$

The extended and sigma Haagerup tensor product  $X \otimes_{eh} Y$  and  $X \otimes_{\sigma} Y$  are respectively an 'enlarged' Haagerup tensor product and a dual space variant.

• E.g.  $H^c \otimes_{\operatorname{eh}} \bar{K}^r \cong B(H, K)$ 

Part II. Operator algebras and their modules

• The characterization of operator algebras (subalgebras of B(H)):

Completely isometric variant (B-Ruan-Sinclair): Up to completely isometric isomorphism the operator algebras are precisely the operator spaces with a multiplication which is a complete contraction  $A \otimes_h A \to A$ . Assuming an identity or approximate identity of norm 1.

Completely bounded variant (B): Up to completely bounded isomorphism the operator algebras are precisely the operator spaces with an associative multiplication which is a completely bounded map  $A \otimes_h A \to A$ .

A concrete left operator A-module is a linear subspace  $X \subset B(K, H)$ , which we take to be norm closed as always, together with a completely contractive homomorphism  $\theta: A \to B(H)$  for which  $\theta(A)X \subset X$ . Such an X is a left A-module via  $\theta$ .

There is also an abstract definition of operator modules.

Theorem [B, C-Effros-S] Operator modules are just the operator spaces which are modules over an operator algebra, such that the module action is completely contractive as a map on the Haagerup tensor product

• My favorite proof of the last theorem and the B-Ruan-Sinclair theorem uses the following:

Theorem [B-Effros-Zarikian] Linear  $T: X \to X$  satisfies that  $T \oplus I : C_2(X) \to C_2(X)$  is completely contractive iff there is a contractive Hilbert space operator S such that  $T(x) = S \cdot x$  for all  $x \in X$ 

## Suppose that A, B are operator algebras

(1) If H, K are Hilbert spaces, and if  $\theta: A \to B(H)$  and  $\pi: B \to B(K)$  are completely contractive homomorphisms, then B(K, H) is an operator A-B-bimodule (with the canonical module actions).

- (2) Submodules of operator modules are clearly operator modules.
- (3) Any operator space X is an operator  $\mathbb{C}$ - $\mathbb{C}$ -bimodule.
- (4) Any operator algebra A is an operator A-A-bimodule of course.

(5) Hilbert A-module: a Hilbert space H which is a left A-module whose associated homomorphism  $\theta: A \to B(H)$  is completely contractive (or sometimes, completely bounded). Then  $H^c$  is an operator A-module.

• We define <sub>A</sub>HMOD to be the category of Hilbert A-modules, with bounded A-module maps as the morphisms (they are automatically completely bounded on the associated column Hilbert spaces).

• There is a theory of Hilbert modules which I will omit here, e.g. always exist Hilbert modules such that  $A'' = \overline{A}^{w*}$  (B-Solel).

• Can take quotients of operator modules, 'opposites', 'prolongations', etc. (latter: X is an operator A-module and  $\theta: B \to A$  is a completely contractive (or completely bounded) homomorphism, and  $bx = \theta(b)x$ ).

• The A-modules that correspond to completely contractive (or completely bounded) homomorphisms  $A \to CB(X)$  we call matrix normed modules. Equivalently, these are exactly the left A-modules X which are also an operator space, such that the module action on X extends to a complete contraction (completely bounded)  $A \otimes X \longrightarrow X$ .

• We will not discuss the latter class much here (used a lot in NC abstract harmonic analysis)

The algebra of a bimodule: If X is an (operator) A-B-bimodule over algebras A and B, set D to be the algebra

$$\left[\begin{array}{cc}a & x\\0 & b\end{array}\right]$$

for  $a \in A$ ,  $b \in B$ ,  $x \in X$ . The product here is the formal product of  $2 \times 2$  matrices, implemented using the module actions and algebra multiplications. This is an (operator) algebra. • Any  $C^*$ -module Z is an operator module (just look at the linking  $C^*$ -algebra  $\mathcal{L}(Z)$ )

$$\|[y_{ij}]\|_n = \|[\sum_{k=1}^n \langle y_{ki} | y_{kj} \rangle]\|^{\frac{1}{2}}$$

• A right  $C^*$ -module Z which is also a left module over a different  $C^*$ algebra A via a nondegenerate \*-homomorphism  $\theta : A \to \mathbb{B}(Z)$ , is also a left operator module. [Indeed by looking at  $M(\mathcal{L}(Z))$  one can see Z is a left operator module over  $\mathbb{B}(Z)$ , the 1-1-corner of  $M(\mathcal{L}(Z))$ .]

• Indeed most of the important modules in  $C^{\ast}-{\rm theory}$  are operator modules

• Any bounded module map between  $C^*$ -modules is completely bounded, with  $||T||_{cb} = ||T||$ . Thus we find ourselves in a situation where we do not have to insist on working only with completely bounded maps, rather we can exploit the fact that our maps already are completely bounded. [One proof: WLOG Y = Z, then look in  $LM(\mathcal{L}(Z))$  where T becomes 'left multiplication by an operator'. Recall also that  $B_A(Z) \cong LM(\mathbb{K}(Z))$ (Lin).]

• There are several methods to 'recover the inner product' from the Banach module structure.

Theorem Suppose Y is a Banach space (resp. operator space) and a right module over a  $C^*$ -algebra A. Then Y is a  $C^*$ -module, and the norm on Y (resp. the matrix norms on Y) coincides with the  $C^*$ -module's norm (resp. canonical operator space structure) if and only if there exists a net of positive integers  $n(\alpha)$ , and contractive (resp. completely contractive) A-module maps  $\phi_{\alpha} : Y \to C_{n(\alpha)}(A)$  and  $\psi_{\alpha} : C_{n(\alpha)}(A) \to Y$  with  $\psi_{\alpha} \circ \phi_{\alpha} \to Id_Y$  strongly (that is, point-norm) on Y. In this case, for  $y, z \in Y$ , the norm limit  $\lim_{\alpha} \phi_{\alpha}(y)^* \phi_{\alpha}(z)$  exists in A and equals the  $C^*$ -module inner-product. Theorem Suppose Y is a Banach space (resp. operator space) and a right module over a  $C^*$ -algebra A. Then Y is a  $C^*$ -module, and the norm on Y (resp. the matrix norms on Y) coincides with the  $C^*$ -module's norm (resp. canonical operator space structure) if and only if there exists a net of positive integers  $n(\alpha)$ , and contractive (resp. completely contractive) A-module maps  $\phi_{\alpha} : Y \to C_{n(\alpha)}(A)$  and  $\psi_{\alpha} : C_{n(\alpha)}(A) \to Y$  with  $\psi_{\alpha} \circ \phi_{\alpha} \to Id_Y$  strongly (that is, point-norm) on Y. In this case, for  $y, z \in Y$ , the norm limit  $\lim_{\alpha} \phi_{\alpha}(y)^* \phi_{\alpha}(z)$  exists in A and equals the  $C^*$ -module inner-product.

• This suggests the following generalization of  $C^*$ -modules: for an operator algebra A and a right A-module Y which is also an operator space, such that there exists a net of positive integers  $n(\alpha)$ , and contractive (resp. completely contractive) A-module maps  $\phi_{\alpha} : Y \to C_{n(\alpha)}(A)$  and  $\psi_{\alpha} : C_{n(\alpha)}(A) \to Y$  with  $\psi_{\alpha} \circ \phi_{\alpha} \to Id_Y$  strongly on Y. This works! These are called A-rigged modules, and their theory generalizes the theory of  $C^*$ -modules (later).

### Part III. Tensor products of operator modules

• If X is a right A-module, Y is a left A-module, Z is a vector space, then a bilinear map  $u: X \times Y \to Z$  is said to be *balanced* if u(xa, y) = u(x, ay)for all  $x \in X, y \in Y, a \in A$ .

• Given such X and Y, we define the algebraic module tensor product  $X \otimes_A Y$  to be the quotient of  $X \otimes Y$  by the subspace spanned by terms of the form  $xa \otimes y - x \otimes ay$ , for  $x \in X, y \in Y, a \in A$ . It is clear that  $X \otimes_A Y$  has the following universal property: given any vector space Z, and any balanced bilinear  $u: X \times Y \to Z$ , then there exists a unique linear  $\tilde{u}: X \otimes_A Y \to Z$  mapping  $x \otimes y$  to u(x, y) for all  $x \in X, y \in Y$ .

• Thus the module tensor product 'linearizes balanced bilinear maps'.

• If X and Y are operator spaces which are, respectively, right and left A-modules, define  $X \otimes_{hA} Y$  (resp.  $X \otimes_A Y$ ) to be the quotient of  $X \otimes_h Y$  (resp.  $X \otimes Y$ ) by the closure of the subspace spanned by terms of the form  $xa \otimes y - x \otimes ay$ .

• These are the module Haagerup tensor product, and the module operator space projective tensor product.

• Thus  $X \otimes_{hA} Y$  'linearizes' balanced completely bounded maps. In fact this property characterizes  $X \otimes_{hA} Y$ .

• Similarly  $X \stackrel{\frown}{\otimes}_A Y$  'linearizes' jointly balanced completely bounded maps.

- $\otimes_{hA}$  (resp.  $\bigotimes_{A}$ ) is functorial for completely bounded module maps
- It is projective (last bullet but everything 'complete quotient')
- It is associative:  $(X_1 \otimes_{hA} X_2) \otimes_{hB} X_3 = X_1 \otimes_{hA} (X_2 \otimes_{hB} X_3).$

• Convenient norm formulae: If  $z \in X \otimes_{hA} Y$  then ||z|| < 1 iff z is a norm convergent sum  $\sum_{k=1}^{\infty} a_k \otimes b_k$ , with  $||\sum_{k=1}^{\infty} a_k a_k^*|| < 1$  and  $||\sum_{k=1}^{\infty} b_k^* b_k|| < 1$ , and where the last two sums converge in *norm*.

• That is,  $\|z\|_n = \inf\{\|x\| \|y\| : z = x \odot_A y\}$ , where  $x \odot_A y$  is defined to be  $[\sum_k x_{ik} \otimes_A y_{kj}]$ 

• 'Change of rings':  $X \otimes_{hA} B$  if X is a right operator A-module, and  $\theta: A \to B$  a completely bdd homomorphism. As expected  $X \otimes_{hA} A \cong X$ .

• In the main cases we are interested in later we also have commutation with direct sums:

$$\oplus_i^c (Y_i \otimes_{hA} Z) \cong (\oplus_i^c Y_i) \otimes_{hA} Z.$$

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• Thus we are seeing that there is a nice 'algebraic calculus', fitting in with the idea in the Appetithäppchen.

• Similar results for m-fold module tensor products.

• Indeed in the 'non-module' case in Part I there is something of an 'algebraic calculus' of module tensor products

Wittstock bimodule extension theorem One may extend completely contractive bimodule maps from a sub-bimodule into B(H), to the containing operator bimodule.

Bilinear module extension theorem Suppose that A is a  $C^*$ -algebra, and that X (resp. Y) is an A-submodule of a right (resp. left) operator Amodule W (resp. Z). If L is a Hilbert space and if  $u: X \times Y \to B(L)$  is a completely contractive A-balanced bilinear map, then u has a completely contractive A-balanced bilinear extension  $\hat{u}: W \times Z \to B(L)$ .

Corollary (Injectivity of module Haagerup tensor product) If A is a  $C^*$ -algebra, then the canonical map  $X \otimes_{hA} Y \to W \otimes_{hA} Z$  is a complete isometry.

### Part IV. Back to $C^*$ -modules

Theorem If Y is a  $C^*$ -module and Z an operator module then  $\mathbb{K}(Y, Z) \cong Z \otimes_{hA} \overline{Y}$  completely isometrically.

Ingredients of proof: The Haagerup norm formula easily gives the can. map  $Z \otimes_{hA} \overline{Y} \to \mathbb{K}(Y, Z)$  contractive. A contractive asymptotic inverse to this comes from the asymptotic factorization maps in the theorem factoring Y through A-modules of the form  $C_n(A)$ 

Corollary  $T \in \mathbb{K}(Y,Z)$  with ||T|| < 1 if and only if we can write  $T = \sum_{k=1}^{\infty} |z_k\rangle \langle y_k|$ , where  $||\sum_{k=1}^{\infty} |z_k\rangle \langle z_k|| < 1$ , and  $||\sum_{k=1}^{\infty} |y_k\rangle \langle y_k|| < 1$ . All the sums here converge in the norm. Equivalently, T factors through  $C_{\infty}(A)$  by compact A-module maps of norm < 1. Theorem The exterior tensor product of C<sup>\*</sup>-modules Y and Z is completely isometrically isomorphic to their spatial operator space tensor product  $Y \otimes_{\min} Z$ .

Theorem The interior tensor product  $Y \otimes_{\theta} Z$  of  $C^*$ -modules is completely isometrically isomorphic to module Haagerup tensor product  $Y \otimes_{hA} Z$ . Here  $\theta : A \to \mathbb{B}(Z)$  is an nondegenerate \*-homomorphism.

• Proofs use the functoriality of  $\otimes_{hA}$ , and the fact that  $C_n(A) \otimes_{hA} Z \cong C_n \otimes_h A \otimes_{hA} Z \cong C_n(Z)$  completely isometrically. Since Y satisfies the Theorem above (i.e. Y factors asymptotically through A-modules of the form  $C_n(A)$ ),  $Y \otimes_{hA} Z$  factors asymptotically through B-modules of the form  $C_n(Z)$ .

• This perspective gave new insights into the tensor products of  $C^*$ -modules. E.g. it gives norm formulae for elements of  $Y \otimes_{\theta} Z$ .

• 
$$\mathbb{K}(Y \otimes_{\theta} Z) \cong Y \otimes_{hA} \mathbb{K}(Z) \otimes_{hA} \bar{Y}$$

Hom-Tensor relations: Let A and B be  $C^*$ -algebras. We have

- (1)  $\mathbb{K}_A(Y, \mathbb{K}_B(Z, N)) \cong \mathbb{K}_B((Y \otimes_{\theta} Z), N)$  completely isometrically, if Y, Z are right  $C^*$ -modules over A and B respectively, and Z, N are left and right operator modules over A and B respectively.
- (2)  $\mathbb{K}_A(X, \mathbb{K}_B(W, M)) \cong \mathbb{K}_B((W \otimes_{\theta} X), M)$  completely isometrically, if X, W are left  $C^*$ -modules over A and B respectively, and W, M are right and left operator modules over A and B respectively.
- (3)  $\mathbb{K}_A(Y, (N \otimes_{hB} M)) \cong N \otimes_{hB} \mathbb{K}_A(Y, M)$  completely isometrically, if Y is a right module over A, if M is a B A operator bimodule and if N is a right A-operator module.
- (4)  $\mathbb{K}_A(X, N \otimes_{hB} M) \cong \mathbb{K}_A(X, N) \otimes_{hB} M$  completely isometrically, if X is a left  $C^*$ -module over A, if N is an A B operator bimodule and if M is a left B-operator module.
- (5)  $\mathbb{K}_B(\mathbb{K}_A(Y, W), M) \cong Y \otimes_{hA} \mathbb{K}_B(W, M)$  completely isometrically, if Y is a right C\*-module over A, M is a left B-operator module, and W is a right A operator module which is a left B C\*-module.

(6) K<sub>B</sub>(K<sub>A</sub>(X, Z), N) ≅ K<sub>B</sub>(Z, N) ⊗<sub>hA</sub> X completely isometrically, if X is a left C\*-module over A, N is a right B operator module, and Z is a left A operator module which is a right B C\*-module.
(7) K<sub>A</sub>(X, K<sub>B</sub>(Z, W)) ≅ K<sub>B</sub>(Z, K<sub>A</sub>(X, W)) completely isometrically, if X, Z are left and right C\*-modules over A and B respectively, and if W is an A - B operator bimodule.

Idea of proof: Use earlier identity  $\mathbb{K}(Y,Z) \cong Z \otimes_{hA} \overline{Y}$  to write everything as a Haagerup tensor product, then use associativity of tensor product Eilenberg-Watts type theorem: If  $C^*MOD_A$  is the category of right  $C^*$ modules over a  $C^*$ -algebra A, with morphisms the bounded A-module maps, then the strongly continuous \*-functors  $C^*MOD_A \to C^*MOD_B$  are precisely (up to natural unitary isomorphism) the interior tensor product  $-\otimes_{\theta} Z$ (or equivalently,  $\mathbb{K}_A((-\bar{N},Z))$  for a right  $C^*$ -module Z over B, and a nondegenerate \*-homomorphism  $\theta : A \to \mathbb{B}(Z)$ .

**Remark.** The 'finitely generated'  $C^*$ -modules are enough in this theorem

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**Remark.** The 'finitely generated'  $C^*$ -modules are enough in this theorem

Strong Morita equivalence: Simplest definition—which also works for nonselfadjoint algebras—the existence of operator bimodules X, Y with  $X \otimes_{hA} Y \cong B$  and  $Y \otimes_{hB} X \cong A$  as operator bimodules.

Theorem A and B are strongly Morita equivalent iff  $OMOD_A$  is equivalent to  $OMOD_B$  as categories. In this case the functor implementing the equivalence 'is'  $- \otimes_{hA} Y$ , with 'inverse functor'  $- \otimes_{hB} X$  for X, Y as above.

• Of course  $Y \otimes_{hA} H^c$  is a column Hilbert space, if H is a Hilbert (space) A-module.

• Seems to me that the 'stable isomorphism theorem' for strongly Morita equivalent  $C^*$ -algebras, and the matching 'Kasparov stabilization' results, become a bit simpler in operator module notation

The main point: the operator space/module Haagerup tensor product approach is supposed to allow one to treat theories involving  $C^*$ -modules much more like pure algebra.

• There is a similar 'extended module Haagerup', and weak\* version of the theory

## Part V. Nonselfadjoint algebras and their modules again

• Recall: a rigged module: right A-module Y which is also an operator space, such that there exists a net of positive integers  $n(\alpha)$ , and contractive (resp. completely contractive) A-module maps  $\phi_{\alpha} : Y \to C_{n(\alpha)}(A)$  and  $\psi_{\alpha} : C_{n(\alpha)}(A) \to Y$  with  $\psi_{\alpha} \circ \phi_{\alpha} \to Id_Y$  strongly on Y.

• There are many alternative characterizations, e.g. in terms of an inner product on a containing  $C^*$ -module (which if you want can be chosen to be expressible in terms of the  $C^*$ -envelope ('minimal'  $C^*$ -algebra/nc Shilov boundary), or in terms of an approximate identity for  $Y \otimes_{hA} X$ , etc.

• Here X is 'adjoint' module  $\tilde{Y}$  of rigged module Y which may be viewed as a submodule of  $CB_A(Y, A)$ .

• To get our 'primary definition' going though one needs the recent theory of hereditary subalgebras of operator algebras due to B-Hay-Neal (using some deep ideas from 'peak interpolation theory').

• The theory in the last section generalizes to this nonselfadjoint setting, with few exceptions. This is mostly because we are using the same key tool, the module Haagerup tensor product, and its 'calculus', i.e. strong algebraic properties.

• Sample exception: module Haagerup tensor product need not be injective now. And now the 'stable isomorphism theorem' is different (best results on this in recent papers of Eleftherakis...).

• And again, there is a weak\* version of all of this ... .

Key point: the Haagerup tensor product facilitates 'continuity' for algebraic (ring-theoretic) structures (Bram). There is a 'calculus' of algebraic formulae (involving tensor products) that is very useful.