Section 7.5 – The Normal Distribution
Section 7.6 – Application of the Normal Distribution

A random variable that may take on infinitely many values is called a **continuous random variable**.

A continuous probability distribution is defined by a function *f* called the **probability density function**.

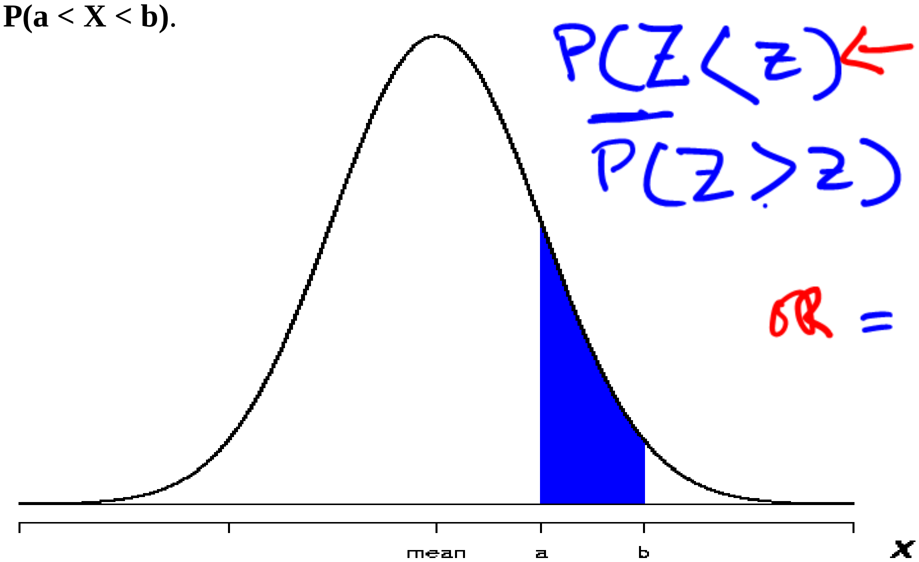
The probability that the random variable *X* associated with a given probability density function assumes a value in an interval $a < x < b$ is given by the area of the region between the graph of *f* and the *x*-axis from $x = a$ to $x = b$.

The following graph is a picture of a normal curve and the shaded region is **P(a < X < b)**.

$P(Z < z) \leftarrow$ z-table

$P(Z > z) = P(Z < -z)$

OR $= 1 - P(Z < z)$



$P(a < Z < b) = P(Z < b) - P(Z < a)$

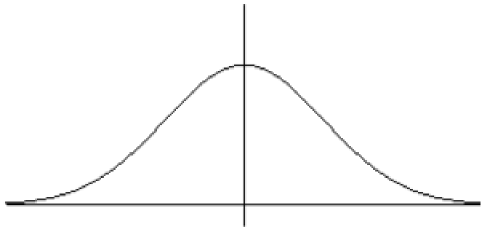
Note: $P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$, since the area under one point is 0. The **area of the region under the standard normal curve to the left** of some value z, i.e. $P(Z < z)$ or $P(Z \leq z)$, is calculated for us in the **Standard Normal Cumulative Probability Table** found in Chapter 7 of the online book.

standard Normal (mean) $\mu = 0$ (sd) $\sigma = 1$

$Z \sim N(0,1)$

Normal distributions have the following characteristics:

1. The graph is a bell-shaped curve.



The curve always lies above the *x*-axis but approaches the *x*-axis as *x* extends indefinitely in either direction.

2. The curve has peak at $x = \mu$. The mean, μ, determines where the center of the curve is located.

3. The curve is symmetric with respect to the vertical line $x = \mu$.

4. The area under the curve is 1.

5. σ determines the sharpness or the flatness of the curve.

6. For any normal curve, 68.27% of the area under the curve lies within 1 standard deviation of the mean (i.e. between $\mu - \sigma$ and $\mu + \sigma$), 95.45% of the area lies within 2 standard deviations of the mean, and 99.73% of the area lies within 3 standard deviations of the mean.

The Standard Normal Variable will commonly be denoted *Z*. The **Standard Normal Curve** has $\mu = 0$ and $\sigma = 1$.

Example 1: Let *Z* be the standard normal variable. Find the values of:

a. $P(Z < -1.91)$ $= 0.0281$

b. P(Z > 0.5) $= 1 - P(Z < 0.5) = 1 - 0.6915$
$= 0.3085$

OR $P(Z < -0.5) = 0.3085$

c. P(-1.65 < Z < 2.02)

$= P(Z < 2.02) - P(Z < -1.65)$

$= 0.9783 - 0.0495$

$= 0.9288$

Example 2: Let *Z* be the standard normal variable. Find the value of *z* if *z* satisfies:

a. P(*Z* < -*z*) = 0.9495 ← table.

$-z = 1.64 \Rightarrow z = -1.64$

b. P(*Z* > *z*) = 0.9115 $P(Z > z) = 1 - P(Z < z)$

$P(Z < z) = 1 - P(Z > z) = 1 - 0.9115 = 0.0885$

$z = -1.35$

OR

$P(Z < -z) = 0.9115 \quad -z = 1.35 \Rightarrow z = -1.35$

c. P(-z < Z < z) = 0.8444

Formula: $P(Z < z) = \frac{1}{2}[1 + P(-z < Z < z)]$

$P(Z < z) = \frac{1}{2}[1 + P(-z < Z < z)] = \frac{1}{2}[1 + 0.8444]$

$P(Z < z) = 0.9222$

$z = 1.42$

When given a normal distribution in which $\mu \neq 0$ and $\sigma \neq 1$, we can transform the normal curve to the standard normal curve by doing whichever of the following applies.

$\mu = E[x]$ mean

$\sigma \Rightarrow$ s.d.

$$P(X < b) = P\left(Z < \frac{b-\mu}{\sigma}\right)$$

$$P(X > a) = P\left(Z > \frac{a-\mu}{\sigma}\right)$$

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

Example 3: Suppose X is a normal variable with $\mu = 7$ and $\sigma = 4$. Find P(X > -1.35).

$$P(X > -1.35) = P\left(Z > \frac{-1.35-7}{4}\right) = P(Z > -2.08)$$

$$P(Z > -2.08) = P(Z < 2.08) = 0.9812$$

OR

$$P(Z > -2.08) = 1 - P(Z < -2.08) = 1 - 0.0188 = 0.9812$$

Applications of the Normal Distribution

Example 4: The heights of a certain species of plant are normally distributed with a mean of 20 cm and standard deviation of 4 cm. What is the probability that a plant chosen at random will be between 10 and 33 cm tall?

$\mu = 20$ $\sigma = 4$

$$P(10 < X < 33) = P\left(\frac{10-20}{4} < Z < \frac{33-20}{4}\right)$$

$$= P(-2.5 < Z < 3.25)$$

$$= P(Z < 3.25) - P(Z < -2.5)$$

$$= 0.9994 - 0.0062 = 0.9932$$

Example 5: Reaction time is normally distributed with a mean of 0.7 second and a standard deviation of 0.1 second. Find the probability that an individual selected at random has a reaction time of less than 0.6 second.

$\mu = 0.7 \quad \sigma = 0.1$

$$P(X < 0.6) = P\left(Z < \frac{0.6 - 0.7}{0.1}\right)$$

$$= P(Z < -1) = 0.1587$$

Approximating the Binomial Distribution Using the Normal Distribution

Theorem

Suppose we are given a binomial distribution associated with a binomial experiment involving n trials, each with probability of success p and probability of failure q. Then if n is large and p is not close to 0 or 1, the binomial distribution may be approximated by a normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$.

Example 6: Consider the following binomial experiment. Use the normal distribution to approximate the binomial distribution. A company claims that 42% of the households in a certain community use their Squeaky Clean All Purpose cleaner. What is the probability that between 15 and 28, inclusive, households out of 50 households use the cleaner?

$n = 50 \quad p = 0.42 \quad q = 1 - p = 0.58$

$\mu = np = 50(0.42) = 21 \quad \sigma = \sqrt{npq} = \sqrt{50(0.42)(0.58)}$

$= 3.49$

$$P(15 \le X \le 28)$$

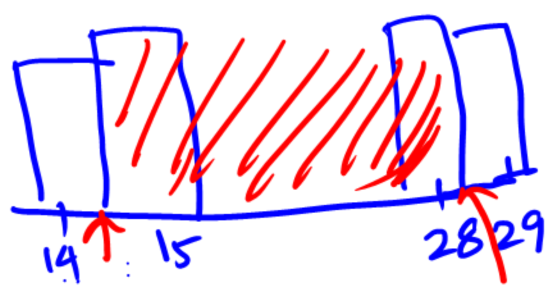
$$\approx P(14.5 \le Y \le 28.5)$$

$$= P\left(\frac{14.5 - 21}{3.49} \le Z \le \frac{28.5 - 21}{3.49}\right)$$

$$= P(-1.86 < Z < 2.15)$$

$$= P(Z < 2.15) - P(Z < -1.86) = 0.9842 - 0.0314$$

$$= 0.9528$$

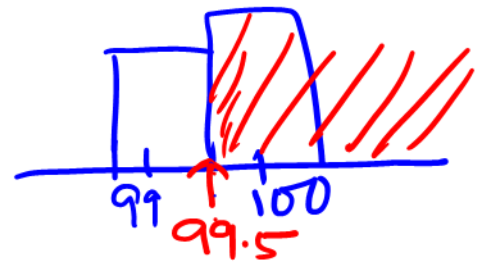


$n = 120 \quad p = 0.75 \quad q = 1 - p = 0.25$

$\mu = np = 120(0.75) = 90 \quad \sigma = \sqrt{npq} = \sqrt{120(0.75)(0.25)} = 4.734$

Example 7: Use the normal distribution to approximate the binomial distribution. A basketball player has a 75% chance of making a free throw. She will make 120 attempts. What is the probability of her making:

a. 100 or more free throws?



$P(X \geq 100)$

$= P(Y \geq 99.5)$

$= P(Y > 99.5)$

$= P\left(Z > \frac{99.5 - 90}{4.734}\right)$

$= P(Z > 2.00)$

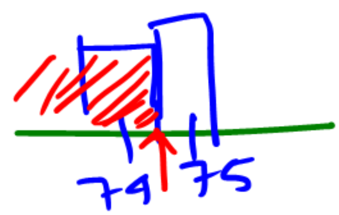
$= P(Z < -2.00)$

$= 0.0228$

OR $1 - P(P < 2)$

$= 1 - 0.9772 = 0.0228$

b. fewer than 75 free throws?



$P(X < 75)$

$= P(Y \leq 74.5)$

$= P(Y < 74.5)$

$= P\left(Y < \frac{74.5 - 90}{4.7434}\right) = P(Z < -3.27) = 0.0005$

Example 8: Use the normal distribution to approximate the binomial distribution. A die is rolled 84 times. What is the probability that the number 2 occurs more than 11 times?

$n = 84 \quad p = 1/6 \quad q = 5/6$

$\mu = np = 84 \cdot \left(\frac{1}{6}\right) = 14 \quad \sigma = \sqrt{84 \cdot \frac{1}{6} \cdot \frac{5}{6}} = 3.4157$

$P(X > 11)$

$= P(Y \geq 11.5) = P(Y > 11.5)$

$= P\left(Z > \frac{11.5 - 14}{3.4157}\right)$

$= P(Z > -0.73) = P(Z < 0.73)$

$= 0.7673$

OR $= 1 - P(Z < -0.73) = 1 - 0.2327 = 0.7673$