Section 3.5
The Inverse of a Square Matrix

In this section we discuss a procedure for finding the inverse of a matrix and show how the inverse can be used to help us solve a system of linear equations.

Let A be a square matrix of size n. A square matrix A^{-1} of size n such that $AA^{-1} = A^{-1}A = I_n$ is called the **inverse of A**.

$2 \cdot 1 = 2$

$2 \cdot \frac{1}{2} = 1$

Note: Not every square matrix has an inverse. A matrix with no inverse is called **singular**.

Example 1: Determine whether the following pairs of matrices are inverses of each other.

a. $\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}$

Check if:

$\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix} = I_2$

$\begin{pmatrix} 5(1) + 4(-1) & 5(-4) + 4(5) \\ 1(1) + 1(-1) & 1(-4) + 1(5) \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

If so, then must also check if:

$\begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} = I_2$

$\begin{pmatrix} 1 \cdot 5 + -4 \cdot 1 & 1 \cdot 4 + -4 \cdot 1 \\ -1 \cdot 5 + 5 \cdot 1 & -1 \cdot 4 + 5 \cdot 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Inverses? YES

b. $\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix}$

Check if:

$\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix} = I_2$

$= \begin{pmatrix} -2\left(-\frac{4}{5}\right) + -3\left(\frac{1}{5}\right) & -2\left(-\frac{6}{5}\right) + -3\left(\frac{4}{5}\right) \\ 1\left(-\frac{4}{5}\right) + 4\left(\frac{1}{5}\right) & 1\left(-\frac{6}{5}\right) + 4\left(\frac{4}{5}\right) \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ ✓ NO

If so, then must also check if:

$\begin{pmatrix} -4/5 & -6/5 \\ 1/5 & 4/5 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 1 & 4 \end{pmatrix} = I_2$

$= \begin{pmatrix} -\frac{4}{5} \cdot -2 + -\frac{6}{5} \cdot 1 & -\frac{4}{5} \cdot -3 + -\frac{6}{5} \cdot 4 \\ \frac{1}{5} \cdot -2 + \frac{4}{5} \cdot 1 & \frac{1}{5} \cdot -3 + \frac{4}{5} \cdot 4 \end{pmatrix}$

$= \begin{pmatrix} 2/5 & -12/5 \\ 2/5 & 13/5 \end{pmatrix}$

Inverses? NO

How to Find the Inverse of a Square Matrix

1. Adjoin the square matrix A with the identity matrix of the same size, $[A \mid I]$.
2. Use the Gauss-Jordan elimination method to reduce $[A \mid I]$ to the form $[I \mid B]$, if possible.

The matrix B is the inverse of the matrix A. It may be verified by checking $AB = BA = I_n$.

If in the process of reducing $[A \mid I]$ to $[I \mid B]$, there is a row of zeros to the left of the vertical line then the matrix does not have an inverse.

Example 2: Find the inverse of each matrix, if possible.

a. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]$$

$R_2 = -3R_1 + R_2$

-3	-6	-3	0
3	4	0	1
0	-2	-3	1

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$R_2 = -\frac{1}{2} R_2$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$R_1 = -2R_2 + R_1$

1	2	1	0
0	-2	-3	1
1	0	-2	1

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad R_2 = -2R_1 + R_2, \quad R_3 = -3R_1 + R_3$$

b. $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 5 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -4 & -2 & 1 & 0 \\ 0 & -3 & -4 & -3 & 0 & 1 \end{array}\right]$$

No inverse

$R_3 = -R_2 + R_3$

B is singular

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -4 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

c. $C = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$R_2 = R_1 + R_2$
$R_3 = -3R_1 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 1 & 1 \\ 0 & -1 & -2 & 1 & 0 & -3 \end{array}\right]$$

$R_3 = R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -2 \end{array}\right] \xrightarrow{R_3 = -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & -1 & -1 & 2 \end{array}\right]$$

$R_2 = -R_3 + R_2$
$R_1 = -R_3 + R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$C^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Formula for the Inverse of a 2X2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Suppose $D = ad - bc$ is not equal to zero. Then. A^{-1} exists and is given by $A^{-1} = \frac{1}{D}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If D = 0 then there is no inverse
the matrix is singular

Example 3: Find the inverse of the following matrix.

a. $E = \begin{pmatrix} -5 & 10 \\ -2 & 7 \end{pmatrix}$

$D = -5 \cdot 7 - (-2) \cdot 10$
$= -35 + 20 = -15 \neq 0$

$E^{-1} = \frac{1}{-15}\begin{pmatrix} 7 & -10 \\ 2 & -5 \end{pmatrix} = \begin{pmatrix} -7/15 & 2/3 \\ -2/15 & 1/3 \end{pmatrix}$

b. $F = \begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix}$

$D = -6 \cdot 2 - -3 \cdot 4$
$= -12 + 12 = 0$

No inverse

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations AX = B, involving the same coefficient matrix, A, and different matrices of constants, B.

The steps are:
1. Write the system of equations in matrix form, i.e., AX= B. *A is the coefficient matrix, X is the variable matrix and B is the constant matrix.*
2. Find the inverse of the coefficient matrix, A.
3. Multiply both sides by A^{-1}.
4. State the answer.

$AX = B$
$A^{-1}AX = A^{-1}B$
$X = A^{-1}B$

Example 4: Write each system of equations as a matrix equation, then solve the system using the inverse of the coefficient matrix.

$2x - 2y = 12$
$-3x + 5y = -8$

$$\underset{A}{\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B_1}{\begin{bmatrix} 12 \\ -8 \end{bmatrix}}$$

$2x - 2y = 0$
$-3x + 5y = 4$

$$\underset{A}{\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B_2}{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}$$

$$A = \begin{pmatrix} 2 & -2 \\ -3 & 5 \end{pmatrix}$$

Now find the inverse of the coefficient matrix.

$$D = ad - bc = 2\cdot 5 - (-2)(-3) = 10 - 6 = 4 \neq 0$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}$$

$$AX = B_1$$
$$X = A^{-1}B_1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 5/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}\begin{pmatrix} 12 \\ -8 \end{pmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{4}\cdot 12 + \frac{1}{2}\cdot -8 \\ \frac{3}{4}\cdot 12 + \frac{1}{2}\cdot -8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 - 4 \\ 9 - 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \end{bmatrix}$$

$x = 11$

$y = 5$

$$AX = B_2$$
$$X = A^{-1}B_2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 5/4 & 1/2 \\ 3/4 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{4}\cdot 0 + \frac{1}{2}\cdot 4 \\ \frac{3}{4}\cdot 0 + \frac{1}{2}\cdot 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$x = 2$

$y = 2$

$A_{n\times n}$ $\quad B_{n\times n}$

$AB = n\times n \qquad BA\ n\times n$

$AB \neq BA$ in general

* Matrix multiplication is not commutative in general

BUT

Matrix addition is commutatie

$$A+B = B+A$$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$\left.\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \\ BA &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \end{aligned}\right\} AB \neq BA$$