Section 3.4
Multiplication of Matrices

If A is a matrix of size *mxn* and B is a matrix of size *nxp* then the product AB is defined and is a matrix of size *mxp*.

AB (mxn) (nxp) = mxp

So, two matrices can be multiplied if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Example 1: Matrix *A* is of size 10X7, matrix *B* is of size 7X2 and matrix C is of size 2X10. Decide if each of the following products are defined. If so, give the size of the product.

	a. CB	b. AB	c. BC	d. CA^T
	(2x10) (7x2)	(10x7) (7x2)	(7x2) (2x10)	(2x10) (7x10)
Product defined?	NO	YES	YES	NO
If so, size?		10 x 2	7x10	

How to Multiply Two Matrices

The element in the *i*th row and *j*th column of AB is found by multiplying each element in the *i*th row of A by the corresponding element in the *j*th column of B and adding the products.

Example 2: Your stock holdings are given by the row matrix (or vector)

$$\begin{array}{cccc} & GM & IBM & BAC \\ A = & (700 & 400 & 200) \end{array}$$

At the close of trading on a certain day, the prices (in dollars per share) of these stocks (GM, IBM, BAC, respectively) are

$$B = \begin{pmatrix} 50 \\ 120 \\ 42 \end{pmatrix}$$

What is the total value of your holdings as of that day?

$$AB = \begin{pmatrix} 700 & 400 & 200 \end{pmatrix} \begin{pmatrix} 50 \\ 120 \\ 42 \end{pmatrix} = (700)(50) + (400)(120) + (200)(42)$$

$$= \$91400$$

Tryit

Example 3: A company manufactures tables and chairs. The following matrix gives the time requirements for each table and each chair in each of the given departments.

	Assembly Dept	*Finishing Dept*
Chair	2 hr	1 hr
Table	4 hr	2 hr

Furthermore, the company has two manufacturing plants, one in California and the other in Florida. The hourly rates for each department in each state are given in the following matrix.

	California	*Florida*
Assembly Dept	\$25	\$22
Finishing Dept	\$18	\$15

$$\begin{array}{cc} CA & FL \end{array}$$
$$\begin{bmatrix} 25 & 22 \\ 18 & 15 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Calculate the labor costs for tables and chair in each state.

$$\begin{array}{c} Ch \\ T \end{array}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{array}{c} \begin{array}{cc} CA & FL \end{array} \\ \begin{bmatrix} 25 & 22 \\ 18 & 15 \end{bmatrix} \end{array} = \begin{bmatrix} 2(25)+1(18) & 2(22)+1(15) \\ 4(25)+2(18) & 4(22)+2(15) \end{bmatrix}$$

$$= \begin{array}{c} Ch \\ T \end{array}\begin{array}{c} \begin{array}{cc} CA & FL \end{array} \\ \begin{bmatrix} 68 & 59 \\ 136 & 118 \end{bmatrix} \end{array}$$

\$68 chair in CA
\$59 chair in FL
\$136 table in CA
\$118 table in FL

Example 4: Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix}$ 2x3, $B = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix}$ 3x3, $C = \begin{pmatrix} -10 & 9 \\ -6 & 4 \end{pmatrix}$ 2x2 and

$D = \begin{pmatrix} -1 & 2 & -3 \end{pmatrix}$ 1x3. Compute, if possible:

a. DB $(1\times 3)(3\times 3) = (1\times 3)$

$$\begin{pmatrix} -1 & 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1\cdot 3+2\cdot 2+-3\cdot 1 & -1\cdot 1+2\cdot 0+-3\cdot 2 & -1\cdot 4+2\cdot 3+-3\cdot -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -7 & 5 \end{pmatrix}$$

$(2\times 3)\quad(3\times 3)\qquad(2\times 3)$

b. AB

$$\begin{pmatrix}1&3&0\\2&4&-1\end{pmatrix}\begin{pmatrix}3&1&4\\2&0&3\\1&2&-1\end{pmatrix}=\begin{pmatrix}9&1&13\\13&0&21\end{pmatrix}$$

$$=\begin{pmatrix}1\cdot3+3\cdot2+0\cdot1 & 1\cdot1+3\cdot0+0\cdot2 & 1\cdot4+3\cdot3+0\cdot-1\\ 2\cdot3+4\cdot2+-1\cdot1 & 2\cdot1+4\cdot0+-1\cdot2 & 2\cdot4+4\cdot3+-1\cdot-1\end{pmatrix}$$

c. BA^T

$3\times 3\quad 3\times 2 = 3\times 2$

$$\begin{pmatrix}3&1&4\\2&0&3\\1&2&-1\end{pmatrix}\begin{pmatrix}1&2\\3&4\\0&-1\end{pmatrix}=\begin{pmatrix}3\cdot1+1\cdot3+4\cdot0 & 3\cdot2+1\cdot4+4\cdot-1\\ 2\cdot1+0\cdot3+3\cdot0 & 2\cdot2+0\cdot4+3\cdot-1\\ 1\cdot1+2\cdot3+-1\cdot0 & 1\cdot2+2\cdot4+-1\cdot-1\end{pmatrix}$$

$$=\begin{pmatrix}6&6\\2&1\\7&11\end{pmatrix}$$

d. $A^T D^T$

$(3\times 2)\quad(3\times 1)$

Not possible

Try this one: CA

$2\times 2\quad 2\times 3\qquad 2\times 3$

$$\begin{pmatrix}-10&9\\-6&4\end{pmatrix}\begin{pmatrix}1&3&0\\2&4&-1\end{pmatrix}=\begin{pmatrix}-10\cdot1+9\cdot2 & -10\cdot3+9\cdot4 & -10\cdot0+9\cdot-1\\ -6\cdot1+4\cdot2 & -6\cdot3+4\cdot4 & -6\cdot0+4\cdot-1\end{pmatrix}$$

$$=\begin{pmatrix}8&6&-9\\2&-2&-4\end{pmatrix}$$

Note: In general, matrix multiplication is not commutative – that is, $AB \neq BA$.

A **square matrix** is a matrix having the same number of rows as columns.

The **identity matrix** is a square matrix that has 1's along its main diagonal (from the upper left corner to the lower right corner) and 0's elsewhere. Since an identity matrix the same number of rows as columns, we simply say an identity matrix is of size *n*.

additive identity

$2+0=2$

$2\cdot 1=2$

multiplicative identity

The **identity matrix** of size *n* is given by $I_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & 1 \end{pmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In general matrix multiplication is not commutative. However, if A is a square matrix of size *n* the identity matrix of size *n* has the following property:

$$AI_n = I_nA = A$$

$G = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \\ 2 & 3 & 1 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2×2

Example 5: Let $G = \begin{pmatrix} 6 & -2 \\ 4 & -8 \end{pmatrix}$.

a. Give the identity matrix of the same size.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. Show that GI = IG = G

$$\begin{pmatrix} 6 & -2 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 6(1)+-2(0) & 6(0)+-2(1) \\ 4(1)+-8(0) & 4(0)+-8(1) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 4 & -8 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ 4 & -8 \end{pmatrix} = \begin{bmatrix} 1(6)+0(4) & 1(-2)+0(-8) \\ 0(6)+1(4) & 0(-2)+1(-8) \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 4 & -8 \end{bmatrix}$$

Multiplication Properties of Matrices

Let A, B and C be matrices whose products and sums are defined. Also let k be a scalar.

1. Associative Property: $A(BC) = (AB)C$ ←
2. Associative Property: $k(AB) = (kA)B$
3. Distributive Property: $A(B + C) = AB + AC$

Example 6: Perform the indicated operations.

$$\overset{k}{\frac{1}{2}}\overset{A}{\begin{pmatrix} 2 & 4 \\ -2 & -10 \end{pmatrix}}\overset{B}{\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}} - 2\begin{pmatrix} 0 & 10 \\ -4 & -12 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & -5 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 20 \\ -8 & -24 \end{pmatrix}$$

$$= \begin{pmatrix} 1(-1) + 2(1) & 1(2) + 2(0) \\ -1(-1) + -5(1) & -1(2) + -5(0) \end{pmatrix} - \begin{pmatrix} 0 & 20 \\ -8 & -24 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -4 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 20 \\ -8 & -24 \end{pmatrix} = \begin{pmatrix} 1 & -18 \\ 4 & 22 \end{pmatrix}$$

Try this one: The following table displays the average grade in each category for an upper level honors course with 5 students.

	Test 1	Tests 2	Test 3	Final Exam	Homework Avg	Quiz Avg
Amy	85	90	100	82	91	99
Rob	87	85	98	90	81	100
Sally	75	81	79	100	80	85
Ted	80	70	88	95	75	95
Julie	70	78	84	92	60	87

=0.15 0.25

If each test is worth 15%, the final exam is worth 25%, the homework average is worth =0.1 10%, and the quiz average is worth 20%, =0.2 what is each student's course average? Use a matrix to display the grades and another to display the percentages. Give the answer in the form of a matrix.

$$\begin{pmatrix} 85 & 90 & 100 & 82 & 91 & 99 \\ 87 & 85 & 98 & 90 & 81 & 100 \\ 75 & 81 & 79 & 100 & 80 & 85 \\ 80 & 70 & 88 & 95 & 75 & 95 \\ 70 & 78 & 84 & 92 & 60 & 87 \end{pmatrix} \begin{pmatrix} 0.15 \\ 0.15 \\ 0.15 \\ 0.25 \\ 0.1 \\ 0.2 \end{pmatrix}$$