Section 3.5
The Inverse of a Square Matrix

In this section we discuss a procedure for finding the inverse of a matrix and show how the inverse can be used to help us solve a system of linear equations.

Let A be a square matrix of size n. A square matrix A^{-1} of size n such that $AA^{-1} = A^{-1}A = I_n$ is called the **inverse of A**.

Note: Not every square matrix has an inverse. A matrix with no inverse is called **singular**.

Example 1: Determine whether the following pairs of matrices are inverses of each other.

a. $\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}$

Check if:

$$\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix} = I_2$$

If so, then must also check if:

$$\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} = I_2$$

Inverses?

b. $\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix}$

Check if:

$$\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix} = I_2$$

If so, then must also check if:

$$\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix} = I_2$$

Inverses?

How to Find the Inverse of a Square Matrix

1. Adjoin the square matrix A with the identity matrix of the same size, $\left[A\,|\,I\right]$.
2. Use the Gauss-Jordan elimination method to reduce $\left[A\,|\,I\right]$ to the form $\left[I\,|\,B\right]$, if possible.

The matrix B is the inverse of the matrix A. It may be verified by checking $AB = BA = I_n$.

If in the process of reducing $\left[A\,|\,I\right]$ to $\left[I\,|\,B\right]$, there is a row of zeros to the left of the vertical line then the matrix does not have an inverse.

Example 2: Find the inverse of each matrix, if possible.

a. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

b. $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 5 \end{pmatrix}$

c. $C = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Formula for the Inverse of a 2X2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $D = ad - bc$ is not equal to zero. Then. A^{-1} exists and is given by $A^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 3: Find the inverse of the following matrix.

a. $E = \begin{pmatrix} -5 & 10 \\ -2 & 7 \end{pmatrix}$

b. $F = \begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix}$

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations AX = B, involving the same coefficient matrix, A, and different matrices of constants, B.

The steps are:

1. Write the system of equations in matrix form, i.e., AX= B. *A is the coefficient matrix, X is the variable matrix and B is the constant matrix.*
2. Find the inverse of the coefficient matrix, A.
3. Multiply both sides by A^{-1}.
4. State the answer.

Example 4: Write each system of equations as a matrix equation, then solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 2y &= 12 \\ -3x + 5y &= -8 \end{aligned}$$

$$\begin{aligned} 2x - 2y &= 0 \\ -3x + 5y &= 4 \end{aligned}$$

Now find the inverse of the coefficient matrix.

D = ad – bc =

$A^{-1} =$

$AX = B_1$
$X = A^{-1}B_1$

$AX = B_2$
$X = A^{-1}B_2$