Contemporary Mathematics

Multipliers, C^*-modules, and algebraic structure in spaces of Hilbert space operators

David P. Blecher

Abstract. Part I is a survey of the author's work (on his own or with several coauthors) on one-sided multipliers and their applications to algebraic structure in spaces of operators. The proofs given here are new for the most part, and the emphasis is on the connections with the theory of Hilbert C^*-modules. In Part II we initiate a theory of one-sided multipliers between two different operator spaces.

Dedicated to J. von Neumann and M. H. Stone.

1. Introduction

The material surveyed in Part I of this article may be regarded as a 'fourth generation' descendant of J. von Neumann and M. H. Stone's powerful and magical framework built around projection lattices, spectral theory, and 'rings of operators'. Both von Neumann and Stone of course were authors of the theory of Hilbert space operators as we know it today. However both were deeply interested in 'algebraic structure' naturally appearing in analytic or topological settings, and the deep relations that then can appear between the algebra, analysis, topology and/or measure theory. Such relations were an abiding theme for example in the theory of selfadjoint algebras of Hilbert space operators developed by von Neumann, on his own and with Murray (see [**51, 48**] and references therein). In the 'second generation', Gelfand and Naimark, Kadison, and the countless others who have followed their work, created the more abstract or 'space-free' setting of abstract C^*-algebras (noncommutative topology) and W^*-algebras (noncommutative measure theory). A crucial role is played by 'abstract projections' in these spaces; which are closely connected to the ideals of these algebras. In the picture which we are trying to draw, we regard the C^*-modules and W^*-modules of Kaplansky, Rieffel, and Paschke (see e.g. [**59, 52**]) as an example of a 'third generation' product, in the sense that it is an example of 'operator algebraic structure' which is at one further remove from concrete algebras of operators. The main object of study here is no longer an algebra, indeed it is a hybrid between a C^*-algebra (resp. W^*-algebra)

*This research was supported in part by a grant from the National Science Foundation.
September 3, 2003. Revised January 16, 2004.

and a Hilbert space. Nonetheless, the ensuing theory here is not too far from the abstract C^*-algebra (resp. W^*-algebra) theory. Indeed much rests on the canonical C^*-algebras (resp. W^*-algebras) which are naturally associated with the module. Another 'third generation' concept is the M-ideals of Alfsen and Effros [**4, 39**], which permit a generalization to Banach spaces of the notions of 'two-sided ideal' and 'orthogonal projection' in C^*-algebras.

Operator spaces (which are defined in Section 2 below) may be viewed as a simultaneous generalization of operator algebras, C^*-modules, and Banach spaces (the latter for example via what is known as the MIN functor - see e.g. [**33**]). This allows us in the 'fourth generation', to go to one further remove. We consider a certain *multiplier C^*-algebra* (resp. *W^*-algebra*) $\mathcal{A}_l(X)$ which one may naturally associate with any operator space (resp. dual operator space) X. The elements of $\mathcal{A}_l(X)$ are maps on X. Generalizing the M-ideal notion mentioned above, we will also consider one sided ideals in X, and certain projections on X, namely the projections in the C^*-algebra (resp. *W^*-algebra*) $\mathcal{A}_l(X)$ which we just mentioned. Our theory may also be regarded as a generalization of part of the module theory mentioned in the last paragraph, as we shall attempt to demonstrate in this paper. However the gap between the third and fourth generation is wider than between other generations: for example our theory is very far from being as ubiquitous or successful as C^*-module theory. Nonetheless some of the magic and power seen in the 'first generation' concept does persist to the fourth, largely because we are still able to utilize von Neumann algebra projection techniques at key points.

A subsidiary theme of our article is *operator algebraic structure*, another prevailing concern of Stone and von Neumann. We will discuss for example characterizations of operator algebras and modules, how rigid such operator algebraic structure is, Banach-Stone-Kadison type theorems, and so on. This turns out to be intimately related to the multiplier algebra discussed in the last paragraph. Indeed the main point is that although $\mathcal{A}_l(X)$ and the related space $\mathcal{M}_l(X)$ are defined purely in terms of the matrix norms and vector space structure on X, they often encode 'operator algebra structure' (see for example Theorem 5.4 or Example 5.6 (3)). Indeed $\mathcal{M}_l(X)$ and $\mathcal{A}_l(X)$ are key to 'latent operator algebra structure in X'. Operator spaces X with trivial (one dimensional) multiplier algebras are exactly the spaces lacking 'operator algebraic structure' in a sense which one can make precise.

The main part of our article, Part I, is devoted to a survey of much of our work on these multipliers and their applications to operator algebraic structure (see e.g. [**10, 20, 14, 22, 12, 13, 25**]). This is a field which admits very many alternative approaches. Because of space constraints we are not able to emphasize all aspects or 'pictures' of the theory, instead we will pick one largely self-contained route through the main results and proofs (the interested reader can consult the papers just listed for aspects not touched upon here). We follow essentially our original route from [**10**], however the proofs (which are postponed mostly to Part II) will be novel. This route has the advantage of highlighting connections with C^*-modules.

Although Part I is essentially a survey, it is designed in part to set us up for proving the new results contained in Part II of the paper. These results may be summarized as constituting the basics of a theory of multipliers between two different operator spaces X and Y (the existing theory only addresses multipliers of a single space). For this to work it seems that there does need to be some relation between X and Y, or between two C^*-algebras that are canonically associated with X and Y. We believe that it is high time that this direction was pursued, since it could have important applications in operator space theory.

We end this introduction with a note on the history of one-sided multipliers of operator spaces. They were introduced by the author in '99; and independently by W. Werner (but in an 'ordered' context–see e.g. [**68**]). We were motivated by some results in the theory of uniform algebras (see [**16**]) to seek to generalize the classical Banach space multipliers [**4, 5**] to operator spaces, with an eye to improving on the existing theorems characterizing operator algebras and their modules. We were able to do this by employing the Arveson-Hamana noncommutative Shilov boundary (see Section 4 below, such techniques have been used in similar situations since [**21**]), and a rather general 'characterization theorem'—this was called the 'oplication theorem' in [**9, 10**]—which had the desired applications to operator algebras and modules. After this, work began towards the tidy and powerful alternative approach to one-sided multipliers that is presented in the joint paper [**20**] with Paulsen. The techniques here were influenced by [**34**]. A year later, the author, Effros, and Zarikian [**14**], partially inspired by an analogous result of W. Werner, found an extremely simple characterization of one-sided multipliers (see Theorem 5.1 (iii)). Because this criterion is so simple, it is extremely useful, and has very numerous consequences; ensuring for example that the theory works well with respect to quotients, duality, tensor products, and many other basic functional analytic constructs. Later still, Werner showed in [**68**] how to view operator space multipliers within his ordered context, and also showed how some parts of Theorem 5.1 may be deduced from his earlier result.

2. Preliminaries

We reserve the symbols H, K, L always for Hilbert spaces. A *concrete operator space* is a subspace X of $B(K, H)$. For such a subspace X we may view the space $M_{m,n}(X)$ of $m \times n$ matrices with entries in X as the corresponding subspace of $B(K^{(n)}, H^{(m)})$. Thus this matrix space has a natural norm, for each $m, n \in \mathbb{N}$. An *abstract operator space* is a vector space X with a norm $\| \cdot \|_n$ on $M_n(X)$, for all $n \in \mathbb{N}$, which is *completely isometric* to a concrete operator space. The latter term, complete isometry, refers to a linear map $T : X \to Y$ such that

$$\|[T(x_{ij})]\|_n = \|[x_{ij}]\|_n$$

for all $n \in \mathbb{N},\ [x_{ij}] \in M_n(X)$. A *completely bounded map* is defined similarly, simply replacing '$= \|[x_{ij}]\|_n$' by '$\leq M\|[x_{ij}]\|_n$' in the last centered equation, for a constant M. The least such M is denoted by $\|T\|_{cb}$. If $\|T\|_{cb} \leq 1$ then T is a *complete contraction*. We write $CB(X, Y)$ for the space of completely bounded maps, and set $CB(X) = CB(X, X)$.

Any concrete operator space is easily seen to satisfy *Ruan's axioms*:

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} , \qquad \text{and} \qquad \|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| ,$$

for all $m, n \in \mathbb{N},\ x \in M_n(X),\ y \in M_m(X)$, and $\alpha, \beta \in M_n$. *Ruan's theorem*, which is the cornerstone of the theory of operator spaces, states that up to complete isometry, abstract operator spaces are the same thing as concrete operator spaces. For more on operator space theory, or on the facts listed in this section, see the texts [**17, 33, 54, 56**].

We saw above that if X is an operator space then so is $M_{m,n}(X)$, if $m, n \in \mathbb{N}$. In particular we will reserve the symbols $C_n(X)$ and $R_n(X)$ for $M_{n,1}(X)$ and $M_{1,n}(X)$ respectively. Much later we will have cause to use infinite matrices and infinite columns. For example if I is a cardinal, then the space $C_I^w(B(K,H))$ of column I-tuples may be identified with $B(K, H^{(I)})$. If $X \subset B(K,H)$, then $C_I^w(X)$ may be identified as the subspace of $C_I^w(B(K,H))$ whose entries lie in X. Also $C_I(X)$ may be identified as the norm closure of the 'finitely supported' columns in $C_I^w(B(K,H))$ with entries from X.

Every operator space X has a dual object which is also an operator space. Namely, we write X^* for the dual Banach space of X, with the norm of a matrix $[f_{ij}] \in M_n(X^*)$ given by

$$\|[f_{ij}]\|_n \ = \ \sup\{\|[f_{ij}(x_{kl})]\| \ : \ [x_{kl}] \in \mathrm{Ball}(M_n(X))\}.$$

We call X^*, equipped with this 'operator space structure', a *dual operator space.* The canonical map from X into X^{**} is a complete isometry, as one would wish. A formula similar to the last centered equation defines matrix norms on $CB(X,Y)$ for any operator space Y, and with these norms, $CB(X,Y)$ is an operator space.

An *injective operator space* X has the property that any completely contractive $T : Y \to X$ has a completely contractive extension $\tilde{T} : Z \to X$. Here Z is any operator space containing Y as a subspace. Alternatively, a subspace $X \subset B(K,H)$ is injective if and only if there is a completely contractive projection of $B(K,H)$ onto X. Although we shall not need this, in the Banach space case there is a similar definition, and in this case it is known that the injective Banach spaces are exactly the spaces $C(K)$ for a *Stonean* space K.

A *concrete operator algebra* is a subalgebra A of $B(H)$. By an (abstract) *operator algebra* we will mean a unital algebra which is also an operator space, which is completely isometrically homomorphic to a concrete operator algebra. Unless stated otherwise *we shall always assume operator algebras to be unital,* that is, possessing an identity of norm 1. An *approximately unital operator algebra* is an operator algebra with a contractive approximate identity. There is an operator algebraic version of Ruan's theorem, reminiscent of the Gelfand-Naimark characterization of C^*-algebras:

THEOREM 2.1. (Blecher, Ruan and Sinclair) *Up to completely isometric homomorphism, the unital operator algebras are precisely the unital algebras A which are operator spaces satisfying $\|1\| = 1$ and $\|ab\|_n \leq \|a\|_n \|b\|_n$, for all $n \in \mathbb{N}$ and $a, b \in M_n(A)$.*

We will sketch a proof of this result in Section 5.

If A is a concrete operator algebra then the *diagonal C^*-algebra* of A is the C^*-algebra $A \cap A^*$. It is easy to see that the diagonal does not depend on the particular representation of A on a Hilbert space, thus it is well defined for abstract operator algebras too. For example if A is unital then its diagonal C^*-algebra is the span of the Hermitians (in the Banach algebra sense) in A.

We end this section by discussing the 'classical' multiplier operator algebras of C^*-algebras and operator algebras. If A is an approximately unital operator algebra one may define the left multiplier operator algebra $LM(A)$ to be the subalgebra of A^{**} defined by $\{G \in A^{**} : GA \subset A\}$. (We recall that the second dual of a C^*-algebra is also a C^*-algebra, and hence the second dual of an operator algebra is an operator algebra). It may also be defined to be the concrete operator algebra $\{T \in$

$B(H) : T\pi(A) \subset \pi(A)\}$, for any nondegenerate completely isometric representation $\pi : A \to B(H)$. It may also be defined to be $CB_A(A)$, the operator space of completely bounded right A-module maps on A. All of these three algebras are completely isometrically isomorphic to each other, as is well known (see e.g. [**13**] Section 6 for proofs and references). If A is a C^*-algebra then inside $LM(A)$ we have the multiplier C^*-algebra $M(A)$ (see e.g. [**55**, Section 3.12]). Indeed $M(A)$ is precisely the diagonal C^*-algebra (defined in the last paragraph) of $LM(A)$.

3. A review of C^*-modules and their maps

In this section we define C^*-modules, and review some of their basic theory, which we will use from time to time in the rest of this paper. We also state some new results about maps on C^*-modules which will motivate the main theorem in Section 5, and which will also illustrate a key theme in this paper, namely that an algebraic condition is often equivalent to a metric one.

Throughout this section B is a C^*-algebra; and the reader is referred to [**44, 67**] for additional clarifications if needed. We recall that a *right C^*-module* over B is a right module Y over B which possesses a B-valued inner product $\langle\cdot|\cdot\rangle : Y \times Y \to B$ satisfying a minor modification of the usual conditions defining a Hilbert space. That is, we want linearity in the second variable; and we require that $\langle y|y\rangle \geq 0$ and $\langle y|y\rangle = 0$ if and only if $y = 0$; and that $\langle y|za\rangle = \langle y|z\rangle a$. Here $y, z \in Y$ and $a \in A$. We note it follows from these and the polarization identity that $\langle y|z\rangle^* = \langle z|y\rangle$. Finally we require that Y be complete in the norm $\|y\| = \|\langle y|y\rangle\|^{\frac{1}{2}}$ (this is a norm by a simple variant of the usual Cauchy-Schwarz argument). A C^*-module over B is called *full* if B is the closed span of the range of the inner product.

There is an analogous definition of *left C^*-modules*. By a *C^*-bimodule* we shall mean an A-B-bimodule over C^*-algebras A and B such that X is both a right and a left C^*-module over B and A respectively, with the left module inner product $[\cdot|\cdot]$ being related to the right module inner product by the formula $x\ \langle y|z\rangle = [x|y]\ z$ for all $x, y, z \in X$.

If Y, Z are two C^*-modules over B then we write $Y \oplus_c Z$ for their C^*-module sum, and we will regard this as a 'column'. Namely $Y \oplus_c Z$ is the usual direct sum, but with inner product

$$\langle \begin{bmatrix} y_1 \\ z_1 \end{bmatrix} \Big| \begin{bmatrix} y_2 \\ z_2 \end{bmatrix} \rangle \ = \ \langle y_1|y_2\rangle + \langle z_1|z_2\rangle, \tag{1}$$

for $y_1, y_2 \in Y, z_1, z_2 \in Z$. This is again a C^*-module over B. The corresponding norm in $Y \oplus_c Z$ is given by the formula

$$\left\| \begin{bmatrix} y \\ z \end{bmatrix} \right\| \ = \ \|\langle y|y\rangle + \langle z|z\rangle\|^{\frac{1}{2}}, \qquad y \in Y, z \in Z.$$

Although we shall not use this, we remark that one may describe the C^*-module direct sum without referring to the 'inner product'. Namely, suppose that B is a nondegenerate $*$-subalgebra of $B(K)$ say. It is possible to describe the natural Hilbert space norms on $Y \otimes_B K$ and $Z \otimes_B K$ without recourse to the inner products (see [**8**, p. 263-264]). Let H be the Hilbert space sum of these two spaces. Then $Y \oplus_c Z$ may be identified with an obvious subspace W of $B(K, H)$. For $S, T \in W$, S^*T equals the quantity (1) above, in $B \subset B(K)$.

A submodule J of a C^*-module Z is *orthogonally complemented* if there is another closed submodule L of Z such that $J + L = Z$, and $\langle x|y\rangle = 0$ for all $x \in J, y \in L$. In this case $J \oplus_c L \cong Z$ isometrically.

We have the following extension of a result of Paschke (see [**52**, Theorem 2.8], where the equivalence of (i) and (ii) may be found). Parts (iii) and (iv) are new. We will not provide a proof here since we will generalize this result later in Part II (see Theorem 11.1).

THEOREM 3.1. *Let $u : Y \to Z$ be a $\mathbb{C}$-linear map between right C^*-modules over B. The following are equivalent:*

(i) *u is a contractive B-module map;*
(ii) *$\langle u(y)|u(y)\rangle \leq \langle y|y\rangle$, for all $y \in Y$;*
(iii) $\left\| \begin{bmatrix} u(y) \\ z \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} y \\ z \end{bmatrix} \right\|$, *for all $y \in Y, z \in Z$.*
(iv) $\left\| \begin{bmatrix} u(y) \\ b \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} y \\ b \end{bmatrix} \right\|$, *for all $y \in Y, b \in B$.*

Remarks. 1) Some characterizations of B-module maps of a different flavor may be found in [**61**].

2) Replacing $z \in Z$ in (iii) by $z \in Y$ does not yield an equivalent statement in general. To see this take $B = Z = \ell^\infty_2$, let Y be the ideal $B(1,0)$ of B, and let u be the map $(a,0) \mapsto (0,a)$ on Y. This map u is not a B-module map, yet satisfies this variant on (iii). Indeed it even satisfies the matricial version of this variant on (iii) (that is, it satisfies the condition in (iii) but with y and z replaced by two matrices $[y_{ij}], [z_{ij}] \in M_n(Y)$, for any $n \in \mathbb{N}$). However the situation is improved if one also insists that Y be 'full' over B; see Theorem 11.3.

Suppose that Y, Z are C^*-modules over B, and that J is a closed ideal of B (or of $M(B)$) containing the span of the ranges of the B-valued inner products on Y and Z. We write $B_B(Y,Z)$ for the set of bounded B-module maps from Y to Z. It clearly follows from Theorem 3.1, although it is also easy to prove directly using Cohen's factorization theorem, that Y and Z are C^*-modules over J, and that

$$B_B(Y,Z) = B_J(Y,Z). \tag{2}$$

Thus there is not a truly essential dependence of $B_B(Y,Z)$ on B.

The most important class of maps between C^*-modules over a C^*-algebra B are the *adjointable maps*, namely those maps $T : Y \to Z$ for which there is a map $S : Z \to Y$ such that

$$\langle Ty|z\rangle \;=\; \langle y|Sz\rangle, \qquad y \in Y, z \in Z.$$

We let $\mathbb{B}(Y,Z)$ denote the set of adjointable maps from Y to Z, and set $\mathbb{B}(Y) = \mathbb{B}(Y,Y)$. The subalgebra $\mathbb{B}(Y)$ of $B(Y)$ is actually a C^*-algebra with the involution above. In fact the proof of this is almost identical to the usual proof that $B(H)$ is a C^*-algebra.

Inside $\mathbb{B}(Y,Z)$ is the space $\mathbb{K}(Y,Z)$ of so-called 'compact' adjointable maps. This is the closure of the span of the operators of the form $y \mapsto z\langle w|y\rangle$ on Y, for $w \in Y, z \in Z$. The latter operator is sometimes written $|z\rangle\langle w|$. It is easy to see that $\mathbb{K}(Y) = \mathbb{K}(Y,Y)$ is a C^*-subalgebra of $\mathbb{B}(Y)$.

For any right or left C^*-module Y, or for any C^*-bimodule which is full at least on one side, we may define a *linking C^*-algebra* $\mathcal{L}(Y)$. This is an extremely useful

C^*-algebra containing Y. Just one wonderful feature of $\mathcal{L}(Y)$ is that it allows us to replace the inner product and module actions on Y by products of appropriate elements of $\mathcal{L}(Y)$. For specificity, suppose that Y is a right C^*-module over a C^*-algebra B. There are two equivalent ways to define $\mathcal{L}(Y)$. The first way is to set it equal to $\mathbb{K}(Y \oplus_c B)$. The second way is to consider the set of 2×2 matrices:

$$\begin{bmatrix} \mathbb{K}(Y) & Y \\ \bar{Y} & B \end{bmatrix}$$

We turn this set into an algebra, using the natural product of 2×2 matrices, and using the inner products and module actions. For example the product $y\bar{z}$, of a term y from the 1-2-corner, and a term $\bar{z}$ from the 2-1-corner, is taken to mean $|y\rangle\langle z| \in \mathbb{K}(Y)$. We define the involution of one of these 2×2 matrices in the obvious way. Define a map $\pi : \mathcal{L}(Y) \to B(Y \oplus_c B)$ by the obvious action (i.e. viewing an element of $Y \oplus_c B$ as a column with two entries, and formally multiplying a 2×2 matrix and such a column). It is easy to check that $\pi(m) \in \mathbb{B}(Y \oplus_c B)$ for each matrix $m \in \mathcal{L}(Y)$, and moreover that π is a *-homomorphism into $\mathbb{B}(Y \oplus_c B)$. Also, one can quickly check that π is 1-1, and that π is an isometry when restricted to each of the four corners of $\mathcal{L}(Y)$. We give $\mathcal{L}(Y)$ a norm by pulling back the norm from $\mathbb{B}(Y \oplus_c B)$ via π, thus $\mathcal{L}(Y)$ is a C^*-algebra *-isomorphic to the range of π. See e.g. [**26**] for more details if needed.

Since any C^*-module Y is a subspace of its linking C^*-algebra $\mathcal{L}(Y)$, it is consequently an operator space. By using the C^*-identity in the C^*-algebra $M_n(\mathcal{L}(Y))$ it is easy to see that the matrix norms on Y are given explicitly by the expression

$$\|[y_{ij}]\|_n = \|[\sum_{k=1}^{n} \langle y_{ki} | y_{kj} \rangle]\|^{\frac{1}{2}}, \qquad [y_{ij}] \in M_n(Y). \tag{3}$$

We call this the *canonical operator space structure* on the C^*-module.

We write $C_I(Y)$ for the C^*-module sum of I copies of Y, for a cardinal I. It is not hard to show that there is no conflict with the operator space notation for columns mentioned in Section 2.

By a result of Kasparov, $\mathbb{B}(Y) = M(\mathbb{K}(Y))$, where the latter is the multiplier C^*-algebra of $\mathbb{K}(Y)$ (see [**44**] for a proof). There is a less well known result along these lines for $B_B(Y)$, the set of bounded B-module maps on Y. We remark that it is an old result due to Wittstock [**70**] that $\|T\| = \|T\|_{cb}$ for module maps between C^*-modules (the proof is an easy exercise using, for example, (3), the equivalence of (i) and (ii) in Theorem 3.1, and a well known criterion for positivity of matrices [**65**, IV.3.2]). Thus $B_B(Y) = CB_B(Y)$ isometrically.

THEOREM 3.2. *If Y is a right C^*-module over a C^*-algebra B, then $B_B(Y)$ is isometrically homomorphic to $LM(\mathbb{K}(Y))$ the left multiplier operator algebra of $\mathbb{K}(Y)$. In particular, $B_B(Y)$ is isometrically homomorphic to an operator algebra. Also, $CB_B(Y)$ is completely isometrically homomorphic to the operator algebra $LM(\mathbb{K}(Y))$.*

The first part of the last theorem is due to H. Lin [**67**, Exercise 15.H]. A more general variant of the second part was first noticed by Paulsen in connection with [**19**]. One (perhaps too slick) way to see it, is to recall that if $A = \mathbb{K}(Y)$, then tensoring by Y is a 'completely isometric functor' F between the categories of right operator modules over A and over B, yielding a complete isometry from $CB_A(A)$

onto $CB_B(F(A)) \cong CB_B(Y)$ (see e.g. [**11**, Lemma 2.2], or [**17**, Chapter 8] for a different proof).

The adjointable maps may also be characterized metrically, just as we saw was the case for bounded module maps in Theorem 3.1. For example the adjointable B-module maps $T: Y \to Y$ on a C^*-module Y over B for which $T = T^*$, are exactly the B-module maps T (which we characterized above) for which T is Hermitian in the Banach algebra sense, in the Banach algebra $B(Y)$. This may be seen by Theorem 3.2. Indeed because $B_B(Y)$ is an operator algebra, any Hermitians which it contains must be in its diagonal C^*-algebra, which in this case is $M(\mathbb{K}(Y)) \cong \mathbb{B}(Y)$. Although we shall not use this, one may deduce from the last argument the following fact. If $u : Y \to Z$ is a map between two different right C^*-modules over B, then u is an adjointable B-module map if and only if there is a map $v: Z \to Y$ such that if R is the map

$$\begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{bmatrix} v(z) \\ u(y) \end{bmatrix}$$

from $Y \oplus_c Z \to Y \oplus_c Z$, then R is Hermitian in $B(Y \oplus_c Z)$, and R is a B-module map. We leave the proof of this as an exercise, using the remarks above. Since B-module maps were characterized above, this fact is in some sense a metric characterization of adjointable maps.

Finally we briefly mention W^*-modules, or as they are more formally known, *self-dual C^*-modules over a W^*-algebra.* These were introduced by Paschke in [**52**], and the basic theory of these objects may be found there (see also [**58, 66**] for example). For example he shows that if Y is a W^*-module, then $\mathbb{B}(Y)$ is a W^*-algebra.

The following result we will only need in Part II. We imagine that this result has been known for decades (see e.g. [**10**, 5.8]).

LEMMA 3.3. *If Z is a C^*-module over a C^*-algebra B, then Z^{**} is a C^*-module over B^{**}. Moreover $(Z \oplus_c B)^{**} \cong Z^{**} \oplus_c B^{**}$ completely isometrically and weak* homeomorphically.*

PROOF. (Sketch) Note that Z (resp. B, $Z \oplus_c B$) may be identified with $p\mathcal{L}(Z)q$ (resp. $q\mathcal{L}(Z)q$, $\mathcal{L}(Z)q$), where $\mathcal{L}(Z)$ is the linking C^*-algebra of Z above, and $p = 1 \oplus 0$ and $q = 0 \oplus 1$ are the canonical 'diagonal projections' in the multiplier C^*-algebra of $\mathcal{L}(Z)$. The second dual of $\mathcal{L}(Z)$ is a W^*-algebra, M say, and by basic duality principles we may identify Z^{**} (resp. B^{**}, $(Z \oplus_c B)^{**}$, $Z^{**} \oplus_c B^{**}$) with the 'corner' pMq (resp. qMq, Mq, Mq). This, on careful inspection, yields the claim. $\square$

Remark. A similar proof shows that for two C^*-modules Y, Z over B we have $(Y \oplus_c Z)^{**} \cong Y^{**} \oplus_c Z^{**}$ completely isometrically and weak* homeomorphically. The main difference in proof is to consider the second dual of the 'augmented linking C^*-algebra' $\mathbb{K}(Y \oplus_c Z \oplus_c B)$. A more general result may be found in [**17**, Section 8.5].

Once or twice towards the end of this paper we will need to appeal to a result due to Zettl, which states that a C^*-module over a W^*-algebra is a W^*-module if and only if it is a dual space (see [**31**] for a modern proof).

Part I. Multipliers and operator algebraic structures

4. The noncommutative Shilov boundary

This slightly technical section will only be used from time to time in the proofs in the sequel. The reader is encouraged to simply browse lightly through this material on a 'first read'.

In [**1, 2**] Arveson introduced the *noncommutative Shilov boundary* of an operator space which happens to contain I_H, and demonstrated its existence in several situations of particular interest. In [**36**] Hamana proved that this boundary always exists for any such X. We find it convenient here to purposely use the phrase 'noncommutative Shilov boundary' for what is usually called the 'C^*-envelope', and to write this object also as ∂X. Although some might object to this usage, it is not hard to justify[1]. In any case we find it convenient and intuitively appealing for our purposes here. More specifically, for an operator space containing I_H, the noncommutative Shilov boundary is taken to be a *smallest* unital C^*-algebra containing a copy $i(X)$ of X. By 'smallest', we mean that it has the following universal property: if B is any other unital C^*-algebra which contains, and is generated by, a copy $j(X)$ of X, then there exists a surjective $*$-homomorphism $\pi : \partial X \to B$ such that $\pi \circ i = j$. Here i and j are unital complete isometries. In the eighties, Hamana showed that *every* operator space X has a *noncommutative Shilov boundary*, which we shall write as ∂X or $(\partial X, i)$. In the nonunital case, the space ∂X may be viewed as a *smallest* C^*-module containing X completely isometrically, via a linear complete isometry $i : X \to \partial X$. By 'smallest', we mean again that ∂X has an appropriate universal property analogous to the one listed several lines above. We will spell out this universal property in the next few paragraph. To be symmetrical one should say 'C^*-bimodule' here instead of C^*-module, but since we are emphasizing left multipliers we shall think of ∂X as a right C^*-module over a C^*-algebra B say. Also, because every right C^*-module is a Morita equivalence *bimodule* in a canonical way, this will not lead to difficulties. The B-valued inner product on ∂X we shall write as $\langle\cdot|\cdot\rangle$; we refer to this as a (right) *Shilov inner product.* The ideal in B densely spanned by the range of this inner product will be written as $\mathcal{F}(X)$, or $\mathcal{F}$ if X is understood. By the universal property below it is easy to see that the noncommutative Shilov boundary, the Shilov inner product, and the C^*-algebra $\mathcal{F}$, are essentially unique up to an appropriate isomorphism.

To explain the universal property of ∂X we will need to introduce the language of 'ternary rings of operators', or *TRO*'s for short. A *concrete TRO* is a norm closed linear subspace Z of a C^*-algebra, or of $B(K, H)$, for two Hilbert spaces H and K, which is closed under the 'ternary product' $(x, y, z) \mapsto xy^*z$. Note that such a space Z is a right C^*-module over the C^*-algebra Z^*Z. The appropriate morphisms between TRO's are the *ternary morphisms*, namely the linear maps satisfying $T(xy^*z) = T(x)T(y)^*T(z)$ for all $x, y, z \in Z$. We remark in passing that ternary morphisms have all the nice properties that we associate with $*$-homomorphisms between C^*-algebras. For example they are automatically completely contractive,

[1] Indeed this choice corresponds to the admittedly inaccurate, but nonetheless quite common and useful, habit of thinking of a C^*-algebra as a 'noncommutative topological space', as opposed to its spectrum. The phrase also has no already established meaning in the literature, although there is an intimately related notion of a 'Shilov boundary ideal' [**1**].

have closed range, and become completely isometric upon quotient by their kernel. A 1-1 linear surjection between TRO's is a ternary morphism if and only if it is completely isometric (see [**37**], this is an old result to which one may attach the names Hamana, Harris, Kirchberg, and Ruan). In fact any ternary morphism $Y \to Z$ is the restriction to the copy of Y in the '1-2-corner', of a $*$-homomorphism between the linking C^*-algebras of Y and Z (see [**37**]).

By an (abstract) TRO we will mean an operator space Z with an abstract 'ternary product' $Z \times Z \times Z \to Z$, written xy^*z for $x, y, z \in Z$, such that there exists a completely isometric ternary morphism from Z onto a concrete TRO. We remark in passing that Neal and Russo have recently given intrinsic characterizations, up to complete isometry, of TRO's C^*-algebras, and one-sided ideals of C^*-algebras, as operator spaces with certain affine geometric properties [**49, 50**]. By a *subTRO* of a TRO we will mean a closed subspace Y which satisfies $YY^*Y \subset Y$. Examples of (abstract) TRO's include C^*-algebras, and also C^*-modules. To see the latter, suppose that Z is a C^*-module, and form the linking C^*-algebra $\mathcal{L}(Z)$ of Z. The copy of Z constituting the 1-2-corner of $\mathcal{L}(Z)$ is clearly a subTRO of $\mathcal{L}(Z)$, and hence it is a TRO. Explicitly, the 'ternary product' on Z is given by the formula $xy^*z = x\langle y|z\rangle$ if Z is a right C^*-module; by $xy^*z = \langle x|y\rangle z$ if Z is a left C^*-module; and by both expressions if Z is a C^*-bimodule. Note that longer expressions such as xy^*zw^* make sense too, for $x, y, z, w \in Z$, when interpreted as products within the linking C^*-algebra $\mathcal{L}(Z)$, and such products obey the usual associative laws, so that for example $xy^*zw^* = (xy^*z)w^* = (xy^*)(zw^*)$, and so on. If S is a subset of a TRO Z, then the *subTRO of Z generated by* S is the smallest subTRO of Z containing S, and it may be described as the closure of the span of expressions of the form $z_1 z_2^* z_3 \cdots z_{2n}^* z_{2n+1}$, for $n \in \mathbb{N}$ and $z_1, \cdots, z_{2n+1} \in S$.

We may now state formally the universal property of the noncommutative Shilov boundary. This property is extremely useful. It is also truly a universal property, in that a space having this property is essentially unique, up to an appropriate completely isometric ternary isomorphism fixing the copy of X.[2]

THEOREM 4.1. (Hamana) *Let X be an operator space and let $(\partial X, i)$ be a noncommutative Shilov boundary of X. If $j : X \to Z$ is a linear complete isometry from X into a TRO Z, such that $j(X)$ generates Z as a TRO, then there exists a surjective ternary morphism $\pi : Z \to \partial X$ such that $\pi \circ j = i$.*

As an example, we show that if X is a Hilbert space H, and if H^c is H thought of as a fixed column in $B(H)$, then we may take $\partial H^c = H^c$ and the inner product is just the usual one. More generally, if X is a right C^*-module then we may take $\partial X = X$, and the Shilov inner product to be the usual one. To see this, take $Z = X, j = Id$ in Theorem 4.1; the fact that $\pi \circ j = i$ forces i to be surjective and π 1-1; which means that (X, Id) also has the universal property in Theorem 4.1.

There are two methods we are aware of to construct a noncommutative Shilov boundary $(\partial X, i)$ of X. One is more recent and is implicit in work of Muhly and Solel [**47**], Woronowicz [**71**], Dritschel and McCullouch [**29**], Arveson [**3**], and, we believe, Kirchberg (although we are not sure if this work is in print). However we will focus on the older method due to Hamana; the reader unsatisfied with the

[2] An equivalent and alternative approach to the noncommutative Shilov boundary, is to *define* ∂X to be any space having this universal property. Of course one still needs to show the *existence* of such a space.

sketch below is referred to [**37**] or [**10**] for further details. In Hamana's method one begins with an *injective envelope* $(I(X), i)$ of the operator space X. By this term, we mean that $I(X)$ is an injective operator space, that $i : X \to I(X)$ is a complete isometry, and that the identity map is the only completely contractive linear map from $I(X)$ to itself extending the identity map on X. This is called the *rigidity property* of the injective envelope. In fact $I(X)$ may be chosen to be a full right C^*-module over a C^*-algebra $\mathcal{D}$ (see e.g. [**37**]; also $\mathcal{D}$ is the algebra denoted $I(X)^*I(X)$ in [**20**, p.3]). Hence $I(X)$ is a TRO. We let ∂X be the subTRO of $I(X)$ generated by $i(X)$. This is a full right C^*-module over a C^*-subalgebra of $\mathcal{D}$ which we shall consistently write as $\mathcal{F}(X)$, or $\mathcal{F}$ if X is understood. In fact by the considerations noted just before Theorem 4.1, it is easy to see that ∂X is the closure in $I(X)$ of the span of 'odd products' $x_1 x_2^* x_3 \cdots x_{2n}^* x_{2n+1}$, for $x_1, \cdots, x_{2n+1} \in X$, whereas $\mathcal{F}(X)$ is the closure in $\mathcal{D}$ of the span of 'even products' $x_1^* x_2 x_3^* \cdots x_{2n}$. Here we have suppressed, as we often do, explicit mention of the map i.

5. One-sided multipliers on operator spaces

The following theorem, which is one of the most important tools in this theory, may be viewed as the generalization to operator spaces of Theorem 3.1. As in Theorem 3.1, we have a 'module condition', an 'order condition', and a 'metric condition', all being equivalent. To explain the notation in this result: the inner product in (ii) is the (right) Shilov inner product, and the matrices there are indexed on rows by i, and on columns by j. The norms in (iii) are just the norm in $M_{2n,n}(X)$. Finally, note that in (i) the module action referred to is the action of the C^*-algebra $\mathcal{F}$ mentioned in the last paragraph (although in this connection see the observation after the Remarks after Theorem 3.1). By facts at the end of the last paragraph the map S in (i) below is necessarily unique.

THEOREM 5.1. [**10, 14**] *Let $T\colon X \to X$ be a linear map on an operator space X. The following are equivalent:*

(i) *T is the restriction to X of a contractive right module map S on ∂X.*
(ii) *$[\langle T(x_i)|T(x_j)\rangle] \leq [\langle x_i|x_j\rangle]$ for all $n \in \mathbb{N}$ and $x_1, \cdots, x_n \in X$.*
(iii) *For all $n \in \mathbb{N}$ and matrices $[x_{ij}], [y_{ij}] \in M_n(X)$ we have*

$$\left\| \begin{bmatrix} Tx_{ij} \\ y_{ij} \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} \right\|.$$

(iv) *There exists a completely isometric linear embedding of X into a C^*-algebra A, and an $a \in \mathrm{Ball}(A)$ with $Tx = ax$ for all $x \in X$.*

PROOF. The equivalence of (i)–(iii) here follows as a special case of more general results later (see e.g. Theorems 12.1 or 15.1), using a new idea, namely the use of Theorem 3.1.

(iv) $\Rightarrow$ (iii) If $x, y \in X$ and σ is the embedding then

$$\left\| \begin{bmatrix} T(x) \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} \sigma(Tx) \\ \sigma(y) \end{bmatrix} \right\| = \left\| \begin{bmatrix} a & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \sigma(x) \\ \sigma(y) \end{bmatrix} \right\| \leq \max\{\|a\|, 1\} \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|.$$

This proves (iii) if $n = 1$, and a similar argument with larger matrices does the general case.

(i) $\Rightarrow$ (iv) By Theorem 3.2 the operator S in (i) corresponds to an element $b \in LM(\mathbb{K}(\partial X))$. We may regard $LM(\mathbb{K}(\partial X)) \subset \mathbb{K}(\partial X)^{**}$. Let $\mathcal{L}$ be the linking C^*-algebra for ∂X constructed in Section 3, thought of as 2×2 matrices as we did there. Set $A = \mathcal{L}^{**}$, this is also a C^*-algebra, and may be viewed (via the canonical projections p, q mentioned in the proof of Lemma 3.3) as also consisting of 2×2 matrices. Let a be the 2×2 matrix in A with b as the 1-1-entry, and all other entries zero. We embed X into A by regarding X as a subspace of the 2×2 matrices in $\mathcal{L}$ 'supported on' the 1-2-entry. Then it is not hard to check that the desired condition in (iv) holds. $\square$

Remark: The C^*-algebra A constructed in the proof of (iv) of the above theorem clearly does not depend on the linear map T. It does depend on the particular noncommutative Shilov boundary chosen, but even then one can show that it does not depend essentially on this choice (that is, it is unique up to an appropriate isomorphism). We will not use this fact though.

There is another very useful equivalent criterion from [**20**], a characterization looking like (iv) of the Theorem, but with the first A there replaced by an injective envelope $I(X)$, and the second A there replaced by an injective C^*-algebra named I_{11} which acts on $I(X)$. We have not emphasized this approach in this paper (except at the very end) because we will not need it here; and because we wish to emphasise C^*-modules, and we can use that theory to prove what we need. Also, this approach has been well promoted elsewhere in the literature (see e.g. [**54**]). In fact we will mainly emphasize condition (iii) in this paper. The simplicity of this criterion (iii) makes it very useful and effective. For example one can show that this criterion is stable under most of the basic functional analytic operations: subspaces, tensor products, duality, interpolation, and so on. This is extremely useful in our theory and applications.

Definition 5.2. *We define a* left multiplier *of an operator space X to be a map $T : X \to X$ such that a positive scalar multiple of T satisfies the equivalent conditions of Theorem 5.1. We write $\mathcal{M}_l(X)$ to be the set of such left multipliers of X.*

It is easy to see from Theorem 5.1 that $\mathcal{M}_l(X)$ is a subalgebra of $B(X)$, and also is a subalgebra of $CB(X)$. Since the set D consisting of the operators T satisfying the equivalent conditions of Theorem 5.1 is a convex set (this may be seen via condition (i) above for example), it is easy to check that there is a norm on $\mathcal{M}_l(X)$ such that D is the closed unit ball of $\mathcal{M}_l(X)$ in this norm. We call this the *multiplier norm* $\|T\|_{\mathcal{M}_l(X)}$ of T. It is also fairly clear (from (iii)) that this norm dominates both the usual 'operator norm' and the 'completely bounded norm' of T. From (i), it follows that $\|T\|_{\mathcal{M}_l(X)}$ is just the norm of the unique (see remark above 5.1) right module map S on ∂X extending T. Thus $\mathcal{M}_l(X)$ is isometrically homomorphic to the subalgebra of $B_{\mathcal{F}}(\partial X)$ consisting of such maps S. By Theorem 3.2 above, $B_{\mathcal{F}}(\partial X) = CB_{\mathcal{F}}(\partial X)$ isometrically, and the latter is an operator algebra, completely isometrically homomorphic to the operator algebra $LM(\mathbb{K}_{\mathcal{F}}(\partial X))$. We may use the last two isometric homomorphisms to assign to $\mathcal{M}_l(X)$ the unique matrix norms making $\mathcal{M}_l(X)$ completely isometrically homomorphic, via the homomorphisms above, to a subalgebra of the operator algebra $CB_{\mathcal{F}}(\partial X)$.

There is a more useful and more intrinsic characterization of the matrix norms on $M_n(\mathcal{M}_l(X))$ for $n \geq 2$, which we now list. The proof that it is equivalent to the formulation in the last paragraph will be omitted since we prove a generalization of this fact in Part II (see Lemma 12.3). Given a matrix $t = [T_{ij}]$ in $M_n(\mathcal{M}_l(X))$, view t as a map $L_t : C_n(X) \to C_n(X)$ via the formula $L_t([x_i]) = [\sum_j T_{ij}x_j]$, for $[x_i] \in C_n(X)$. Then $\|[T_{ij}]\|_{M_n(\mathcal{M}_l(X))}$ is the 'multiplier norm' of L_t. That is,

$$M_n(\mathcal{M}_l(X)) \cong \mathcal{M}_l(C_n(X)) \tag{4}$$

isometrically, via the map $t \mapsto L_t$ above.

COROLLARY 5.3. *For any operator space X, $\mathcal{M}_l(X)$ is an operator algebra.*

PROOF. The discussion two paragraphs above showed that $\mathcal{M}_l(X)$ is completely isometrically homomorphic to a unital subalgebra of an operator algebra. □

Not surprisingly, one may also prove this last result using Theorem 2.1. However one of the attractive features of the left multiplier approach, and this was our main motivation for the introduction of operator space multipliers [**10**], is that it automatically yields theorems such as Theorem 2.1 as *corollaries*, and moreover often yields stronger versions of such theorems. We sketch the idea for this (which comes from [**10**], and was inspired by the paper [**16**] where the author and Le Merdy combined Kaijser and Tonge's work from the 1970's on function algebras with the classical theory of Banach space multipliers). Suppose that A is an algebra which is also an operator space, with identity of norm 1, and consider the 'regular representation' $\lambda : A \to B(A)$ given by

$$\lambda(a)(b) = ab, \qquad a, b \in A.$$

If we knew that:

$$\lambda \text{ has range inside } \mathcal{M}_l(A), \text{ and is a complete contraction,} \tag{5}$$

then it clearly follows that λ is a complete isometry, and A is an operator algebra. This is because $\mathcal{M}_l(A)$ is an operator algebra (see Corollary 5.3); and the other inequality needed to show that λ is completely isometric is obvious. For example to see that λ is isometric note that for $a \in A$ we have

$$\|a\| = \|\lambda(a)(1)\| \leq \|\lambda(a)\|_{B(A)} \leq \|\lambda(a)\|_{\mathcal{M}_l(A)}.$$

If A satisfies the norm inequality condition in Theorem 2.1 then in fact (5) follows rather quickly from the criterion in Theorem 5.1 (iii), as the reader can see by inspecting the first paragraph of the proof of 5.4 below (this deduction of Theorem 2.1 from criterion Theorem 5.1 (iii) was noticed independently by Paulsen).

The notions and results above have obvious modifications for *right multipliers* and the *right multiplier algebra* $\mathcal{M}_r(X)$. A *one-sided* multiplier is of course one that is either a left or a right multiplier. Note that any right multiplier commutes with every left multiplier. To see this we will use the formulation in (i) of Theorem 5.1. One approach is to take an element $w \in \partial X$ of the form xy^*z for $x, z \in X$ and $y \in \partial X$. As we said when we gave the construction of ∂X at the end of Section 4, the span of elements of this form are dense in ∂X. If S is a bounded right module map on ∂X, and if R is a bounded left module map on ∂X, then

$$R(S(xy^*z)) = R(S(x)y^*z) = S(x)y^*R(z).$$

Similarly, $S(R(xy^*z))$ equals this same quantity. Thus $RS = SR$.

PROPOSITION 5.4. *If A is a unital operator algebra, then $\mathcal{M}_l(A) \cong A$ completely isometrically homomorphically. Indeed the 'regular representation' $\lambda : A \to B(A)$*

given by $\lambda(a)(b) = ab$ for $a, b \in A$, is a completely isometric homomorphism onto $\mathcal{M}_l(A)$.

PROOF. We may suppose that A satisfies the norm conditions in Theorem 2.1 (it is easy to see that any operator algebra does satisfy these conditions). We first will check (5), and to do that we will verify the condition in Theorem 5.1 (iii). To that end, fix $a \in \text{Ball}(A)$, and choose $x, y \in A$. We have

$$\left\|\left[\begin{array}{c} \lambda(a)(x) \\ y \end{array}\right]\right\| = \left\|\left[\begin{array}{c} ax \\ y \end{array}\right]\right\| = \left\|\left[\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right]\left[\begin{array}{cc} x & 0 \\ y & 0 \end{array}\right]\right\| \leq \left\|\left[\begin{array}{cc} a & 0 \\ 0 & 1 \end{array}\right]\right\|\left\|\left[\begin{array}{c} x \\ y \end{array}\right]\right\|$$
$$\leq 1\left\|\left[\begin{array}{c} x \\ y \end{array}\right]\right\|.$$

The first '$\leq$' here follows from the norm inequality condition in Theorem 2.1, whereas the second follows from Ruan's axioms. A similar calculation prevails with x, y replaced by matrices in $M_n(X)$, and thus by Theorem 5.1 (iii), $\lambda(a) \in \text{Ball}(\mathcal{M}_l(A))$. Thus λ is a contraction into $\mathcal{M}_l(A)$. A similar calculation using larger matrices shows that λ is a complete contraction into $\mathcal{M}_l(A)$, verifying (5).

This, by the observation after (5), proves that λ is a complete isometry. To complete the proof it remains to check that λ maps onto $\mathcal{M}_l(A)$. For this it suffices to show that every map T in $\mathcal{M}_l(A)$ is a right A-module map on A. To achieve this, we first observe that by symmetry, the 'right regular representation' $\rho : A \to B(A)$ must map into the space of right multipliers. Since left and right multipliers commute, as we noted above Proposition 5.4, we have $T\rho(a) = \rho(a)T$ for $a \in A$. This shows that T is a right A-module map. $\square$

We remark in passing that, more generally, if A is an approximately unital operator algebra then $\mathcal{M}_l(A)$ is completely isometrically homomorphic to $LM(A)$ (this was defined at the end of Section 2). This was shown in [**10**], but in fact follows from the ideas in the last proof. We also remark that very recently there have been some breakthroughs related to operator algebras without any kind of identity, via the notion of *quasimultipliers* [**43**]. These quasimultipliers are intimately related to the one-sided multipliers above.

We now turn to the C^*-algebra $\mathcal{A}_l(X)$, which also has several equivalent definitions. For example it may be defined to be the class of maps satisfying the following equivalent conditions:

THEOREM 5.5. *Let X be an operator space and let $T : X \to X$ be a linear map. The following are equivalent:*

(i) *T is in the diagonal C^*-algebra of $\mathcal{M}_l(X)$ (see end of Section 2),*
(ii) *T is the restriction to X of an adjointable (in the usual C^*-module sense) map $R : \partial X \to \partial X$ such that $R(X) \subset X$ and $R^*(X) \subset X$,*
(iii) *There exists a map $S : X \to X$ such that $\langle T(x)|y\rangle = \langle x|S(y)\rangle$ (this is the (right) Shilov inner product), for all $x, y \in X$.*

Moreover the set of such maps T is a closed subalgebra of $B(X)$ which is a C^*-algebra with respect to the involution $T^* = S$, where S is related to T as in (iii).

PROOF. Recall from the paragraph after Definition 5.2 that $\mathcal{M}_l(X)$ is isomorphic to a unital subalgebra of the operator algebra $CB_{\mathcal{F}}(\partial X)$. By the argument in the paragraph after Theorem 3.2, the Hermitians in the latter algebra are exactly the operators $T \in \mathbb{B}(\partial X)$ with $T = T^*$. Since the 'diagonal C^*-algebra' is the span of the Hermitians it contains, it is clear that (i) is equivalent to (ii). That (ii) implies (iii) is obvious. For the proof that (iii) implies the other conditions see [**10**] or the later result Theorem 13.1. □

The following follows immediately from the last result, Theorem 5.1, and the example after Theorem 4.1:

EXAMPLE 5.6. (1) *For a Hilbert space H, $\mathcal{M}_l(H^c) = \mathcal{A}_l(H^c) = B(H)$ (recall H^c is H thought of as a fixed column in $B(H)$).*
(2) *For a C^*-algebra A, $\mathcal{A}_l(A)$ is the usual C^*-algebra multiplier algebra $M(A)$ of A, whereas $\mathcal{M}_l(A)$ is the usual left multiplier algebra $LM(A)$ (see Section 2).*
(3) *Generalizing* (1) *and* (2)*, if Y is a C^*-module, then $\mathcal{M}_l(Y)$ is the space of bounded module maps on Y, whereas $\mathcal{A}_l(Y)$ is the C^*-algebra of* adjointable maps *in the usual C^*-module sense mentioned in Section 3.*

These examples show that in certain cases, the spaces $\mathcal{A}_l(X)$ and $\mathcal{M}_l(X)$ do consist precisely of the maps on X of most interest.

It is clear from Theorem 5.5 (iii) that adjointable maps on operator spaces are modeled on adjointable maps on C^*-modules. It is therefore interesting to ask which of the 'classical' results for adjointable maps on C^*-modules are also valid for adjointable maps on operator spaces. We list a few examples (mostly from [**25**]) of results which do transfer in this way:

(1) It is striking that, for example, of the four sections of Chapter 15 of [**67**] devoted to the basic theory of C^*-modules, almost all of the results in the first three sections concerning adjointable maps go through with the same proofs! It was particularly surprising that Section 15.3 of that text, which develops the polar decomposition for adjointable maps, is valid with *identical proofs*.
(2) The behavior of one sided multipliers and adjointable maps with respect to duality often mimics closely the duality properties of maps on C^*-modules/TRO's and W^*-modules/weak* closed TRO's. See for example results in Section 8 below which may be viewed as generalizations of Paschke's important results that $\mathbb{B}(Z)$ is a W^*-algebra, and that any $T \in \mathbb{B}(Z)$ is weak* continuous, if Z is a W^*-module [**52, 53**].
(3) Generalizing the earlier equation (4), if X is a dual operator space and if I is a cardinal, then $\mathcal{M}_l(C_I^w(X)) \cong M_I(\mathcal{M}_l(X))$; and also $\mathcal{A}_l(C_I^w(X)) \cong M_I(\mathcal{A}_l(X))$ as von Neumann algebras. This generalizes the useful relation $\mathbb{B}(C_I^w(Z)) \cong M_I \bar{\otimes} \mathbb{B}(Z)$ as von Neumann algebras, for any W^*-module Z.

Another example of course is Theorem 5.1, which in the case that X is a C^*-module collapses to Theorem 3.1.

The following table is a very basic dictionary for comparing operator space notions in the remainder of this paper with well known C^*-module notions.

Operator space X	C^***-module** Z **over** B
adjointable multiplier	adjointable map
$\mathcal{A}_l(X)$	$\mathbb{B}(Z)$
left multiplier	bounded module map
$\mathcal{M}_l(X)$	$B_B(Z)$
left M-projection	projection in $\mathbb{B}(Z)$
right M-ideal	closed submodule
right M-summand	complemented submodule
Dual operator space	W^*-module/weak*-TRO
Column space $C_I(X)$	C^*-module sum

Note that the entries in the first column *generalize* those in the second (although this statement needs to be qualified for the entries in the last row). That is if X in the first column is taken to be a C^*-module, then the appropriate entry in each row in the first column coincides precisely with the entry in the second column.

Sadly, there are some useful results about adjointable maps on C^*-modules which are not valid adjointable maps on operator spaces. One example of this is the fact that $p\mathcal{A}_l(X)p$ is a C^*-subalgebra of, but may not always equal, $\mathcal{A}_l(pX)$. Here p is a projection in $\mathcal{A}_l(X)$.

More results along the lines of the list above will be mentioned in another list in Section 9.

6. Banach-Stone-Kadison theorems, and recovery of the product

In [**63**], Stone characterized isometries between general $C(K)$ spaces, generalizing an earlier result due to Banach. In [**42**], Kadison generalized the 'Banach-Stone' theorem by characterizing surjective isometries between C^*-algebras. This result has inspired very many functional analysts; see for example [**35**] for a recent collection of such results, together with their history. In [**1, 2**], Arveson used his 'noncommutative Shilov boundary' to obtain theorems of the Banach-Stone type for unital complete isometries between certain operator algebras [**1, 2**]. Later work along the lines indicated by Arveson was done by Blecher, Effros and Ruan, and others (see for example [**15, 13**] and references therein). In light of the fact that the left multiplier algebra from Section 5 is only defined in terms of the linear structure and matrix norms, and yet encodes the product in a unital operator algebra (see Proposition 5.4), one would expect to obtain Banach-Stone type theorems almost for free. And indeed this is the case, as we remarked above [**10**, Corollary 4.12]. For suppose, for example, that $\nu : A \to B$ is a linear surjective complete isometry between unital operator algebras. From Theorem 5.1 and the relation (4), it is clear that the map $T \mapsto \nu T \nu^{-1}$ is a completely isometric surjective homomorphism from $\mathcal{M}_l(A) \to \mathcal{M}_l(B)$. By Proposition 5.4, the left regular representation λ of A on itself, and 'its inverse', the map taking S to $S(1)$, are completely isometric surjective homomorphisms from $A \to \mathcal{M}_l(A)$ and $\mathcal{M}_l(B) \to B$ respectively. Composing these three maps, we get a surjective completely isometric homomorphism $B \to A$ which takes b to $\nu^{-1}(b\nu(1_A))$. Letting π be the inverse of this homomorphism, and setting $b = \pi(a)$ gives $\nu(a) = \pi(a)U$, where $U = \nu(1_A) \in B$. Since ν is a surjective isometry, there exists an $a_0 \in \mathrm{Ball}(A)$ with $1_B = \nu(a_0) = \pi(a_0)U$. A well known result from operator theory states that if $ST = I_H$ for two contractive operators, then $T^* = S$. Thus $U^* = \pi(a_0) \in B$, and U is an isometry in $B \cap B^*$. By symmetry,

$U = \nu(1_A)$ must also be a coisometry, and therefore a unitary, in $B \cap B^*$. We have therefore proved the following 'Banach-Stone' type theorem, at least in the case of unital operator algebras. The result in the generality stated here is from [**10**, Appendix B].

THEOREM 6.1. *Let A and B be approximately unital operator algebras. A linear surjection $\nu : A \to B$ is completely isometric if and only if there exists a completely isometric surjective homomorphism $\pi : A \to B$, and a unitary U with U in the diagonal C^*-algebra of $M(B)$ (see Section 2), such that $\nu(a) = \pi(a)U$ for all $a \in A$.*

We next address the question of whether the product on an operator algebra A may be recovered from its operator space structure alone. Suppose that $\mathcal{U}$ is the group of unitaries $U \in A$, with $U^{-1} \in A$. For $U \in \mathcal{U}$ the expression $m(x, y) = xU^{-1}y$ defines a new product on A with identity U, and with this product A is completely isometrically homomorphic, via the homomorphism $x \mapsto xU^{-1}$, to the A with its old product. These two operator algebras have the same operator space structure. Conversely, suppose that m is a new operator algebra product on A, such that A with product m is completely isometrically homomorphic to an operator algebra. Applying Theorem 6.1 to the identity map from A with product m to A with its old product, it is evident that there exists a $U \in \mathcal{U}$ with

$$m(x, y) = xU^{-1}y, \qquad x, y \in A. \tag{6}$$

Thus there is a bijective correspondence between $\mathcal{U}$, and the unital operator algebra products on A compatible with the given operator space structure of A. Moreover given one such product and the set $\mathcal{U}$ we may recover all other such products from the equation (6). In any case from the operator space structure of A alone one may only hope to recover the product up to this group $\mathcal{U}$ of unitaries.

Let us first suppose that we have forgotten the product on A, but that we do remember the identity element e. Form $\mathcal{M}_l(A)$ using for example the criterion in Theorem 5.1 (iii), and define $\theta \colon \mathcal{M}_l(A) \to A$ by $\theta(T) = T(e)$. Then it follows from Proposition 5.4 that the forgotten product on A is $ab = \theta(\theta^{-1}(a)\theta^{-1}(b))$, for $a, b \in A$. If however we have also forgotten the identity element e of A, then one may proceed as follows. Although there are no doubt better ways, one may identify $\mathcal{U}$ by Theorem 6.1 as the set comprised precisely of the elements $\rho(Id)$ for surjective linear complete isometries $\rho : \mathcal{M}_l(A) \to A$, where Id is the identity map on A. Given $U \in \mathcal{U}$, it follows from Proposition 5.4 again that the unique operator algebra product on A which has U as its identity element is given by the formula $ab = \theta(\theta^{-1}(a)\theta^{-1}(b))$, where $\theta \colon \mathcal{M}_l(A) \to A$ is the map $\theta(T) = T(U)$. These arguments are taken from [**12, 14**].

7. Multipliers and module actions

Modules of course represent another important aspect of algebraic structure. We recall first the trivial fact that *Banach module* actions $A \times X \to X$ of a Banach algebra A on a Banach space X, are in an obvious 1-1 correspondence with contractive unital homomorphisms $\theta : A \to B(X)$, via the relation

$$ax = \theta(a)(x), \qquad a \in A, x \in X. \tag{7}$$

Although this fact is trivial one should not underestimate its practicality.

A *concrete operator module* is a subspace $X \subset B(H)$, for which there is a completely contractive unital homomorphism $\pi : A \to B(H)$ with $\pi(A)X \subset X$. Here A is traditionally an operator algebra, and if A is a C*-algebra then it is well known that π as above must necessarily also be a $*$-homomorphism. In any case, such X is a left A-module with the module action $ax = \pi(a)x$, for $a \in A, x \in X$. An *abstract (left) operator module* is an operator space X which is also a left A-module over an operator algebra A, such that $\|ax\|_n \leq \|a\|_n\|x\|_n$, for $n \in \mathbb{N}, a \in M_n(A), x \in M_n(X)$. An important theorem of Christensen, Effros and Sinclair [**28**] states that up to the appropriate notion of completely isometric isomorphism of modules, abstract operator modules are the same thing as concrete operator modules. Similar definitions and results hold for right operator modules, or operator bimodules.

Examples of operator modules include the modules that are of most interest to C^*-algebraists, namely Hilbert modules (the Hilbert spaces that the algebras act on), and C^*-modules and the bimodules of the kind discussed towards the end of this section. The fact that C^*-modules are operator modules follows almost immediately from the existence of the linking C^*-algebra (defined in Section 3).

The next result is reminiscent of (but is much less obvious than) the correspondence in the lines above Equation (7); it gives a useful characterization of the class of operator modules in terms of homomorphisms. It also shows that the algebras $\mathcal{M}_l(X)$ and $\mathcal{A}_l(X)$ are key to any latent operator module structure in X.

THEOREM 7.1. *There is a bijective correspondence between operator module actions of an operator algebra A on an operator space X, and completely contractive unital homomorphisms $\theta : A \to \mathcal{M}_l(X)$. Indeed if A is a unital C^*-algebra then the correspondence is also with unital $*$-homomorphisms $A \to \mathcal{A}_l(X)$.*

In fact this result, first found by the author [**10**] (see [**20**] for an alternative approach) is now easy to prove using Theorem 5.1 (iii). To prove the easy direction we need only to know that every operator space X is a left operator $\mathcal{M}_l(X)$-module, which may be seen for example by essentially the proof of the implication (i) $\Rightarrow$ (iv) in Theorem 5.1, and the simple observation that if E and B are respectively a a subspace and a subalgebra of a C^*-algebra C, and if $BE \subset E$, with $1_B x = x$ for all $x \in E$, then E is an operator B-module. To prove the more difficult direction one simply follows the proof we gave in Section 5 for Theorem 2.1, but replacing λ by the canonical map from A to $B(X)$ given by the correspondence above (7). We remark that this proof requires nothing of A beyond that it be an operator space and unital algebra. This observation allows one to extend the Christensen, Effros and Sinclair theorem that we mentioned a few paragraphs back, to such algebras (see 5.3 in [**10**]).

Another connection to C^*-modules occurs as follows. The interesting class of bimodules found in C^*-module theory are right C^*-modules Z over a C^*-algebra B, for which there is a left action of another C^*-algebra A via a 'nondegenerate' $*$-homomorphism $A \to \mathbb{B}(Z)$. In fact such bimodules are operator A-B-bimodules. Conversely, a right C^*-module Z over B which is also a nondegenerate left operator module over A, is one of the bimodules just introduced. We give a quick proof of the last assertion in the case that A is unital: By Theorem 7.1, the left action corresponds to a unital $*$-homomorphisms $A \to \mathcal{A}_l(Z)$. By Example 5.6 (3), this is a unital $*$-homomorphism into $\mathbb{B}(Z)$. Note that this proof shows that the bimodule 'associativity' condition $(az)b = a(zb)$ is automatic. In fact this 'associativity' is automatic for any operator bimodule, as we showed in [**10**] (it follows immediately from Theorem 7.1 and the remarks above 5.4).

8. Duality, and a von Neumann algebra

We devote this brief section to duality, simply listing a few main results without proof.

Unlike its near relatives the noncommutative Shilov boundary and the injective envelope, the one-sided multiplier algebras above work extremely well with respect to duality. This is mostly because of the existence of the criterion in Theorem 5.1 (iii). Using this criterion we were able to prove the following fact from [**14**], which is another cornerstone of our theory:

Theorem 8.1. (Blecher, Effros and Zarikian) *If X is a dual operator space then $\mathcal{A}_l(X)$ is a W^*-algebra. Moreover every map in $\mathcal{A}_l(X)$ is automatically weak* continuous.*

This result opens a door to the use of von Neumann algebra techniques. Indeed as we hinted at in the introduction, many of our deepest results are only possible because of this theorem. For example, the type decomposition, or Murray-von Neumann equivalences, in $\mathcal{A}_l(X)$, have consequences for the structure of X. At the very least the theorem implies that for a dual space X, the algebra $\mathcal{A}_l(X)$ will have lots of orthogonal projections (if it is nontrivial), and that these projections may be utilized in the ways von Neumann algebraists are familiar with. These projections will be studied in the next Section.

As remarked earlier, Theorem 8.1 may be viewed as a generalization of Paschke's result that $\mathbb{B}(Z)$ is a W^*-algebra for any W^*-module Z (in this connection recall Zettl's result stated at the end of Section 3).

Another useful duality result is the following fact from [**25**]. Via the map $T \mapsto T^{**}$, it is not hard to show that $\mathcal{A}_l(X)$ is completely isometric to a C^*-subalgebra of the W^*-algebra $\mathcal{A}_l(X^{**})$. More is true for dual spaces:

Theorem 8.2. *For a dual operator space X there is a conditional expectation from the W^*-algebra $\mathcal{A}_l(X^{**})$ onto the C^*-subalgebra $\mathcal{A}_l(X)$, and this conditional expectation is weak* continuous with respect to the (unique) weak* topologies making these spaces W^*-algebras.*

Using duality properties of multipliers, one may prove versions of many of the basic theorems characterizing operator algebras and their modules which are appropriate to dual spaces and the weak* topology (see [**12**]). For example, Theorems 5.1 and 5.4 were key to the first proof (see [**12**], together with [**45**]) of the following 'nonselfadjoint version' of Sakai's characterization of von Neumann algebras as the C^*-algebras which have a predual.

Theorem 8.3. (Le Merdy-Blecher) *An operator algebra A is completely isometrically isomorphic via a weak* homeomorphism to a weak* closed unital subalgebra of $B(H)$, if and only if A is a dual operator space.*

9. One-sided ideals and projections in operator spaces

In this section we describe, very briefly and again without proofs, a few facts about the noncommutative variant (due to the author, Effros, and Zarikian), of Alfsen and Effros' notion of *M-ideals* in a Banach space [**4**]. Although the definition of an M-ideal (given below) is stated in purely Banach space language, the M-ideals in a C^*-algebra are exactly the closed two-sided ideals (see e.g. [**4, 62**]). In a given Banach space there may not exist any nontrivial M-ideals, but if there do then they are often very significant. There is by now a vast and important theory of these 'M-ideals' [**39**]. We therefore seek a generalization of the classical M-ideal notion to operator spaces, which has the property that the classical M-ideals are the left M-ideals in $MIN(X)$,[3] and such that the left M-ideals in C^*-algebras are the left ideals.

The key to our approach is the observation that in the classical case, the M-ideals in X correspond to members of a certain family of commuting projections on X^{**}, and this family may be viewed as orthogonal projections in a certain commutative von Neumann algebra. This is explained well in the text [**5**] (for example see the use of boolean algebras there related to Stone's representation theorem for such algebras). This commutative von Neumann algebra happens to be the 'classical variant' of our algebra $\mathcal{A}_l(\cdot)$ of adjointable maps (see Corollary 4.22 in [**10**]). It is natural then to think of defining *right M-ideals* of an operator space X in terms of a certain family of noncommuting projections[4] on X^{**}, namely certain projections in the W^*-algebra $\mathcal{A}_l(X^{**})$ (see Theorem 8.1). This all suggests the following definition:

DEFINITION 9.1. [**14**] *A* left M-projection *on an operator space X is an orthogonal projection in $\mathcal{A}_l(X)$. A* right M-summand *is the range of a left M-projection. A* right M-ideal *in X is a closed subspace J of X such that $J^{\perp\perp}$ is a right M-summand in X^{**}.*

Actually we have several equivalent definitions of left M-projections [**14**]. For example they are the linear idempotent maps P on X for which there is some completely isometric embedding $X \hookrightarrow B(H)$ with $x^*y = 0$ for all $x \in P(X), y \in (I-P)(X)$. Another useful equivalent condition is that the map

$$x \mapsto \begin{bmatrix} P(x) \\ x - P(x) \end{bmatrix}$$

be a complete isometry from $X \to C_2(X)$. Loosely speaking, this is saying that X is a 'column sum' of $P(X)$ and $(I-P)(X)$. Thus for example $X \oplus 0$ is a right M-summand in $C_2(X)$.

The definitions in 9.1 are directly inspired by the classical definitions, which we briefly recall: An M-projection on a Banach space X is an idempotent linear map $P : X \to X$ such that the map $x \mapsto (P(x), (I-P)(x))$ from $X \to X \oplus_\infty X$ is an isometry. A closed subspace $J \subset X$ is an *M-ideal* if $J^{\perp\perp}$ is the range of an M-projection on X^{**}.

[3] We recall that MIN here denotes the simplest way of turning a Banach space X into an operator space. Namely since every Banach space X is embedded isometrically into a commutative C^*-algebra $C(K)$, it will inherit a matrix norm from the norm on $M_n(C(K)) \cong C(K, M_n)$, and it is not hard to see that these norms are independent of the particular embedding in a $C(K)$). It is very nice when a given 'noncommutative/operator space theory', applied to MIN(X) yields just the associated classical theory. This will be the case for us here.

[4] The *complete M-ideals* of Effros and Ruan [**32**], while very important, do correspond to a 'commutative theory' in a sense on can make precise—see [**25**, Section 7]. These ideals turn out to be exactly the left M-ideals which are also right M-ideals [**14**].

Again we note that the definitions above are only in terms of the linear structure and matrix norms of X, and yet often encode important algebraic information, as the following examples from [**14**] show:

EXAMPLE 9.2. (1) *The M-ideals in a Banach space X are exactly the right M-ideals in $MIN(X)$.*
(2) *The right M-ideals in a C^*-algebra are exactly the closed right ideals.*
(3) *The right M-ideals in a C^*-module are exactly the closed submodules.*
(4) *The right M-ideals in H^c for a Hilbert space H, are just the closed subspaces.*
(5) *The right M-ideals in an operator algebra are exactly the closed right ideals with a left contractive approximate identity.*

Thus again we see that in a given operator space these one-sided M-ideals are often highly significant. So too are the one-sided M-summands; which for example in a C^*-algebra are principal ideals, and in a C^*-module are exactly the orthogonally complemented submodules.

In early '00, Effros proposed generalizing the basic calculus of Banach space M-ideals (as may be found in the first couple of chapters of [**39**] say) to 'one-sided ideals' in operator spaces, in order to forge a useful and appropriate tool for 'noncommutative functional analysis'. He was motivated in part by his earlier work [**30**]. This program was initiated in [**14, 72**]. Recently the author and Zarikian have essentially completed this [**24, 25**]. With very few exceptions we now know exactly which of the basic results generalize, which do not, and why. It seems that there are perfect (nontrivial) analogues of approximately half the results in the classical M-ideal and M-summand calculus. For the other half, some additional and natural hypotheses usually need to be imposed in order to obtain a correct statement of the result. We refer the reader to [**24**] for an accessible short survey of this topic. As we said earlier, our deepest positive results are mainly rooted in the fact that $\mathcal{A}_l(X)$ is a W^*-algebra if X is a dual operator space. Instead of acting on a Hilbert space, $\mathcal{A}_l(X)$ acts on X; but nonetheless the same spatial principles and intuitions apply. We can use basic von Neumann algebra projection techniques and spectral theory to deduce structural conclusions about the operator space X. Thus we generalize the classical theory of M-ideals in a truly noncommutative way. Another main technique of ours is to first prove that left multipliers have a certain properties, and then to use these properties to deduce facts about right M-ideals.

Again we end this section by emphasizing that many of our results may be viewed as generalizations of facts about submodules of C^*-modules. To justify this claim we list a few parallels between these two classes:

(1) It is shown in [**14**] that a right M-ideal which is a dual space is a right M-summand. The analogous result for C^*-modules states that a submodule of a C^*-module which is a dual space is orthogonally complemented.
(2) In [**14**] we observed that there is at most one contractive linear projection onto a right M-ideal J. If there does exist one, then J is a right M-summand. Thus one may conclude that if a submodule W of a C^*-module is linearly contractively complemented, then it is orthogonally complemented. Another application along these lines: if W does not contain an isomorphic copy of c_0, then it is orthogonally complemented. See [**25**, Section 2.3] for details.

(3) In [**25**] it is shown that one-sided M-ideals satisfy a Kaplansky density theorem. That is the unit ball of a one-sided M-ideal in a dual operator space is weak* dense in the unit ball of its weak* closure. An analogous result holds for submodules of W^*-modules (see e.g. [**40**, Theorem 3.6]).

10. Multipliers, Morita equivalence, and double commutant

In a series of papers, with the help of operator space theory, we have made several contributions to the general theory of nonselfadjoint operator algebras (see e.g. the survey [**7**], although this dates to 1997). This field definitely lacks certain fundamental tools which are available in the selfadjoint case, such as von Neumann's double commutant theorem. Recently B. Solel and the author [**23**] found the following double commutant theorem:

THEOREM 10.1. *Every approximately unital operator algebra A possesses canonical classes of completely isometric nondegenerate representations π of A on a Hilbert space, satisfying a 'double commutant theorem':*

$$\pi(A)'' = \overline{\pi(A)}^{w^*}.$$

Moreover for a subclass of these representations π, we have $\pi(A)'' = \overline{\pi(A)}^{w^*} \cong A^{**}$ completely isometrically.

If A is a 'dual operator algebra', that is if A is one of the algebras characterized in Theorem 8.3, then the 'weak* closure' in the Theorem above is not necessary.

Since every $*$-representation of a C^*-algebra is in one of our 'canonical classes', Theorem 10.1 formally generalizes von Neumann's double commutant theorem. However we warn the reader that in the nonselfadjoint case these representations may be rather large, and therefore impractical for most purposes. In this section we will indicate briefly one application of Theorem 10.1 to a context which is also related to the multiplier algebras mentioned in earlier sections. This application comes from a sequel paper to [**23**] in preparation, and the idea comes from a situation in the Morita theory of C^*-algebras where von Neumann's double commutant theorem is pivotal. In the theory of C^*-algebraic Morita equivalence, or in the related theory of C^*-modules, one often proves a result about a right module X over B by taking *any* representation (we don't care how 'big' it is) of B, or of the linking C^*-algebra of X, on a Hilbert space. One then views all objects as acting on that Hilbert space, and then one performs a computation there, often using von Neumann's double commutant theorem. See for example [**58**, Theorem 6.5].

The afore-mentioned application from the sequel to [**23**] shows that the same sort of thing can be done in the framework of the nonselfadjoint Morita equivalence of the author, Muhly and Paulsen [**19**]. If X is an equivalence A-B-bimodule in the sense of that paper, then as in the C^*-algebraic situation it is often important to know that X^{**} may be made into some kind of equivalence bimodule too, in a way that is functorial. This is where Theorem 10.1 comes in. Just as in the C^*-algebra case, one represents the linking operator algebra $\mathcal{L}(X)$ of X (see [**19**, Chapter 5]) completely isometrically and nondegenerately on a Hilbert space, via one of the representations from Theorem 10.1 in which $\mathcal{L}(X)^{**} = \mathcal{L}(X)''$. By carefully computing the commutants $\mathcal{L}(X)'$ and $\mathcal{L}(X)''$, as in [**46**, Theorems 4.1 and 4.2] (see also the discussion in [**10**, Lemma 5.8]), one can show that X^{**} is an A^{**}-B^{**}-equivalence bimodule in a good sense.

The main point is that Theorem 10.1 allows one to make the passage smoothly (and functorially) to the second dual of the equivalence bimodule.

We will next relate these equivalence bimodules, and the discussion in the last paragraph, to the multiplier algebras from earlier sections. In fact the remainder of this section may be considered to be a long and technical example giving another illustration of many of the points in this paper, and giving some applications of these points to equivalence bimodules. The reader unfamiliar with the concepts in [**19**] should perhaps just browse the remaining results in this Section. We refer to that paper and [**18**] for all definitions. We will also rely heavily on facts in [**37**] (which are repeated in Appendix A of [**10**]), such as the universal property of ∂X, and its consequences. In this Section we write $C^*_e(\cdot)$ for the C^*-envelope of an approximately unital operator algebra (see e.g. [**10**, p. 310]).

We thank Baruch Solel for permission to publish the following background facts (which the author proved during exchanges with him in '00) here instead.

Lemma 10.2. *Suppose that X is a strong Morita equivalence A-B-bimodule in the sense of* [**19**]. *Then the canonical strong Morita equivalence $C^*_e(A)$-$C^*_e(B)$-bimodule W induced by X as in* [**18**], *is a noncommutative Shilov boundary ∂X of X.*

Proof. We consider the linking operator algebra L (see [**19**, Section 5]) for X. Its C^*-envelope may be identified with the linking C$^*-$algebra L' for the strong Morita equivalence of $C^*_e(A)$ and $C^*_e(B)$ with equivalence bimodule W. This, and other facts quoted below, may be found in [**18**] (see also [**6**, p. 407]). We can think of W as the closure of the span of $C^*_e(A)X$ (or $XC^*_e(B)$) in $B(K,H)$, where L is represented completely isometrically and nondegenerately on $H \oplus K$. We claim that the copy of X in the corner of the linking algebra L for X, generates L'. Indeed notice that if e_α is a c.a.i. for A of the form $\sum_k x_k y_k$, with $\sum_k x_k x_k^* \leq 1$ and $\sum_k y_k^* y_k \leq 1$, then $e_\alpha e_\alpha^*$ is a c.a.i. for $C^*_e(A)$ by [**7**] Lemma 8.1. Write $v_\alpha = \sum_k x_k x_k^*$ (it depends on α). Note that $e_\alpha e_\alpha^* \leq v_\alpha \leq 1$, from which it follows that v_α is a c.a.i. for $C^*_e(A)$ which lies in XX^*. Hence $A \subset C^*(XX^*)$ (since any $a \in A$ satisfies $av_\alpha \in XX^* \subset C^*(XX^*)$). Hence $C^*(XX^*) = C^*_e(A)$. Therefore $C^*_e(A)X \subset C^*(XX^*)X$, so that W is generated by X as a TRO.

By the universal property 4.1, there is a surjective ternary morphism $\psi : W \to \partial X$ such that $\psi_{|X} = i$. By [**37**, Proposition 2.1], or [**10**, A.2 or A.6], there is a surjective $*$-homomorphism θ from L' onto the linking C^*-algebra for ∂X, whose '1-2-corner' is ψ. The 2-2-corner of θ is a surjective $*$-homomorphism $\rho : C^*_e(B) \to \mathcal{F}(X)$. It is easy to see that since θ is a homomorphism we have $i(x)\rho(b) = i(xb)$ for all $b \in B, x \in X$. From this we see that ρ is actually completely isometric when restricted to B (since the associated homomorphism $B \to CB(X)$ is a complete isometry [**19**, Theorem 4.1]).

Next note that $\rho(B)$ generates $\mathcal{F}(X)$ as a C^*-algebra, since B generates $C^*_e(B)$ and $\rho : C^*_e(B) \to \mathcal{F}(X)$ is surjective. So by the universal property of $C^*_e(B)$ there is a $*$-homomorphism $\pi : \mathcal{F}(X) \to C^*_e(B)$ such that $\pi(\rho(b)) = b$ for all $b \in B$. It follows that $\pi \circ \rho = Id$ on $C^*_e(B)$. Thus ρ is 1-1. From this it is easy to see that ψ is 1-1 (note that $\psi(w)^*\psi(w) = \rho(w^*w)$). Thus W may be taken to be the noncommutative Shilov boundary of X. $\square$

Corollary 10.3. *Suppose that X is a strong Morita equivalence A-B-bimodule in the sense of* [**19**]. *Then $\mathcal{M}_l(X) = CB_B(X) \cong LM(A)$ completely isometrically and homomorphically.*

Proof. With the notation of Lemma 10.2, and by basic C^*-algebraic Morita theory, $\mathbb{K}(W) \cong C_e^*(A)$. By the discussion after Definition 5.2, and by Theorem 3.2, we have $\mathcal{M}_l(X) \cong \{d \in LM(C_e^*(A)) : dX \subset X\}$. However $dX \subset X$ if and only if $dA \subset A$ (since A is the closure of XY, and $X = AX$). Thus $\mathcal{M}_l(X) \cong \mathcal{M}_l(A) = LM(A)$. From functorial considerations from the basic theory of strong Morita equivalence (see [**19**] say), $CB_B(X) \cong CB_A(A) \cong LM(A)$. □

Such results will be useful in the study of strong Morita equivalence of operator algebras. For example, it follows from facts in [**12**] that if in addition to the hypothesis in the last result, X is a dual operator space, then $LM(A) \cong CB_B(X)$ is a dual operator algebra, and moreover the left actions of these last two isomorphic algebras on X are weak*-continuous in the first variable. This may be viewed as a variant of Paschke's result that $B_B(Z)$ is a W^*-algebra if Z is a W^*-module.

Another application of this result is that it allows one to characterize the complete M-summands (in either the sense of [**32**], or in the 'one-sided' sense of [**14**]) in such a Morita equivalence bimodule. For example, the left M-projections on X, which as we said in Section 9 may be defined be the contractive projections in $\mathcal{M}_l(X)$, are by the last result exactly the idempotent completely contractive B-module maps on X. The ranges of such maps, the right M-summands, are thus exactly what was called in [**6**] the 'c-complemented' B-submodules of X. The complete M-summands (in the sense of [**32**]) in X are, by [**14**, 6.2], the c-complemented A-B-submodules of X, that is, the range of an idempotent completely contractive A-B-module map on X. To treat the right M-ideals, one needs to consider X^{**}, and then to apply there the summand case just discussed. However to do this one needs to be able to view X^{**} as an equivalence bimodule too, and this is precisely where Theorem 10.1 comes in, and the discussion in the fifth paragraph of this section. In that paragraph we indicated how one checks that X^{**} has an 'equivalence bimodule structure'. From this it is not hard to compute the one-sided M-ideals of X.

The following result, and the related final paragraph of this section, were mentioned without proof at the end of Appendix A of [**10**]. They shows that the *operator space structure* of an equivalence bimodule essentially encodes all the information. Indeed the next result is really a Banach-Stone-Kadison type theorem for equivalence bimodules over nonselfadjoint algebras:

Corollary 10.4. *If T is a surjective linear completely isometry between two strong Morita equivalence bimodules (over nonselfadjoint operator algebras), then T is an "equivalence bimodule" isomorphism.*

Before we begin the proof, we remark that what we mean by saying that T is an "equivalence bimodule" isomorphism, is that T is the '1-2-corner' of a completely isometric isomorphism between the linking operator algebras (see [**19**, Section 5]); an isomorphism which takes each corner onto the matching corner. This way of stating the definition of an equivalence bimodule isomorphism is shorthand for a long list of algebraic conditions, some of which will appear explicitly in the proof below, such as $\lambda(a)T(x) = T(ax)$, etc.. The interested reader can write down the entire list.

PROOF. We follow the proof of [**10**, Corollary A.5]. Suppose that (A_i, B_i, X_i, Y_i) are the Morita contexts, for $i = 1, 2$, and suppose that $T : X_1 \to X_2$ is a surjective complete isometry. For $i = 1, 2$, suppose that W_i is the induced C^*-algebraic $C^*_e(A_i)$-$C^*_e(B_i)$-bimodule discussed in Lemma 10.2, with $J_i : X_i \to W_i$ the canonical embedding. Then by a standard diagram chase one sees that $(W_1, J_1 \circ T^{-1})$ has the desired universal property for, and hence it is, a noncommutative Shilov boundary of X_2. Thus W_1 and W_2 are isomorphic as ternary systems, via a ternary morphism R say extending T. As in the proof of 10.2 there exists a matching $*$-isomorphism θ from the linking C$^*-$algebra of W_1 to the linking C$^*-$algebra of W_2, which preserves corners, and whose 1-2-corner is R. We will show that θ restricted to the linking algebra of X_1 yields the required isomorphisms. Let $\lambda : C^*_e(A_1) \to C^*_e(A_2)$ be the 1-1-corner of θ. Then $T(ax) = R(ax) = \lambda(a)R(x) = \lambda(a)T(x)$ for any $a \in A, x \in X$. Hence $\lambda(a)X_2 \subset X_2$, so that $\lambda(a)A_2 \subset A_2$ since $A_2 = X_2Y_2$. Since A_2 contains an approximate identity for $C^*_e(A_2)$ (see [**7**, 8.1]) we have $\lambda(a) \in A_2$. By looking at θ^{-1}, and using a similar argument to the above we see that λ is a completely isometric isomorphism from A_1 onto A_2. Similarly, there is a completely isometric isomorphism $\rho : B_1 \to B_2$ such that $T(xb) = T(x)\rho(b)$. Defining $S : W_1^* \to W_2^*$ to be the 2-1 corner of θ, we have $S(y) = R(y^*)^*$ for $y \in Y_1$. Using the fact that θ is a homomorphism, we have for $x \in X, y \in Y$ that $\theta(xy) = T(x)S(y)$. Since T is onto, we have $X_2S(y) \subset A_2$, so that $B_2S(y) = Y_2X_2S(y) \subset Y_2A_2 \subset Y_2$. Hence $S(y) \in Y_2$, since B_2 acts nondegenerately on W_2. By looking at θ^{-1} and using a similar argument to the above, we see that S is a complete isometry from Y_1 onto Y_2. The rest of the assertion is clear. $\square$

In the spirit of Section 6 above, we discuss next how one can essentially recapture all the 'Morita equivalence data' from the operator space structure of X (this was remarked on without proof in [**10**, p. 338]). In fact suppose that we are given a naked operator space X, and are told only that it is the underlying operator space for some strong Morita equivalence bimodule in the sense of [**19**] (we are not told what the associated algebras are). Form the noncommutative Shilov boundary ∂X, this will be a C^*-algebraic strong Morita equivalence bimodule over two C^*-algebras $\mathcal{E}$ and $\mathcal{F}$. Let $A = \{a \in \mathcal{E} : aX \subset X\}$, and $B = \{b \in \mathcal{F} : Xb \subset X\}$. Let $Y = \{\bar{z} \in \overline{\partial X} : X\bar{z} \subset A\}$. The Morita pairings $X \times Y \to A$ and $Y \times X \to B$ are then the canonical ones obtained by restricting the canonical inner products on ∂X to X and Y. The reason why this works is that one can check that it certainly works in the case that ∂X is the containing C^*-algebra equivalence bimodule W of Lemma 10.2. Hence by the essential uniqueness of the noncommutative Shilov boundary (which follows from its universal property), it works for any other noncommutative Shilov boundary (W_2, j). Basically the idea is similar to the idea for the proof of Corollary 10.4, one obtains a $*$-isomorphism between the linking C^*-algebras for W and W_2 which in the 1-2-corner fixes the copies of X, and takes the linking operator algebra (with corners A, B, X, Y) to another operator algebra whose four corners are a Morita context for $j(X)$, making $j(X)$ "equivalence bimodule" isomorphic to X in the sense used in Corollary 10.4.

Part II. Multipliers between two operator spaces

11. The goal, the obstacle, and the C^*-module case

The main goal of this Part is to introduce one-sided multipliers between two different operator spaces X and Y, and to initiate a study of such maps. Recalling that (adjointable, say) C^*-module maps are a good model for these multipliers, it seems quite apparent that we need some relation between X and Y to have any hope of such maps making sense. This is because it seems unrealistic to expect to have an interesting class of maps between general C^*-modules over two quite unrelated algebras. But in fact there is another obstacle which haunts the entire theory presented in this paper, namely that although we can take 'column sums' of copies of the same space X (this is just the sum $C_n(X)$ met in Section 2), the 'column sum' of two unrelated operator spaces is usually not well defined in any useful way. This is similar to the obstacle one meets in trying to define C^*-module sums of C^*-modules over quite unrelated algebras. In this Part we propose some methods to overcome these obstacles; namely we define the 'relative column sum', and we define multipliers from X to Y relative to a containing operator space V, or relative to two fixed containing C^*-modules over two C^*-algebras which are related to each other.

We begin by looking at the C^*-module case, and by considering a generalization of Theorem 3.1. This will be used in later sections to prove generalizations of Theorem 5.1.

Suppose that W and Z are right C^*-modules over C^*-algebras B and C respectively, where B is a C^*-subalgebra of C. Then we may define an analogue of the C^*-module direct sum. Namely we endow the algebraic sum $W \oplus Z$ with the norm

$$\left|\left|\left| \begin{bmatrix} w \\ z \end{bmatrix} \right|\right|\right| = \|\langle w|w\rangle + \langle z|z\rangle\|^{\frac{1}{2}}. \tag{8}$$

There are many ways to see that this is a norm; one way that will be useful for us later is as follows. We will identify $W \oplus Z$ with a subspace of a C^*-module direct sum $E \oplus_c Z$, for a certain C^*-module E over C, in such a way that the induced norm is just the one in the displayed equation above. Namely, we define E to be the 'C^*-module interior tensor product' (see e.g. [**44**]) of W and BC, where the latter is the closure in C of the span of products bc for $b \in B, c \in C$. The obvious map from W into E taking wb to $w \otimes b$, for $w \in W, b \in B$, is easily checked to be a well defined isometry, and so we may regard W as a subspace of E, and $W \oplus Z$ as a subspace of $E \oplus_c Z$. The norm on $W \oplus Z$ inherited from $E \oplus_c Z$ is then just the norm above. More is true (although we shall not need this in this section), the operator space matrix norms on $W \oplus Z$ inherited from the canonical operator space structure on the C^*-module $E \oplus_c Z$, are easily calculated from Equation (3) to be:

$$\left|\left|\left| \begin{bmatrix} w_{ij} \\ z_{ij} \end{bmatrix} \right|\right|\right| = \left|\left|\left[\sum_{k=1}^{n} \langle w_{ki}|w_{kj}\rangle + \langle z_{ki}|z_{kj}\rangle\right]\right|\right|^{\frac{1}{2}}, \qquad [w_{ij}] \in M_n(W), [z_{ij}] \in M_n(Z).$$

We write $W \oplus^C Z$ for the resulting operator space structure on $W \oplus Z$.

The following is a mild generalization of Theorem 3.1. The norm on the right hand side of (iii) and (iv) below is the norm defined in (8):

THEOREM 11.1. *Suppose that W and Z are right C^*-modules over C^*-algebras B and C respectively, where B is a C^*-subalgebra of C. If $u : W \to Z$ is a $\mathbb{C}$-linear map, then the following are equivalent:*

(i) *u is a contractive B-module map;*
(ii) *$\langle u(w)|u(w)\rangle \leq \langle w|w\rangle$, for all $w \in W$;*
(iii) $\left\|\begin{bmatrix} u(w) \\ z \end{bmatrix}\right\| \leq \left\|\begin{bmatrix} w \\ z \end{bmatrix}\right\|$, *for all $w \in W, z \in Z$.*
(iv) $\left\|\begin{bmatrix} u(w) \\ c \end{bmatrix}\right\| \leq \left\|\begin{bmatrix} w \\ c \end{bmatrix}\right\|$, *for all $w \in W, c \in C$.*

PROOF. (i) $\Rightarrow$ (ii) We follow Paschke's argument. Without loss of generality we may assume that B and C are both unital (otherwise replace them both by their unitizations, and note that u is still a module map over these unitizations). For $x \in W$, set $h_n = (\langle x|x\rangle + \frac{1}{n})^{-\frac{1}{2}}$, and set $x_n = xh_n$. Then

$$\langle x_n|x_n\rangle = h_n\langle x|x\rangle h_n \leq h_n\left(\langle x|x\rangle + \frac{1}{n}\right)h_n = 1,$$

so that $\|x_n\| \leq 1$. Hence $\|u(x_n)\| \leq 1$, so that $\langle u(x_n)|u(x_n)\rangle \leq 1$. Since u is a module map it follows that $\langle u(x)|u(x)\rangle \leq \langle x|x\rangle + \frac{1}{n}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ we obtain $\langle u(x)|u(x)\rangle \leq \langle x|x\rangle$.

In the remainder of the proof we use the following notation: if $u\colon W \to Z$ is any map, and if X is another C^*-module over C, then we will write τ_u or τ_u^X for the map $u \oplus I$ between the module sums $W \oplus^C X \to Z \oplus_c X$. That is:

$$\tau_u\left(\begin{bmatrix} w \\ x \end{bmatrix}\right) = \begin{bmatrix} u(w) \\ x \end{bmatrix},$$

for $w \in W, x \in X$.

(ii) $\Rightarrow$ (iii) Assuming (ii), we have

$$\left\|\tau_u\left(\begin{bmatrix} w \\ z \end{bmatrix}\right)\right\|^2 = \|\langle u(w)|u(w)\rangle + \langle z|z\rangle\|^2 \leq \|\langle w|w\rangle + \langle z|z\rangle\|^2 = \left\|\begin{bmatrix} w \\ z \end{bmatrix}\right\|^2.$$

(ii) $\Rightarrow$ (iv) This is similar to the last calculation.

(iv) $\Rightarrow$ (iii) Set $c = \langle z|z\rangle^{\frac{1}{2}}$.

(iii) $\Rightarrow$ (i) We first note that it suffices to prove this implication in the case that $W = B$. To see this, fix $w \in \mathrm{Ball}(W)$, define $\epsilon_w : b \mapsto wb$ on B, and set $S = u \circ \epsilon_w$. If S is a B-module map for each $w \in W$ then it is easy to see that u is a B-module map. However S satisfies the analogue of hypothesis (iii). Indeed τ_S is a contraction, since it is the composition of the contractions τ_u and τ_{ϵ_w}. The latter map is a contraction by the proof of the implication (ii) $\Rightarrow$ (iii), since ϵ_w is a B-module map.

Next we note that it suffices to prove the implication in the case that $W = B$ is a von Neumann algebra. To see this, the idea is to take the second dual of the map $\tau_u\colon B \oplus^C Z \to C_2(Z)$. By Lemma 3.3 we have that Z^{**} is a right C^*-module over C^{**}, and that $C^{**} \oplus_c Z^{**} \cong (C \oplus_c Z)^{**}$ via a weak* homeomorphic isometry σ say. Similarly, by the remark after 3.3, or alternatively by basic operator space theory, $C_2(Z)^{**} \cong C_2(Z^{**})$ via a weak* homeomorphic isometry ν say. Let us suppose first that $B = C$ (which is the only case we need for applications to results in Part I). Since the two maps $\tau_{u^{**}}$ and $\nu \circ \tau_u^{**} \circ \sigma$ are weak* continuous and agree on the weak* dense subspace $B \oplus Z$, they must be equal everywhere. Thus $\tau_{u^{**}}$ is completely contractive. A similar argument prevails if $B \neq C$, the main difference being that we also need to use the fact that the map σ above restricts to a weak* continuous isomorphism $B^{**} \oplus^{C^{**}} Z^{**} \to (B \oplus^C Z)^{**}$. This follows because $B \oplus Z$ is weak* dense in both $B^{**} \oplus^{C^{**}} Z^{**}$ and $(B \oplus^C Z)^{**}$.

Thus whether or not $B = C$, we have shown that $\tau_{u^{**}}$ is completely contractive. If the result were true in the von Neumann algebra case then it follows that u^{**} is a B^{**}-module map. Restricting to B we have that $u = (u^{**})_{|B}$ is a B-module map, proving (i).

Thus we may assume henceforth that $W = B$ is a von Neumann algebra. We claim that

$$u(bq) = u(bq)q, \qquad \text{for any } b \in B, \text{ and any orthogonal projection } q \in B. \tag{9}$$

Supposing that (9) were true, then for any $b \in B$ we have using (9) twice (once with $1 - q$) that

$$u(b)q = u(b(q + (1 - q))q = u(bq)q + u(b(1 - q))q = u(bq)q = u(bq).$$

Since $u(bq) = u(b)q$ for all projections q in B, and since B is the closed span of its projections, it follows that u is a B-module map.

It remains to check (9), or equivalently that $u(bq)(1_C - q) = 0$ for any projection q in B. Set $K = \|u(bq)(1_C - q)\|$, and suppose by way of contradiction that $K \neq 0$. Let $v = \frac{1}{K} u(bq)(1 - q)$. Note that v has norm 1, and hence so does $\langle v|v\rangle$. If $b \in \text{Ball}(B)$ then qb^*bq and $\langle v|v\rangle = (1 - q)\langle v|v\rangle(1 - q)$ are mutually orthogonal elements in $\text{Ball}(B)$ and $\text{Ball}(C)$ respectively, and hence

$$\left\| \begin{bmatrix} bq \\ v \end{bmatrix} \right\|^2 = \|qb^*bq + \langle v|v\rangle\| \leq 1.$$

Thus we may conclude that

$$\sqrt{K^2 + 1} = \left\| \begin{bmatrix} Kv \\ v \end{bmatrix} \right\| = \left\| \begin{bmatrix} u(bq)(1 - q) \\ v(1 - q) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} u(bq) \\ v \end{bmatrix} \right\| \leq 1,$$

the last inequality by hypothesis (iii). Thus $K = 0$, which is a contradiction. This proves (9). $\square$

Remarks: 1) The last proof was inspired in part by an idea in the proof of Theorem 1.1 of [**22**].

2) We have been able to generalize the last result to the situation that B is not a subalgebra of C, but rather that there is a $*$-homomorphism $\theta : B \to C$. In this case the variant of (i) in Theorem 11.1 is that $u(wb) = u(w)\theta(b)$ for $w \in W, b \in B$; and the variant of (ii) is that $\langle u(w)|u(w)\rangle \leq \theta(\langle w|w\rangle)$ for $w \in W$. In (iii) one needs to replace the norm on the right side by $\|\theta(\langle w|w\rangle) + \langle z|z\rangle\|^{\frac{1}{2}}$. A similar modification needs to be made in (iv). Since we do not need this result we omit its proof.

The following is obvious from the obvious proof of (ii) $\Rightarrow$ (iii) in the last theorem.

Corollary 11.2. *If X, Y, Z, W are right C^*-modules over B, and if $R : X \to Y$ and $S : W \to Z$ are contractive B-module maps, then the map $R \oplus S : X \oplus_c W \to Y \oplus_c Z$ is a contraction.*

Theorem 11.3. *Suppose that W and Z are right C^*-modules over a C^*-algebra B, and suppose that W is 'full' over B (See Section 3). If $u : W \to Z$ is a $\mathbb{C}$-linear* *map, then the following statements are also equivalent to the equivalent statements* (i)–(iv) *of Theorem* 11.1*:*

(v) $$\left\| \begin{bmatrix} u(w) \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} w \\ x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \right\|, \text{ for all } m \in \mathbb{N}, w, x_1, x_2, \dots, x_m \in W.$$

(vi) *The map* $u \oplus Id_W : C_2(W) \to Z \oplus_c W$ *is completely contractive.*

Proof. (vi) $\Rightarrow$ (v) This is clear.

(v) $\Rightarrow$ (i) The criterion is saying that the map $u \oplus Id : W \oplus_c C_m(W) \to Z \oplus_c C_m(W)$ is contractive. We follow the idea in the second and third paragraph of the implication (iii) $\Rightarrow$ (i) in Theorem 11.1, taking the second dual of this map. By the remark after Lemma 3.3, $(W \oplus_c C_m(W))^{**} \cong W^{**} \oplus_c C_m(W^{**})$, and $(Z \oplus_c C_m(W))^{**} \cong Z^{**} \oplus_c C_m(W^{**})$. By Lemma 3.3, W^{**} is a right W^*-module over B^{**}. Thus, as in Theorem 11.1, we may reduce the theorem to the case that B is a von Neumann algebra and W is a right W^*-module over B.

By the W^*-module variant of Kasparov's stable isomorphism theorem (see e.g. [**10**, 5.8] (2)), there is an infinite cardinal I, and contractive right B-module maps $\epsilon : B \to C_I^w(W)$ and $P : C_I^w(W) \to B$, with $P \circ \epsilon = Id_B$. Next we observe that the criterion in (v) is equivalent to the map $u \oplus Id : W \oplus_c C_I^w(W) \to Z \oplus_c C_I^w(W)$ being contractive. By these last two facts, and by Corollary 11.2 used twice, we have for $w \in W$ and $b \in B$ that

$$\left\| \begin{bmatrix} u(w) \\ b \end{bmatrix} \right\| = \left\| \begin{bmatrix} u(w) \\ P(\epsilon(b)) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} u(w) \\ \epsilon(b) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} w \\ \epsilon(b) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} w \\ b \end{bmatrix} \right\|.$$

By Theorem 3.1 we conclude that u is a module map.

(i) $\Rightarrow$ (vi) By Wittstock's observation recorded above Theorem 3.2, it suffices to show that $u \oplus Id_W$ is contractive. But this follows from Corollary 11.2. $\square$

Remarks: 1) Examination of the proof above shows that one may replace the criterion $x_1, \cdots, x_m \in W$ in (v) (resp. the map $u \oplus Id_W$ in (vi)) by $x_1, \cdots, x_m \in Y$ (resp. $u \oplus Id_Y$) for any full right C^*-module Y over B.

2) If W and Z are right C^*-modules over C^*-algebras B and C respectively, then one might at times be interested in the situation that there exists a 1-1 $*$-homomorphism θ from B to C. However this situation often may be reduced to the situation when B is a subalgebra of C, and θ is the inclusion map. To see this we note that W is also a right C^*-module over $\theta(B)$, with new module action $w \circ \theta(b) = wb$, and new inner product $\langle\langle w_1 | w_2 \rangle\rangle = \theta(\langle w_1 | w_2 \rangle)$. Moreover this new C^*-module structure yields the same canonical operator space matrix norms (see Equation (3)) that we had before. Thus if we can prove a result (such as Theorem 11.1) about such objects in the case that θ is an inclusion of C^*-subalgebras, then a similar result (but sometimes slightly modified, such as in the Remark 2 after Theorem 11.1) for general θ is usually easily derivable from this special case.

A similar trick works for operator spaces. There, we might be interested in the case that operator spaces X and Y possess noncommutative Shilov boundaries $(\partial X, i)$ and $(\partial Y, j)$ respectively, which are right C^*-modules over C^*-algebras B and C say, such that there exists a 1-1 $*$-homomorphism $\theta : B \to C$. As above, ∂X may be given a new right C^*-module structure so that B becomes a C^*-subalgebra of C, *without changing the ternary operation or operator space structure* of ∂X. This is an easy but tedious exercise.

We end this section with a technical result which will be used once in the next section, and which is unrelated to results in Part I:

LEMMA 11.4. *Let Z be a right C^*-module over a C^*-algebra B. Then there is an injective envelope $I(Z)$ of Z which is a right C^*-module over a C^*-algebra $\mathcal{R}$, with the following properties: $\mathcal{R}$ contains B as a C^*-subalgebra, and the module action of $\mathcal{R}$ on $I(Z)$, restricted to an action of B on Z, is the original one.*

PROOF. We may suppose that B is unital. Form the linking C^*-algebra $\mathcal{L}(Z)$ as described in Section 3, and suppose that it is suitably represented nondegenerately as a C^*-subalgebra of $B(H \oplus K)$. Thus $1_B = I_K$. The following is a mild variant of the Hamana-Ruan construction of the injective envelope, and the reader may want to follow along with this construction in any of the sources [**37, 60, 20, 54**]. Consider the operator system

$$\mathcal{S}_B(Z) \quad = \quad \begin{bmatrix} \mathbb{C}\, I_H & Z \\ \bar{Z} & B \end{bmatrix} \ \subset B(H \oplus K).$$

Let Φ be a *minimal $\mathcal{S}_B(Z)$ projection* on $B(H \oplus K)$. This is a completely positive idempotent map whose range is an injective envelope $I(\mathcal{S}_B(Z))$ of $\mathcal{S}_B(Z)$ (see the cited references). By a result of Choi and Effros [**27**], $I(\mathcal{S}_B(Z))$ is a C^*-algebra with a new product $x \circ y = \Phi(xy)$. Also $I(\mathcal{S}_B(Z))$ may be regarded as a 2×2 matrix algebra with respect to the canonical diagonal projections $p = I_H \oplus 0$ and $q = 0 \oplus I_K$. Let $\mathcal{R}$ be the 2-2-corner $qI(\mathcal{S}_B(Z))q$, this is a C^*-subalgebra of $I(\mathcal{S}_B(Z))$. By definition of the new product it is easy to see that B is a C^*-subalgebra of $\mathcal{R}$. Let E be the 1-2-corner $pI(\mathcal{S}_B(Z))q$. Clearly E is injective, and is also a right C^*-module over $\mathcal{R}$. In fact by definition of the new product it is easy to see that the right action of $\mathcal{R}$ on E extends the action of B on Z. By [**20**, Theorem 2.6], E is injective in the category of operator B-modules. We wish to show that E is an injective envelope of Z. To do this we first show that Id_E is the only completely contractive B-module map $u : E \to E$ extending the identity map on Z. For by Suen's variant on Paulsen's lemma [**64**], such u is the corner of a completely positive map Ψ on the subspace $\mathcal{S}_B(E)$ of $I(\mathcal{S}_B(Z))$, such that Ψ extends the identity map on $\mathcal{S}_B(Z)$. Extend Ψ further to a complete contraction from $I(\mathcal{S}_B(Z))$ to itself. By the rigidity property of the injective envelope, this latter map and hence also u must be the identity map.

The result is completed with an appeal to the fact from that the injective envelope of Z is also the B-module injective envelope of Z [**20**, Theorem 2.6]. The idea for this is as follows: by that result in [**20**], any injective envelope $I(Z)$ of Z can be made into a B-module which is injective as an operator B-module. A routine diagram chase, using facts from the last paragraph, shows that $I(Z) \cong E$ completely isometrically and as B-modules. $\square$

12. Left multipliers between two operator spaces

In this section X and Y are operator spaces possessing noncommutative Shilov boundaries $(\partial X, i)$ and $(\partial Y, j)$ respectively, which will be fixed for the remainder of this section. We will assume also that ∂X and ∂Y are right C^*-modules over C^*-algebras B and C respectively, where B is C^*-subalgebra of C. In fact for simplicity the reader may want to only consider the case that $B = C$ which is all we need for example for the results stated in Part I.

The next result generalizes Theorem 5.1, which was a result of great importance in the theory described in Part I. To explain the notation in this result: the inner products in (ii) are the (right) Shilov inner products on Y and X respectively, and the matrices there are indexed on rows by i, and on columns by j. The first norm in (iii) is just the norm in $M_{2n,n}(Y)$, whereas the second is the norm on $M_n(X \oplus Y)$ inherited from $M_n(\partial X \oplus^C \partial Y)$. An explicit formula for this norm was given just above Theorem 11.1. Finally, note that in (i) the map S is necessarily unique again.

THEOREM 12.1. *Let X, $Y, \partial X, \partial Y, B$ and C be as above, where $B \subset C$, and let $T\colon X \to Y$ be a linear map. The following are equivalent:*

(i) *T is the restriction to X of a completely contractive right B-module map $S : \partial X \to \partial Y$.*
(ii) *$[\langle T(x_i)|T(x_j)\rangle] \leq [\langle x_i|x_j\rangle]$ for all $m \in \mathbb{N}$ and $x_1, \cdots, x_m \in X$.*
(iii) *For all $n \in \mathbb{N}$ and matrices $[x_{ij}] \in M_n(X)$, $[y_{ij}] \in M_n(Y)$ we have*

$$\left|\left|\left| \begin{bmatrix} Tx_{ij} \\ y_{ij} \end{bmatrix} \right|\right|\right| \leq \left|\left|\left| \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} \right|\right|\right|.$$

If $B = C$ then we may replace 'completely contractive' by 'contractive' in (i).

PROOF. (i) $\Rightarrow$ (ii) Firstly note that without loss of generality we may suppose that X is a C^*-module, by replacing X by ∂X and T by S. Then this implication is seen to be a matricial version of the implication (i) $\Rightarrow$ (ii) in Theorem 11.1. For simplicity we suppose first that $B = C$, as we have in the special case of Theorem 5.1. In this case (ii) follows easily from Theorem 3.1 (ii) and a well known criterion for matrix positivity [**65**, IV.3.2]. In this case we do not need T completely contractive, contractive will suffice.

If $B \neq C$, we sketch another argument. The point is that Paschke's proof of the implication (i) $\Rightarrow$ (ii) in Theorem 11.1 has a matricial version that works here. We will use the notation used in the proof of Theorem 11.1. We need to replace h_n there by the matrix $H = ([\langle x_i|x_j\rangle] + \frac{1}{n}I_m)^{-\frac{1}{2}}$, x_n by the row matrix $v = [x_1, \cdots, x_m]H$, and the expression $\langle x|x\rangle$ by $[\langle x_i|x_j\rangle]$. One shows analogously to the quoted proof that $\|v\| \leq 1$. Since T is completely contractive, T applied entrywise to v has norm ≤ 1; and then one proceeds along the earlier line. We leave the details as an exercise.

(ii) $\Rightarrow$ (iii) Let $[x_{ij}] \in M_n(X)$, $[y_{ij}] \in M_n(Y)$, and fix $k \in \{1, \cdots, n\}$. By (ii) we have

$$[\langle T(x_{ki})|T(x_{kj})\rangle + \langle y_{ki}|y_{kj}\rangle] \leq [\langle x_{ki}|x_{kj}\rangle + \langle y_{ki}|y_{kj}\rangle] .$$

Taking the sum over k of these last matrices we obtain

$$\Big|\Big|[\sum_k \langle T(x_{ki})|T(x_{kj})\rangle + \langle y_{ki}|y_{kj}\rangle]\Big|\Big| \leq \Big|\Big|[\sum_k \langle x_{ki}|x_{kj}\rangle + \langle y_{ki}|y_{kj}\rangle]\Big|\Big|.$$

This is equivalent to (iii), by the formula just above Theorem 11.1.

(iii) $\Rightarrow$ (i) We prove this assertion first in the case that $\mathcal{F}(Y) = C$, in the notation of the second paragraph after Theorem 4.1. This is all we need for the promised results in Part I. Note that (iii) says that the map $T \oplus Id : X \oplus Y \to C_2(Y)$ is completely contractive, when $X \oplus Y$ is viewed as a subspace of $\partial X \oplus^C \partial Y$. The latter space is a subspace of $\partial X \oplus^{\mathcal{D}} I(Y)$, where $I(Y)$ and $\mathcal{D} = \mathcal{D}(Y)$ are as in the second paragraph after Theorem 4.1. We remark that the C^*-module sum $W \oplus_c Z$ of two right C^*-modules over a C^*-algebra B is a left operator ℓ^∞_2-module (this can be seen by noting for example that there is a copy of ℓ^∞_2 inside $\mathbb{B}(W \oplus_c Z)$). Since $\partial X \oplus^{\mathcal{D}} I(Y)$ is a ℓ^∞_2-submodule of a C^*-module sum of two right C^*-modules over $\mathcal{D}$ (as we noted below (8)), it is also a left operator ℓ^∞_2-module. We regard $C_2(Y) \subset C_2(I(Y))$. We will use Wittstock's module map extension theorem [**70, 64**] (together with the fact that $C_2(I(Y))$ is a complemented subspace of $M_2(I(Y))$, which in turn is a complemented subspace of $M_2(B(H))$, if $I(Y)$ is a complemented subspace of $B(H)$). By Wittstock's theorem, we can extend the ℓ^∞_2-module map $T \oplus Id : X \oplus Y \to C_2(I(Y))$, to a completely contractive left ℓ^∞_2-module $u : \partial X \oplus^{\mathcal{D}} I(Y) \to C_2(I(Y))$. Thus u is of the form $(x, y) \mapsto (\tilde{T}(x), G(y))$ for complete contractions $\tilde{T} : \partial X \to I(Y)$ and $G : I(Y) \to I(Y)$. Since G restricts to the identity on Y, by the rigidity property mentioned in the second paragraph after Theorem 4.1 we must have that G is the identity map. Thus $u = \tilde{T} \oplus Id$, and we may apply Theorem 11.1. We conclude that $\tilde{T}$ is a right B-module map. Thus

$$\tilde{T}(xb) = \tilde{T}(x)b = T(x)b \in YC \subset \partial Y, \qquad x \in X, b \in B.$$

This forces the range of $\tilde{T}$ to be contained in ∂Y, thus proving (i).

In the general case we first note that it follows from a routine diagram chase that any injective envelope $I(\partial Y)$ of ∂Y is an injective envelope $I(Y)$ of Y. Thus we may write $I(Y)$ for the injective envelope of ∂Y in Lemma 11.4, this is a C^*-module over $\mathcal{R}$, where C is a C^*-subalgebra of $\mathcal{R}$. Then $\partial X \oplus^C \partial Y$ is a subspace of $\partial X \oplus^{\mathcal{R}} I(Y)$. We may now follow the proof in the last paragraph, but with $\mathcal{D}$ replaced by $\mathcal{R}$, to extend T to a right $\mathcal{R}$-module map $\tilde{T} : I(X) \to I(Y)$. Since $\tilde{T}$ is also a right B-module map we may conclude the proof as in the last paragraph. $\square$

Definition 12.2. *We define a* relative left multiplier *to be a map $T : X \to Y$ such that a positive scalar multiple of T satisfies the equivalent conditions of Theorem 12.1. We write $\mathcal{M}^{rel}_l(X, Y)$ for the set of such relative left multipliers.*

Note that $\mathcal{M}^{rel}_l(X, X)$ is simply the space $\mathcal{M}_l(X)$ from Section 5. It is easy to see that $\mathcal{M}^{rel}_l(X, Y)$ is a vector space, and that with respect to operator composition we have a well defined bilinear map $\mathcal{M}^{rel}_l(X, Y) \times \mathcal{M}^{rel}_l(V, X) \to \mathcal{M}^{rel}_l(V, Y)$. This is assuming that the C^*-algebras acting on the right of X, Y, V are appropriately compatible.

Let X, Y be as in Theorem 12.1. To define an operator space structure on $\mathcal{M}^{rel}_l(X, Y)$ we first observe that there is a canonical linear isomorphism

$$\{S \in CB_B(\partial X, \partial Y) : S(X) \subset Y\} \;\cong\; \mathcal{M}^{rel}_l(X, Y), \tag{10}$$

given by the map $S \mapsto S_{|X}$. Since $CB(\partial X, \partial Y)$ is an operator space so is the set on the left side of (10). We may therefore use the linear isomorphism above to give $\mathcal{M}^{rel}_l(X, Y)$ an operator space structure. As in Section 5, we see that Theorem 12.1 gives alternative descriptions of the unit ball of $\mathcal{M}^{rel}_l(X, Y)$. We also have the following generalization of the useful formula (4):

Lemma 12.3. *If X, Y are as in Theorem* 12.1*, and if $m, n \in \mathbb{N}$ then we have $M_{m,n}(\mathcal{M}^{rel}_l(X, Y)) \cong \mathcal{M}^{rel}_l(C_n(X), C_m(Y))$ completely isometrically.*

Proof. The first point is that $\partial C_n(X) = C_n(\partial X)$; and similarly for Y. This follows from [**10**, Theorem A.13]. Then the relation follows from (10) and a trivial variant of the relation $M_n(CB_B(W, Z)) \cong CB_B(C_n(W), C_n(Z))$, valid for C^*-modules W, Z over B. See e.g. the proof of [**8**, Lemma 8.3]. The complete isometry here follows from the isometry used twice:

$$M_k(M_{m,n}(\mathcal{M}_l^{rel}(X,Y))) \cong \mathcal{M}_l^{rel}(C_{kn}(X), C_{km}(Y)) \cong M_k(\mathcal{M}_l^{rel}(C_n(X), C_m(Y))),$$

for any $k \in \mathbb{N}$. $\square$

We are not quite sure yet exactly in what way $\mathcal{M}_l^{rel}(X,Y)$ depends on the particular noncommutative Shilov boundaries chosen. However in the final two sections we will discuss a fairly general framework in which there is no such dependence.

We turn to an interesting example of relative left multipliers between different spaces, using facts proved in Section 10. Let (A, B, X, Y) be a strong Morita equivalence context in the sense of [**19**]. We claim that for a fixed $x \in X$ the maps $Y \to A$ and $B \to X$, given by $y \mapsto (x, y)$ and $b \mapsto xb$ respectively, are relative left multipliers. To see this we note that in this case one can show that $\partial A = C_e^*(A), \partial B = C_e^*(B)$ (see [**10**, Theorem 4.17]). Lemma 10.2 may be phrased as $\partial Y = Y \otimes_{hA} C_e^*(A)$ and $\partial X = X \otimes_{hB} C_e^*(B)$, in the notation of [**18**]. The map $y \otimes c \mapsto (x, y)c$ from $Y \otimes_{hA} C_e^*(A) \to C_e^*(A)$ is a right $C_e^*(A)$-module map which restricts to the map $y \mapsto (x, y)$ above. Similarly the map $d \mapsto x \otimes d$ from $C_e^*(B) \to X \otimes_{hB} C_e^*(B)$ is a right $C_e^*(B)$-module map, which restricts to the map $b \mapsto xb$ above. Thus by Theorem 12.1 these are relative left multipliers.

COROLLARY 12.4. *Let (A, B, X, Y) be a strong Morita equivalence context in the sense of* [**19**]*, with A, B unital. Then $\mathcal{M}^{rel}(Y, A) \cong X$ and $\mathcal{M}^{rel}(B, X) \cong X$ completely isometrically.*

PROOF. By the discussion in Section 11, we may identify $\mathcal{M}^{rel}(B, X)$ with the subspace of $CB_{C_e^*(B)}(C_e^*(B), W)$ of maps taking B into X, where W is as in the proof of Lemma 10.2. However if B is unital then so is $C_e^*(B)$, and so $CB_{C_e^*(B)}(C_e^*(B), W) \cong W$. This easily yields the isomorphism $\mathcal{M}^{rel}(B, X) \cong X$. Similarly, we may identify $\mathcal{M}^{rel}(Y, A)$ with the subspace of $CB_{C_e^*(A)}(W, C_e^*(A))$ of maps taking Y into A, where W is as in the proof of Lemma 10.2. However since W is an equivalence bimodule between unital C^*-algebras $C_e^*(A)$ and $C_e^*(B)$, it is well known that $CB_{C_e^*(A)}(W, C_e^*(A))$ is identifiable with the 'conjugate C^*-module' W^* of W (see e.g. [**19**, Corollary 4.2]). Thus $\mathcal{M}^{rel}(Y, A)$ is completely isometric to

$$\{w^* \in W^* : \langle w|y\rangle \in A \ \text{ for all } y \in Y\}.$$

By the discussion on the bottom of page 405 in [**6**], the last space is just a copy of X. $\square$

Remark: There is a version of the last Corollary which is valid in the general (nonunital) case. One needs to replace $\mathcal{M}^{rel}(\cdot, \cdot)$ with a certain subset of this space, called the *compact relative multipliers*. For general operator spaces X, Y these compact relative multipliers are just the relative multipliers which considered as module maps from ∂X to ∂Y lie in $\mathbb{K}(\partial X, \partial Y)$.

13. Adjointable maps between two operator spaces

In this section we consider two operator spaces X, Y with fixed noncommutative Shilov boundaries ∂X and ∂Y which are right C^*-modules over the same C^*-algebra B.

THEOREM 13.1. *Let X, Y be as above and suppose that $T : X \to Y$. The following are equivalent:*

(i) *T is the restriction to X of an adjointable (in the usual C^*-module sense) B-module map $R : \partial X \to \partial Y$ such that $R(X) \subset Y$ and $R^*(Y) \subset X$,*

(ii) *There exists a map $S : Y \to X$ such that $\langle T(x)|y\rangle = \langle x|S(y)\rangle$ (these are the (right) Shilov inner products) for all $x, y \in X$.*

Moreover the set $\mathcal{A}_l(X,Y)$ consisting of maps T satisfying condition (ii) *above, is a closed subspace of $B(X,Y)$ which is a C^*-bimodule over the algebras $\mathcal{A}_l(X)$ and $\mathcal{A}_l(Y)$. The module actions here on $\mathcal{A}_l(X,Y)$ are simply composition of operators. The $\mathcal{A}_l(X)$-valued inner product on $\mathcal{A}_l(X,Y)$ is $\langle T|R\rangle = SR$, for $T, R \in \mathcal{A}_l(X,Y)$ where S is related to T as in* (ii) *above.*

Proof. We leave it to the interested reader to check that any $T \in \mathcal{A}_l(X,Y)$ is linear; that the map S in (ii) is necessarily unique and linear; that $\mathcal{A}_l(X,Y)$ is an $\mathcal{A}_l(Y)$-$\mathcal{A}_l(X)$-bimodule; and that the $\mathcal{A}_l(X)$-valued inner product specified above does indeed take values in $\mathcal{A}_l(X)$. In fact the only nontrivial part of the proof that $\mathcal{A}_l(X,Y)$ is a right C^*-module consists in showing that for $T \in \mathcal{A}_l(X,Y)$, (a) $T^*T \geq 0$ in $\mathcal{A}_l(X)$, and (b) $\|T^*T\| = \|T\|^2$. Here T^* denotes the map S in (ii). In fact (a) follows from Theorem 4.10 (2) in [**10**], since

$$\langle T^*Tx|x\rangle = \langle Tx|Tx\rangle \geq 0, \qquad x \in X.$$

To prove (b) we first note that if $R \in \mathcal{A}_l(X)_+$, with $R = V^*V$ for a $V \in \mathcal{A}_l(X)$, then

$$\sup\{\|\langle Rx|x\rangle\| : x \in \mathrm{Ball}(X)\} = \sup\{\|\langle Vx|Vx\rangle\| : x \in \mathrm{Ball}(X)\} = \|V\|^2 = \|R\|.$$

Setting $R = T^*T$ and using (a) we see that $\|T^*T\|$ equals

$$\sup\{\|\langle T^*Tx|x\rangle\| : x \in \mathrm{Ball}(X)\} = \sup\{\|\langle Tx|Tx\rangle\| : x \in \mathrm{Ball}(X)\} = \|T\|^2.$$

It follows that $\|T\| = \|T^*\|$ as in the Hilbert space case.

(i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) Suppose that T satisfies (ii). Then $T^*T \in \mathcal{A}_l(X)$ by the first part of the proof. For $x_1, \cdots, x_n \in X$ and $b_1, \cdots, b_n \in B$, we define $R(\sum_k x_k b_k) = \sum_k T(x_k)b_k$. To see that R is well defined and bounded, set $u = \sum_k x_k b_k$, take $y_1, \cdots, y_m \in Y$ and $c_1, \cdots, c_m \in B$, and set $v = \sum_k y_k c_k$. Then

$$\langle v|\sum_k T(x_k)b_k\rangle = \sum_{i,j} c_j^*\langle y_j|T(x_i)\rangle b_i = \sum_j c_j^*\langle T^*(y_j)|u\rangle = \langle \sum_k T^*(y_k)c_k|u\rangle. \tag{11}$$

Setting $y_k = T(x_k)$ and $c_k = b_k$, we obtain from (11) and a Cauchy-Schwarz inequality:

$$\|\sum_k T(x_k)b_k\|^2 \leq \|\sum_k T^*T(x_k)b_k\| \, \|u\|.$$

Now the mapping $\sum_k x_k b_k \mapsto \sum_k T^*T(x_k)b_k$ is simply the unique B-module map on ∂X extending $T^*T \in \mathcal{A}_l(X)$ (see [**10**] Theorem 4.10, in conjunction with the observation in equation (2)), and this extension has the same norm. Thus

$$\|\sum_k T(x_k)b_k\|^2 \leq \|T^*T\| \, \|u\|^2 = \|T\|^2 \, \|u\|^2.$$

Thus R is bounded and well defined.

Since R is bounded, it extends by density to a unique bounded B-module map $R : \partial X \to \partial Y$. Similarly T^* extends to a bounded map $S : \partial Y \to \partial X$. We leave it as an exercise, using (11), that R is adjointable with adjoint S, and satisfies (i). $\square$

As in the last proof we write T^* for the unique S related to T in (ii) of the Theorem. We call such maps T *relatively adjointable.* Strictly speaking we should write $\mathcal{A}_l^{rel}(X,Y)$ for what we wrote as $\mathcal{A}_l(X,Y)$ above, but for simplicity we have used the shorter notation in this section. As was the case for $\mathcal{M}_l^{rel}(X,Y)$, the space $\mathcal{A}_l(X,Y)$ is only defined relative to fixed noncommutative Shilov boundaries ∂X and ∂Y. There are frameworks in which one may remove this 'relative' nature, as we shall see in the next sections.

Remarks: 1) It is easy to see that the set

$$\{R \in \mathbb{B}_B(\partial X, \partial Y) : R(X) \subset Y, R^*(Y) \subset X\}$$

is a right C^*-module over the C^*-algebra

$$\{R \in \mathbb{B}(\partial X) : R(X) \subset X, R^*(X) \subset X\}.$$

By basic properties of 'ternary morphisms' mentioned in Section 4, the restriction map from this C^*-module onto $\mathcal{A}_l(X,Y)$ is a completely isometric surjective ternary isomorphism.

2) If W is a third operator operator space whose noncommutative Shilov boundary ∂W is also a right C^*-module over the same algebra B as above, then 'composition of operators' is a well defined bilinear map $\mathcal{A}_l(X,Y) \times \mathcal{A}_l(W,X) \to \mathcal{A}_l(W,Y)$.

14. Multipliers relative to a superspace

In this section we consider a fairly general situation in which we can remove some of the relative nature of spaces $\mathcal{M}_l^{rel}(X,Y)$ and $\mathcal{A}_l^{rel}(X,Y)$ considered above.

DEFINITION 14.1. *We say that a closed subspace X of an operator space V is a* Shilov submodule *if there is a noncommutative Shilov boundary $(\partial V, i)$ of V such that the subTRO Z of ∂V generated by $i(X)$ is a noncommutative Shilov boundary of X, and also Z is a $\mathcal{F}(V)$-submodule of ∂V. Here $\mathcal{F}(V)$ is as defined in the second paragraph of Section 4.*

If X and Y are Shilov submodules of a third operator space V, then we say that (X,Y) is a ∂-compatible V-pair.

We will not use this, but it is not hard to see that saying that X is a Shilov submodule of V is equivalent to saying that there is a noncommutative Shilov boundary $(\partial V, i)$ of V such that the smallest closed $\mathcal{F}(V)$-submodule of ∂V containing $i(X)$ is a noncommutative Shilov boundary of X. By the universal property in Theorem 4.1, and a routine diagram chase, it is easy to see that the notion of Shilov submodule does not depend on the particular Shilov boundary of V considered above. If X is a Shilov submodule of V, then we will reserve the symbol ∂X for the space Z above. Note that ∂X is a right C^*-module over $\mathcal{F}(V)$.

If (X,Y) is a ∂-compatible V-pair, then we define $X \oplus_V Y$ to be the algebraic sum $X \oplus Y$ endowed with an operator space structure by identifying it with a subspace of $C_2(V)$ in the canonical way. For any noncommutative Shilov boundary ∂V of V we have canonical complete isometric embeddings

$$X \oplus_V Y \hookrightarrow \partial X \oplus_c \partial Y \hookrightarrow C_2(\partial V).$$

Here $\partial X \oplus_c \partial Y$ is the C^*-module sum of ∂X and ∂Y as C^*-modules over $\mathcal{F}(V)$.

The second matrix norm in (iii) below is the norm on $M_n(X \oplus_V Y)$.

Corollary 14.2. *Let (X, Y) be a ∂-compatible V-pair, let $\partial X, \partial Y, \mathcal{F}(V)$ be as above, and set $C = \mathcal{F}(V)$. If $T \colon X \to Y$ is a linear map then the following are equivalent:*

(i) *T is the restriction to X of a contractive right C-module map $S : \partial X \to \partial Y$.*
(ii) *$[\langle T(x_i)|T(x_j)\rangle] \leq [\langle x_i|x_j\rangle]$ for all $m \in \mathbb{N}$ and $x_1, \cdots, x_m \in X$.*
(iii) *For all $n \in \mathbb{N}$ and matrices $[x_{ij}] \in M_n(X)$, $[y_{ij}] \in M_n(Y)$ we have*

$$\left|\left|\left| \begin{bmatrix} Tx_{ij} \\ y_{ij} \end{bmatrix} \right|\right|\right| \leq \left|\left|\left| \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} \right|\right|\right|.$$

Proof. Follows from Theorem 12.1 with $B = C = \mathcal{F}(V)$. $\square$

If (X, Y) is a ∂-compatible V-pair, then we write $\mathcal{M}_l^V(X, Y)$, or $\mathcal{M}_l(X, Y)$ when V is understood, for the operator space $\mathcal{M}_l^{rel}(X, Y)$ from the last section, in the case that the noncommutative Shilov boundaries of X and Y are taken to be subTRO's of ∂V (as in Definition 14.1). A map in $\mathcal{M}_l^V(X, Y)$ will be called a *left V-multiplier* from X to Y. The last theorem then identifies the unit ball of $\mathcal{M}_l^V(X, Y)$. We write $\mathcal{A}_l^V(X, Y)$ for the C^*-bimodule in Theorem 13.1 (taking $B = \mathcal{F}(V)$ there).

The following shows that for C^*-modules, the V-multipliers coincide with the usual important classes of maps:

Proposition 14.3. *Let Y, Z be right C^*-modules over a C^*-algebra B. Set $V = Y \oplus_c Z$, and regard Y, Z as subspaces of V. Then (Y, Z) is a ∂-comparable V-pair, and $\mathcal{M}_l^V(Y, Z) \cong B_B(Y, Z)$ and $\mathcal{A}_l^V(Y, Z) \cong \mathbb{B}(Y, Z)$.*

Proof. Denote the closed span of the range of the canonical B-valued inner product on $Y \oplus_c Z$ by $\mathcal{F}$. In this case, as we said after Theorem 4.1, one can take $\partial V = V$, viewed as a right C^*-module $\mathcal{F}$. Then Y, Z are also C^*-modules over $\mathcal{F}$, (Y, Z) is a ∂-comparable V-pair, and $\mathcal{M}_l^V(Y, Z) \cong B_{\mathcal{F}}(Y, Z)$ and $\mathcal{A}_l^V(Y, Z) \cong \mathbb{B}_{\mathcal{F}}(Y, Z)$. We now may appeal to the principle in equation (2). $\square$

Lemma 14.4. *If (X, Y) is a ∂-compatible V-pair then $\partial X \oplus_c \partial Y$ is a noncommutative Shilov boundary of $X \oplus_V Y$.*

Proof. First observe that $\partial X \oplus_c \partial Y$ is a left operator ℓ^∞_2-submodule of $C_2(\partial V)$. The canonical map $X \oplus_V Y \to \partial X \oplus_c \partial Y$ is a complete isometry as noted above. Inside $\mathbb{B}(\partial X \oplus_c \partial Y)$ there is a copy of ℓ^∞_2 (this is true for the sum of any two C^*-modules). We may follow the proof in [**10**, Theorem A.13]: one supposes that W is a 'ternary ideal' in $\partial X \oplus_c \partial Y$ such that the canonical map $X \oplus_V Y \to (\partial X \oplus_c \partial Y)/W$ is a complete isometry, and then one needs to show that $W = (0)$. This is accomplished by letting $W_1 = e_1 W \subset \partial X$, and $W_2 = e_2 W \subset \partial Y$, where e_i is the 'standard basis' for ℓ^∞_2, and showing that the canonical maps $X \to (\partial X)/W_1$ and $Y \to (\partial X)/W_2$ are complete isometries. Since the reasoning is identical to that in [**10**, Theorem A.13] we omit the details. $\square$

We write ϵ_X and P_X for the canonical inclusion and projection maps between X and $X \oplus_V Y$. Similarly for ϵ_Y and P_Y. These maps are restrictions of the canonical adjointable inclusion and projection maps between the C^*-module $\partial X \oplus_c \partial Y$ and its summands. It is clear from the definitions in Section 9 that $p = \epsilon_X \circ P_X$ is a left M-projection on $X \oplus_V Y$, onto the right M-summand $X \oplus 0$. Similarly $q = \epsilon_Y \circ P_Y$ is the left M-projection onto $0 \oplus Y$.

For X, Y, V as above, $X \oplus \{0\}$ and $\{0\} \oplus Y$ are a ∂-compatible $X \oplus_V Y$-pair, as may be seen using Lemma 14.4. Thus it seems that in most situations we may assume without loss of generality that X, Y are complementary right M-summands in V (by 'replacing' V by $X \oplus_V Y$, and using the observation in the last paragraph).

Conversely, if (X, Y) is a ∂-compatible V-pair, and if also X and Y are 'complementary' right M-summands in V, then $V \cong X \oplus_V Y$ completely isometrically. This is because, by one of the definitions of a right M-summand, the map

$$V \rightarrow C_2(V) \; : \; x + y \; \mapsto \; \begin{bmatrix} x \\ y \end{bmatrix}$$

is a complete isometry, and its range is the space $X \oplus_V Y$ defined earlier.

The next observation we make is that since the operator algebra $\mathcal{M}_l(X \oplus_V Y)$ (resp. C^*-algebra $\mathcal{A}_l(X \oplus_V Y)$) contains the two canonical complementary projections p, q mentioned a few paragraphs above, it splits as a 2×2 matrix algebra (resp. C^*-algebra). We first claim that the 1-1-corner is completely isometrically homomorphic to $\mathcal{M}_l(X)$ (resp. $\mathcal{A}_l(X)$). To see this consider the map $\theta : \mathcal{M}_l(X) \rightarrow \mathcal{M}_l(X \oplus_V Y)$ taking T to $\epsilon_X \circ T \circ P_X$. This may be viewed as the restriction to $\mathcal{M}_l(X)$ of the map $R \rightarrow \epsilon_{\partial X} \circ R \circ P_{\partial X}$ from the space of bounded module maps on X, to the space of bounded module maps on $X \oplus_c \partial Y$. Thus it is a well defined completely contractive homomorphism. From this argument, or directly, it is easy to see that θ is completely isometric. If $S = pS'p$ for a map $S' \in \mathcal{M}_l(X \oplus_V Y)$, then by the Lemma 14.4 and Theorem 12.1, S' is the restriction to $X \oplus_V Y$ of a bounded $\mathcal{F}(V)$-module map R' on $\partial X \oplus_c \partial Y$. Then S is the restriction to $X \oplus_V Y$ of $\epsilon_{\partial X} \circ P_{\partial X} \circ R' \circ \epsilon_{\partial X} \circ P_{\partial X}$. From this it is clear that $\theta(P_X \circ S' \circ \epsilon_X) = S$. Thus $\theta(\mathcal{M}_l(X)) = p \, \mathcal{M}_l(X \oplus_V Y)p$. If $R \in \mathbb{B}(X)$ then $\epsilon_{\partial X} \circ R \circ P_{\partial X} \in \mathbb{B}.(\partial X \oplus_c \partial Y)$. Thus it is easy to argue that θ induces a $*$-monomorphism from $\mathcal{A}_l(X)$ onto $p \, \mathcal{A}_l(X \oplus_V Y)p$.

By identical reasoning we have completely isometries from $\mathcal{M}_l(Y)$, $\mathcal{M}_l(X, Y)$ and $\mathcal{M}_l(Y, X)$ into the other three corners of $\mathcal{M}_l(X \oplus_V Y)$. Similar assertions hold for the $\mathcal{A}_l(\cdot)$ spaces.

The following is the analogue of Theorem 5.1 (iv) for left V-multipliers. It can be stated in many forms, but perhaps the following is the most concise:

Proposition 14.5. *If X, Y are complementary right M-summands of an operator space V (see the discussion after Lemma 14.4), and if (X, Y) is a ∂-compatible V-pair, then a linear map $T : X \rightarrow Y$ is a left V-multiplier if and only if there is a completely isometric linear embedding of V into a C^*-algebra A, and an $a \in \mathrm{Ball}(A)$ with $Tx = ax$ for all $x \in X$.*

Proof. $(\Rightarrow)$ Suppose that $T : X \rightarrow Y$ is a left V-multiplier. Since $\mathcal{M}_l^V(X, Y)$ may be regarded as a corner of $\mathcal{M}_l(X \oplus_V Y)$, and since $X \oplus_V Y \cong V$ by the discussion after Lemma 14.4, we may regard T as a left multiplier R of V. Thus by Theorem 5.1 (iv), there is a completely isometric linear embedding $\sigma : V \rightarrow A$, and an $a \in \mathrm{Ball}(A)$ with $\sigma(R(x)) = a\sigma(x)$, for all $x \in X$. However $R(x) = T(x)$.

$(\Leftarrow)$ Similar to the corresponding part of the proof for Theorem 5.1. $\square$

The next result generalizes Theorem 8.1, and it should be useful in the way that Theorem 8.1 was. For example structural properties in a W^*-module (for example those considered in [**40**]) should have implications for the pair X, Y.

Corollary 14.6. *If X, Y are weak* closed Shilov submodules of a dual operator space V, then $\mathcal{A}_l^V(X,Y)$ is a W^*-module. Moreover, every $T \in \mathcal{A}_l^V(X,Y)$ is automatically weak* continuous.*

Proof. To see that $\mathcal{A}_l^V(X,Y)$ is a W^*-module it suffices to show that $\mathcal{A}_l^V(X,Y)$ is a dual space. However as we just saw, $\mathcal{A}_l^V(X,Y)$ is a 'corner' in $\mathcal{A}_l(X \oplus_V Y)$. Thus by Theorem 8.1 it suffices to show that $X \oplus_V Y$ is a dual operator space. However this is clear since $C_2(V)$ is a dual operator space, and $X \oplus_V Y$ is easily seen to be weak* closed in $C_2(V)$.

The last assertion follows from the analogous fact in Theorem 8.1, together with the fact that the canonical inclusion and projection maps between $X \oplus_V Y$ and its summands are weak* continuous in this case (which follows from basic operator space theory). ☐

It is often useful that the adjointable maps on an operator space, or even on a Hilbert space, are characterizable as the span of the Hermitian (i.e. self-adjoint) ones. The following, which is essentially the principle remarked on in the paragraph after the proof of Theorem 3.2, may be viewed as a 'two-space' analogue of this fact.

Corollary 14.7. *Let (X, Y) be a ∂-compatible V-pair, set $B = \mathcal{F}(V)$, and let $T : X \to Y$ be a linear map. Then T satisfies the equivalent conditions in Theorem 13.1 if and only if there is a map $S : Y \to X$ such that the map $(x, y) \mapsto (S(y), T(x))$ is a Hermitian in the Banach algebra $\mathcal{M}_l(X \oplus_V Y)$. In this case $T^* = S$.*

Proof. We leave this as an exercise. The idea is very similar to the last proof and the discussion above it. That is, using the canonical inclusion and projection maps, we transfer the desired statement to a statement about maps between the C^*-modules $\partial X, \partial Y$ and $\partial X \oplus_c \partial Y$. ☐

15. Multipliers and the injective envelope

We conclude this paper with some remarks and results concerning two questions that may have occurred to the reader. The first question arises if one contrasts Theorems 11.1 (iii) and 11.3 (vi). These results are both about maps $u : W \to Z$, but the main difference is in whether we consider $u \oplus Id_W$ or $u \oplus Id_Z$. It is thus natural to ask for a variant of Theorem 14.2 (iii) involving $T \oplus Id_X$ as opposed to $T \oplus Id_Y$. The second question is if there is a formulation of the results in this Part in terms of the injective envelope.

In connection with the second question, we first note that Theorem 12.1 has a simple variant in terms of the injective envelope, valid for operator spaces X and Y whose injective envelopes are right C^*-modules over the same injective C^*-algebra D say. In (i) of 12.1 one simply replaces '∂' by $I(\cdot)$, in (ii) the inner products are valued in D, and the second norm in (iii) is the canonical norm on $M_n(I(X) \oplus_c I(Y))$, where the direct sum is that of C^*-modules over D. The proof of the equivalences of these revised statements (i)–(iii) is just as before (in fact simpler, since we are not considering the case $B \neq C$).

There is a framework similar to the one considered in Section 14 which addresses both questions mentioned at the start of the present section simultaneously. We begin with the following variations on some of the definitions above. We consider an operator space V, and fix an injective envelope $(I(V), i)$ of V, which is a right

C^*-module over a C^*-algebra $\mathcal{D} = \mathcal{D}(V)$ (see the second paragraph after Theorem 4.1). We say that a pair (X, Y) of subspaces of V is an *I-compatible V-pair*, if there are two $\mathcal{D}$-submodules W, Z of $I(V)$, such that (W, i) and (Z, i) are injective envelopes of X.

We remark that it is easy to see that W and Z above are subTRO's of $I(V)$, and are also right C^*-modules over $\mathcal{D}(V)$. We will write W as $I(X)$, and $\mathcal{D}(X)$ for the C^*-subalgebra W^*W of $\mathcal{D}(V)$. Similar notations hold for Y. Thus $I(X) \oplus_c I(Y)$ is a right C^*-module over $\mathcal{D}(V)$, and it is a subspace of $C_2(I(V))$. We define $X \oplus_V Y$ as we did before, and note that also $X \oplus_V Y \subset I(X) \oplus_c I(Y) \subset C_2(I(V))$ completely isometrically. Combining the facts that $C_2(I(V))$ is an injective $\mathcal{D}$-module, Theorem 2.6 in [**20**], and Corollary 11.2, one may deduce that $I(X) \oplus_c I(Y)$ is also injective as an operator space.

In (i) of the next theorem, the notation $I_{11}(V)$ is used precisely in the sense of [**20**].

Theorem 15.1. *Let (X, Y) be an I-compatible V-pair. If $T : X \to Y$ is a linear map, then the following are equivalent:*

(i) *There is an $a \in \mathrm{Ball}(I_{11}(V))$ such that $i(Tx) = ai(x)$ for all $x \in X$.*
(ii) *$T \oplus Id_V : X \oplus_V V \to Y \oplus_V V$ is completely contractive.*
(iii) *$T \oplus Id_Y : X \oplus_V Y \to C_2(Y)$ is completely contractive.*
(iv) *There is a C^*-algebra A, a completely isometric embedding $V \hookrightarrow A$, and an $a \in \mathrm{Ball}(A)$ such that $Tx = ax$ for all $x \in X$.*
(v) *T is the restriction to X of a contractive $\mathcal{D}(V)$-module map $S : I(X) \to I(Y)$.*

If, further, $\mathcal{D}(Y) \subset \mathcal{D}(X)$, where $\mathcal{D}(\cdot)$ is as defined above, then the above conditions are equivalent to:

(vi) *$T \oplus Id_X : C_2(X) \to Y \oplus_V X$ is completely contractive.*

*On the other hand, if (X, Y) is also a ∂-*compatible V-pair*, then conditions (i)–(v) above are also equivalent to conditions* (i)–(iii) *in Theorem 12.1.*

Proof. That (ii) implies (v) follows just as in the proof of 12.1. Namely, we first extend $T \oplus Id_V$ to a complete contraction $I(X) \oplus_c I(V) \to I(Y) \oplus_c I(V)$. We then argue as in 12.1 that an appeal to Theorem 11.3 is justified. By this result, T is the restriction of a $\mathcal{D}(V)$-module map $S : I(X) \to I(Y)$.

Given (v), since $I(V)$ is injective as a $\mathcal{D}(V)$-module (see [**20**]), we may extend S to a completely contractive $\mathcal{D}(V)$-module map on $I(V)$. This yields (i) by [**20**, Corollary 1.8]. That (i) implies (iv) is easy (take $A = I(\mathcal{S}(V))$ in the notation of [**20**]). Similarly, that (iv) implies (vi) and (iii) is easy, as in the proof that (iv) implied (iii) in Theorem 5.1. To show that (iii) implies (v) we may proceed as in the proof that (iii) implied (i) in Theorem 12.1 to extend $T \oplus Id_Y$ to a completely contractive left ℓ^∞_2-module map $I(X) \oplus_c I(Y) \to C_2(I(Y))$. Using rigidity as before it follows that this map is of the form $S \oplus Id_{I(Y)}$ for a map $S : I(X) \to I(Y)$. By Theorem 11.1, S is a $\mathcal{D}(V)$-module map as in the last paragraph.

That (vi) implies (v) follows just as in the proof of 12.1. Namely, we extend $T \oplus Id_X$ to a complete contraction $C_2(I(X)) \to I(Y) \oplus_c I(X)$, and argue as in 12.1 that an appeal to Theorem 11.3 is justified. By this result T is the restriction of a $\mathcal{D}(X)$-module map $S : I(X) \to I(Y)$. By the observation in equation (2), S is a $\mathcal{D}(V)$-module map.

Clearly S above is also a $\mathcal{F}(V)$-module map, where $\mathcal{F}(V)$ is as in the second paragraph below Theorem 4.1. If further (X, Y) are a ∂-compatible V-pair, then

$$S(xf) = S(x)f \subset Y\mathcal{F}(V) \subset \partial Y,$$

for $x \in X, f \in \mathcal{F}(V)$. Thus the restriction of S to ∂X maps into ∂Y, giving (i) of Theorem 12.1. Conversely, small variation of the proof that (i) implies (iii) in Theorem 12.1 also shows that (i) of Theorem 12.1 implies (iii) of the present theorem. □

The following is a special case.

Corollary 15.2. *Let X, Y be subspaces of an operator space V. Consider the properties:* (a) *Some noncommutative Shilov boundary $(\partial V, i)$ of V is a noncommutative Shilov boundary of X and of Y too; and* (b) *Some injective envelope $(I(V), i)$ of V is an injective envelope of X and of Y too. If $T : X \to Y$ is linear, and if* (b) *holds, then conditions* (i)–(v) *in the previous theorem are equivalent for T. If* (a) *holds then* (b) *holds, and the equivalent conditions in the previous theorem are equivalent to*

(vi) *There is a contractive right module map $R : \partial V \to \partial V$ such that $R(i(x)) = i(T(x))$ for all $x \in X$.*

Proof. It is easy to check by routine diagram chases that the word 'Some' in (a) and (b) may be replaced by 'Every'.

If (b) holds then (X, Y) are an I-compatible V-pair and we obtain the equivalence of (i)-(v) from the last theorem.

Suppose that (a) holds. In this case (X, Y) is clearly a ∂-compatible V-pair. If $(I(V), i)$ is any injective envelope of V, and if $\mathcal{T}(V)$ is the subTRO of $I(V)$ generated by V, then we know from Section 4 that $(\mathcal{T}(V), i)$ is a noncommutative Shilov boundary of V. By the 'rigidity property' mentioned below Theorem 4.1 it is fairly clear that $I(V)$ is an injective envelope of $\mathcal{T}(V)$. Thus $(\mathcal{T}(V), i)$ is a noncommutative Shilov boundary of X. If we repeat this for X, we see that any injective envelope $I(X)$ of X is the injective envelope of the subTRO $\mathcal{T}(X)$. By the universal property of the noncommutative Shilov boundary $\mathcal{T}(X)$ is completely isometric via a ternary isomorphism to $\mathcal{T}(V)$, and by a routine diagram chase this isomorphism extends to a completely isometric ternary isomorphism between $I(X)$ and $I(V)$. It follows again by a routine diagram chase that $I(V)$ is an injective envelope for X, yielding (b). Then the equivalence with (iv) follows from Theorem 14.2. □

We next claim that the discussion in the paragraphs between Lemma 14.4 and Proposition 14.5 above, is also valid for I-compatible V-pairs. To see this one needs the following result:

Lemma 15.3. *If (X, Y) is an I-compatible V-pair then $I(X) \oplus_c I(Y)$ is an injective envelope for $X \oplus_V Y$.*

Proof. We remarked earlier (above Theorem 15.1) that $I(X) \oplus_c I(Y)$ is injective. It suffices, by one of the equivalent definitions of the injective envelope [**37, 60**], to show that if P is a completely contractive projection on $I(X) \oplus_c I(Y)$ which restricts to the identity on $X \oplus_V Y$, then P is the identity map. If ϵ_X, P_X are as in the discussion below Lemma 14.4, then $P_{I(X)} \circ P \circ \epsilon_{I(X)}$ is a complete contraction on $I(X)$ which restricts to Id_X. By the rigidity property of the injective envelope (see Section 4), $P_{I(X)} \circ P \circ \epsilon_{I(X)} = Id_{I(X)}$. Similarly $P_{I(Y)} \circ P \circ \epsilon_{I(Y)} = Id_{I(Y)}$. Since $P^2 = P$, by pure algebra we must conclude that $P_{I(Y)} \circ P \circ \epsilon_{I(X)}$ and $P_{I(X)} \circ P \circ \epsilon_{I(Y)}$ are zero. Thus $P = Id$. $\square$

Since (by the Lemma) the discussion in the paragraphs after Lemma 14.4 transfers to the present setting, it is easy to check that the conclusions of Corollary 14.6 are true for I-compatible V-pairs too.

There is another characterization of left V-multipliers which is also analogous to the formulation of left multipliers in [**20**]. To state this characterization we suppose that (X, Y) is an I-compatible V-pair. For simplicity we also suppose that $\mathcal{D}(Y) \subset \mathcal{D}(X)$ (if this is not the case, then replace all occurrences of $\mathcal{D}(X)$ below by $\mathcal{D}(V)$). We then have as above that $I(X)$ and $I(Y)$ are right C^*-modules over $\mathcal{D}(X)$, and hence also over the C^*-algebra multiplier algebra $M(\mathcal{D}(X))$. The latter C^*-algebra is injective too, by [**20**, Corollary 1.8]. Indeed $M(\mathcal{D}(X)) \cong I_{22}(X)$ in the language of [**20**]; and henceforth we shall just write I_{22} for $M(\mathcal{D}(X))$. We consider the 'generalized linking C^*-algebra' $A = \mathbb{B}_{I_{22}}(I(Y) \oplus_c I(X) \oplus_c I_{22})$. With respect to the canonical diagonal projections corresponding to the identities of $\mathbb{B}(I(Y)), \mathbb{B}(I(X))$ and I_{22} respectively, A may be written as a 3×3 matrix C^*-algebra, whose k-ℓ-corner we write as $I_{k\ell}$, for $k, \ell \in \{0, 1, 2\}$. Clearly $I_{02} = I(Y)$ and $I_{12} = I(X)$. In fact one can show using facts in [**38**] that A, and consequently also each I_{ij}, is injective. We will not use this here however, and therefore we omit the proof of this. We write i and j for the canonical maps from Y and X into I_{02} and I_{12} respectively.

THEOREM 15.4. *Suppose that (X, Y) is an I-compatible V-pair. Then a linear map $T : X \to Y$ satisfies conditions* (i)–(v) *in Theorem 15.1 if and only if there exists an element $a \in I_{01}$ such that $i(Tx) = aj(x)$ for all $x \in X$.*

PROOF. By [**20**, Corollary 2.7 (iii)] and the principle in equation (2),

$$I_{01} \cong \mathbb{B}_{I_{22}}(I(X), I(Y)) = B_{I_{22}}(I(X), I(Y)) = B_{\mathcal{D}(V)}(I(X), I(Y)).$$

The result is clear from this and Theorem 15.1 (v). $\square$

This result, and the matching part of Theorem 15.1, may also be proved by a variation of the proof given in [**54**] of the analogous assertion for $\mathcal{M}_l(X)$.

It should be interesting and useful to extend other known results about $\mathcal{M}_l(X)$ and $\mathcal{A}_l(X)$ (for example those in [**25**]) to the case of two spaces X and Y.

Acknowledgments: I am grateful to Georges Skandalis for helpful discussions related to some of these results. For example he pointed out to me that the implication (iv) $\Rightarrow$ (ii) in Theorem 3.1 is quite obvious, and he showed me a clever proof of the implication (ii) $\Rightarrow$ (i) which avoids duality considerations. It is conceivable that by variants on his argument, and some effort, one could remove the passage to the second dual in the proofs in Section 10. I also thank Baruch Solel for permission to publish here the results in Section 10 on Morita equivalence of operator algebras (which the author proved during exchanges with him in '00); and Vrej Zarikian and Damon Hay for comments on preliminary drafts of this manuscript.

References

[1] W. B. Arveson, *Subalgebras of C^*-algebras,* Acta Math. **123** (1969), 141-224.

[2] W. B. Arveson, *Subalgebras of C^*-algebras II,* Acta Math. **128** (1972), 271-308.

[3] W. B. Arveson, *Notes on the unique extension property,* Unpublished note, 2003.

[4] E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces, I and II*, Ann. of Math. **96** (1972), 98–173.

[5] E. Behrends, *M-structure and the Banach-Stone theorem*, Lecture Notes in Mathematics 736, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[6] D. P. Blecher, *A generalization of Hilbert modules,* J. Funct. Anal. **136** (1996), 365-421.

[7] D. P. Blecher, *Some general theory of operator algebras and their modules,* in *Operator algebras and applications*, A. Katavalos (editor), NATO ASIC, Vol. 495, Kluwer , Dordrecht, 1997.

[8] D. P. Blecher, *A new approach to C^*-modules,* Math Annalen **307** (1997), 253-290.

[9] D. P. Blecher, *The Shilov boundary of an operator space - and applications to the characterization theorems and Hilbert C^*-modules,* June 1999, ArXiV: math.OA/9906083v1

[10] D. P. Blecher, *The Shilov boundary of an operator space and the characterization theorems*, J. Funct. Anal. **182** (2001), 280–343.

[11] D. P. Blecher, *On Morita's fundamental theorem for C^*-algebras*, Math. Scand. **88** (2001), 137-153.

[12] D. P. Blecher, *Multipliers and dual operator algebras*, J. Funct. Anal. **183** (2001), 498–525.

[13] D. P. Blecher, *One-sided ideals and approximate identities in operator algebras,* To appear J. Australian Math. Soc.

[14] D. P. Blecher, E. G. Effros, and V. Zarikian, *One-sided M-ideals and multipliers in operator spaces, I*, Pacific J. Math. **206** (2002), 287–319.

[15] D. P. Blecher and D. M. Hay, *Complete isometries - an illustration of noncommutative functional analysis,* Proceedings of 4th Conference on Function Spaces, Contemp. Math. Vol. **328**, Amer. Math. Soc. (2003).

[16] D. P. Blecher and C. Le Merdy, *On function and operator modules,* Proceedings of the American Mathematical Society **129** (2001), 833-844.

[17] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules - an operator space approach,* To appear, Oxford Univ. Press.

[18] D. P. Blecher, P. S. Muhly and Q. Na, *Morita equivalence of operator algebras and their C^*-envelopes,* Bull. London Math. Soc. **31** (1999), 581-591.

[19] D. P. Blecher, P. S. Muhly and V. I. Paulsen, *Categories of operator modules - Morita equivalence and projective modules*, Memoirs of the Amer. Math. Soc., Vol. 143, number 681, January 2000.

[20] D. P. Blecher and V. I. Paulsen, *Multipliers of operator spaces and the injective envelope*, Pacific J. Math. **200** (2001), 1–17.

[21] D. P. Blecher, Z. J. Ruan, and A. M. Sinclair, *A characterization of operator algebras*, J. Funct. Anal. **89** (1990), 188-201.

[22] D. P. Blecher, R. R. Smith, and V. Zarikian, *One-sided projections on C^*-algebras*, To appear J. Op. Theory.

[23] D. P. Blecher and B. Solel, *A double commutant theorem for operator algebras,* To appear J. of Operator Theory.

[24] D. P. Blecher and V. Zarikian, *Multiplier operator algebras and applications,* Proc. Nat. Acad. Sci. U.S.A. **101** (2004), 727-731.

[25] D. P. Blecher and V. Zarikian, *The calculus of one-sided M-ideals and multipliers in operator spaces,* Submitted.

[26] L. G. Brown, J. A. Mingo, and N-T. Shen, *Quasi-multipliers and embeddings of Hilbert C^*-bimodules,* Canad. J. Math. **46** (1994), 1150-1174.

[27] M-D. Choi and E. G. Effros, *Injectivity and operator spaces,* J. Funct. Anal. **24** (1977), 156-209.

[28] E. Christensen, E. G. Effros, and A. M. Sinclair, *Completely bounded multilinear maps and C^*-algebraic cohomology,* Inv. Math. **90** (1987), 279-296.

[29] M. Dritschel and S. McCullouch, *Boundary representations for operator algebras,* J. Oper. Th. (To appear).

[30] E. G. Effros, *Order ideals in a C^*-algebra and its dual*, Duke Math. J. **30** (1963), 391–412.

[31] E. G. Effros, N. Ozawa, and Z. J. Ruan, *On injectivity and nuclearity for operator spaces,* Duke Math. J. **110** (2001), 489-521.
[32] E. G. Effros and Z-J. Ruan, *Mapping spaces and liftings for operator spaces*, Proc. London Math. Soc. **69** (1994), 171–197.
[33] E. G. Effros and Z. J. Ruan, *Operator Spaces*, Oxford University Press, Oxford, 2000.
[34] M. Frank and V. I. Paulsen, *Injective envelopes of C^*-algebras as operator modules,* Pacific J. Math. **212** (2003), 57-69.
[35] R. J. Fleming and J. E. Jamison, *Isometries on Banach spaces: function spaces,* Monographs and Surveys in Pure and Appl. Math. 129, Chapman and Hall/CRC press (2003).
[36] M. Hamana, *Injective envelopes of operator systems,* Publ. R.I.M.S. Kyoto Univ. **15** (1979), 773-785.
[37] M. Hamana, *Triple envelopes and Silov boundaries of operator spaces,* Math. J. Toyama University **22** (1999), 77-93.
[38] M. Hamana, *Modules over monotone complete C^*-algebras,* Internat. J. Math. **3** (1992), 185-204.
[39] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics 1547, Springer-Verlag, Berlin-Heidelberg-New York (1993).
[40] L. A. Harris, *A generalization of C^*-algebras*, Proc. London Math. Soc. (3) 42 (1981), no. 2, 331–361.
[41] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vols. I–IV*, Amer. Math. Soc., Providence (1997).
[42] R. V. Kadison, *Isometries of operator algebras,* Ann. of Math. **54** (1951), 325-338.
[43] M. Kaneda, *Multipliers and algebrizations of operator spaces,* Ph. D. Thesis University of Houston (2003).
[44] E. C. Lance, *Hilbert C^*-modules–A toolkit for operator algebraists,* London Math. Soc. Lecture Notes, Cambridge University Press (1995).
[45] C. Le Merdy, *An operator space characterization of dual operator algebras,* Amer. J. Math., **121** (1999), 55-63.
[46] B. Magajna, *Hilbert modules and tensor products of operator spaces,* Linear Operators, Banach Center Publ. Vol. 38, Inst. of Math. Polish Acad. Sci. (1997), 227-246.
[47] P. S. Muhly and B. Solel, *An algebraic characterization of boundary representations, Nonselfadjoint operator algebras, operator theory, and related topics,* p. 189-196, Oper. Th. Adv. Appl., 104, Birkhäuser, Basel, 1998.
[48] F. J. Murray and J. von Neumann, *On rings of operators,* IV, Ann. of Math. **44** (1943), 716-808.
[49] M. Neal and B. Russo, *Operator space characterizations of C^*-algebras and ternary rings,* Pacific J. Math. **209** (2003), 339-364.
[50] M. Neal and B. Russo, *State spaces of JB^*-triples,* Preprint (2003).
[51] J. von Neumann, *Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren,* Math. Ann. **102** (1929), 370-427.
[52] W. L. Paschke, *Inner product modules over B^*-algebras,* Trans. Amer. Math. Soc., **182** (1973), 443-468.
[53] W. L. Paschke, *Inner product modules arising from compact groups of a von Neumann algebra,* Trans. Amer. Math. Soc. **224** (1976), 87-102.
[54] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press (2002).
[55] G. K. Pedersen, *C^*-algebras and their automorphism groups,* Academic Press (1979).
[56] G. Pisier, *Introduction to operator space theory,* London Math. Soc. Lecture Note Series 294, Cambridge University Press (2003).
[57] Y-T. Poon and Z-J. Ruan, *Operator algebras with contractive approximate identities,* Canadian J. Math. **46** (1994), 397-414.
[58] M. A. Rieffel, *Morita equivalence for C^*-algebras and W^*-algebras,* J. Pure Appl. Algebra **5** (1974), 51-96.
[59] M. A. Rieffel, *Morita equivalence for operator algebras,* Proceedings of Symposia in Pure Mathematics **38** Part 1 (1982), 285-298.
[60] Z. J. Ruan, *Injectivity of operator spaces,* Trans. Amer. Math. Soc. **315** (1989), 89-104.
[61] J. Schweizer, *Interplay between noncommutative topology and operators on C^*-algebras*, Dissertation Eberhard-Karls-Universitat, Tubingen (1996).

[62] R. R. Smith and J. D. Ward, *M-ideal structure in Banach algebras*, J. Funct. Anal. **27** (1978), 337–349.
[63] M. H. Stone, *Applications of the theory of boolean rings in topology,* Trans. Amer. Math. Soc. **41** (1937), 375-481.
[64] C-Y. Suen, *Completely bounded maps on C^*-algebras,* Proc. Amer. Math. Soc. **93** (1985), 81-87.
[65] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York (1979).
[66] N. Weaver, *Mathematical Quantization*, CRC Press (2001).
[67] N. E. Wegge-Olsen, *K-theory and C^*-algebras,* Oxford Univ. Press (1993).
[68] W. Werner, *Multipliers on matrix ordered operator spaces and some K-groups,* J. Funct. Anal. **206** (2004), 356-378.
[69] G. Wittstock, *Ein operatorwertiger Hahn-Banach Satz,* J. Funct. Anal. **40** (1981), 127-150.
[70] G. Wittstock, *Extensions of completely bounded module morphisms*, Proceedings of conference on operator algebras and group representations, Neptum, 238-250, Pitman (1983).
[71] S. L. Woronowicz, *Nonextendible positive maps,* Commun. Math. Phys. **51** (1976), 243-282.
[72] V. Zarikian, *Complete one-sided M-ideals in operator spaces*, Ph.D. thesis, UCLA, 2001.
[73] V. Zarikian, *Local characterizations of one-sided M-ideals (working title)*, in preparation.
[74] C. Zhang, *Representations of operator spaces,* J. Oper. Th. **33** (1995), 327-351.

Department of Mathematics, University of Houston, Houston, TX 77204-3008
E-mail address, David P. Blecher: `dblecher@math.uh.edu`