Math 3321

Linear Differential Equations

University of Houston

Lecture 03

Outline

1. First Order Differential Equations
2. Linear Equations
3. Strategy to Solve Linear Equations
4. Solution Method for First Order Linear Equations
5. Examples of Solving Linear Equations
6. Linearity

First Order Differential Equations

Definition

A *first order differential equation* is an ODE which can be written

$$F(x, y, y') = 0.$$

As stated in Chapter 1, we can write our equation in this form by moving all the non-zero terms to the left-hand side of the equation.

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Fact

In this course, we will make an additional assumption, which is that we can solve for y' explicitly in the equation. That is, we will be able to write the equation in the form:

$$y' = f(x, y).$$

Background Material

We assume familiarity with the basics of integration, including the following techniques:

- Substitution (the most common technique, often called u-sub)
- Integration-by-parts
- Trigonometric integrals
- Partial fraction decomposition

Linear Equations

Our first new strategy for solving differential equations will be used for first order ODEs, $y' = f(x, y)$, where the function f can be written as

$$f(x, y) = P(x)y + q(x).$$

Our strategy will require us to write this as

$$y' - P(x)y = q(x).$$

Setting $p(x) = -P(x)$, we have

$$y' + p(x)y = q(x)$$

which will be our preferred form.

Linear Equations

Definitions

A first order differential equation $y' = f(x, y)$ is a *linear equation* if the differential equation can be written in the form

$$y' + p(x)y = q(x) \tag{1}$$

where p and q are continuous functions on some interval I. We will refer to the above equation as the *standard form* for first order linear equations.

Differential equations that are not linear are called *nonlinear differential equations.*

Linear Equations

Example:

1. For the following differential equations, write the equation in standard form and identify the functions p and q.

   a. $y' = 2y$

   b. $\frac{1}{t}x' = \frac{2x}{t} + t\cos(t)$

   c. $(4 - x^2)\frac{dy}{dx} - x^2y = x^2 - 4$

Linear Equations

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1. For the following differential equations, write the equation in standard form and identify the functions p and q.

   a. $y' = 2y$

      $y' - 2y = 0$
      Hence $p(x) = -2, q(x) = 0$

   b. $\dfrac{1}{t}x' = \dfrac{2x}{t} + t\cos(t)$

      $x' - 2x = t^2\cos(t)$
      Hence $p(t) = -2, q(t) = t^2\cos(t)$

   c. $(4 - x^2)\dfrac{dy}{dx} - x^2y = x^2 - 4$

      $y' - \frac{x^2}{4-x^2}y = \frac{x^2-4}{4-x^2} = -1$
      Hence $p(x) = -\frac{x^2}{4-x^2}, q(x) = -1$

Strategy to Solve Linear Equations

Our strategy is born of an observation about the standard form

$$y' + p(x)y = q(x).$$

We see the sum on the left side of the equation features the unknown function y and the derivative of this function y' and this reminds us of the product rule for differentiation. Note the following

$$\frac{d}{dx}\left(u(x)y\right) = u(x)y' + u'(x)y$$

and see that the expression on the right side of this equation is very similar to the left side of the standard form equation.

Strategy to Solve Linear Equations

Our Goal

Our goal when solving first order linear equations is to multiply both sides of the equation in standard form by the correct function u so that we can form

$$u(x)\left(y' + p(x)y\right) = u(x)y' + u'(x)y \tag{2}$$

on the left side of our equation. That is, we wish to write the equation as

$$u(x)y' + u'(x)y = u(x)q(x).$$

Realizing the left hand side of our equation is now

$$\frac{d}{dx}\left(u(x)y\right) = u(x)q(x)$$

will allow us to integrate (assuming $u(x)q(x)$ can be integrated) and solve the differential equation.

Finding $u(x)$

In order for equation (2) to be true, we need $u'(x) = u(x)p(x)$. Let h be an anti-derivative for p. That is, $h'(x) = p(x)$ and $h(x)$ is found by solving

$$h(x) = \int p(x)dx$$

with the integration constant set to be 0. Our function u will be

$$u(x) = e^{h(x)}.$$

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Definition

The function u formed above, which is used to multiply both sides of $y' + p(x)y = q(x)$, is called an *integrating factor*. Once again, this will enable us to write the left side of the equation as the derivative of a product.

Solution Method for First Order Linear Equations

Solving $y' + p(x)y = q(x)$:

Our steps are as follows:

1. Identify: Can we write the given equation in the form (1): $y' + p(x)y = q(x)$? If yes, do so.

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Note: $u'(x) = \left[e^{h(x)}\right]' = e^{h(x)}h'(x) = e^{h(x)}p(x) = u(x)p(x).$

Solution Method for First Order Linear Equations

Solving $y' + p(x)y = q(x)$:

3. Multiply equation (1) by the integrating factor to obtain

$$u(x)y' + u(x)p(x)y = u(x)q(x).$$

   This gives us

$$\frac{d}{dx}\left(u(x)y\right) = u(x)q(x) \tag{3}$$

   so that we can now turn our attention to integrating both sides.

Solution Method for First Order Linear Equations

Solving $y' + p(x)y = q(x)$:

4. Integrating (3) gives

$$u(x)y = e^{h(x)}y = \int e^{h(x)}q(x)dx + C$$

so that

$$y = e^{-h(x)}\left[\int e^{h(x)}q(x)dx + C\right] = e^{-h(x)}\int e^{h(x)}q(x)dx + Ce^{-h(x)}.$$

It follows that the general solution of (1) is

$$y = e^{-h(x)}\int e^{h(x)}q(x)dx + Ce^{-h(x)}.$$

Examples of Solving Linear Equations

1. $y' - 2xy = x$

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1. We identify

$$p(x) = -2x, \quad q(x) = x$$

2. We compute the integrating factor

$$h(x) = \int p(x)dx = -x^2$$

3. We write the general solution

$$\begin{aligned} y(x) &= e^{-h(x)} \int e^{h(x)} q(x)dx + Ce^{-h(x)} \\ &= e^{x^2} \int xe^{-x^2}\, dx + Ce^{x^2} \\ &= e^{x^2}(-\frac{1}{2}e^{-x^2}) + Ce^{x^2} \\ &= -\frac{1}{2} + Ce^{x^2} \end{aligned}$$

Examples of Solving Linear Equations

2. $\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{2y}{x^2} = x\cos(x),\ x > 0$

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2. $\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x),\ x > 0$

1. We write the ODE as: $y'(x) - \frac{2y}{x} = x^2\cos(x)$ *and we identify*

$$p(x) = -2/x, \quad q(x) = x^2\cos(x)$$

2. We compute the integrating factor

$$h(x) = \int p(x)dx = -2\int \frac{1}{x}dx = -2\ln x = -\ln x^2$$

3. We write the general solution

$$\begin{aligned} y(x) &= e^{-h(x)}\int e^{h(x)}q(x)dx + Ce^{-h(x)} \\ &= x^2\int x^{-2}x^2\cos(x)\,dx + Cx^2 \\ &= x^2\sin(x) + Cx^2 \end{aligned}$$

Examples of Solving Linear Equations

3. $xy' + 3y = \dfrac{\ln(x)}{x}$

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1. We write the ODE as: $y'(x) + \frac{3y}{x} = \frac{\ln(x)}{x^2}$ *and we identify*

$$p(x) = \frac{3}{x}, \quad q(x) = \frac{\ln(x)}{x^2}$$

2. We compute the integrating factor

$$h(x) = \int p(x)dx = 3\int \frac{1}{x}dx = 3\ln x = \ln x^3$$

3. We write the general solution

$$\begin{aligned} y(x) &= e^{-h(x)}\int e^{h(x)}q(x)dx + Ce^{-h(x)} \\ &= x^{-3}\int x^3x^{-2}\ln(x)\,dx + Cx^{-3} \\ &= x^{-3}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + Cx^{-3} = \frac{1}{2x}\ln x - \frac{1}{4x} + Cx^{-3} \end{aligned}$$

Linearity

Definition

An operation L is *linear* if the operation satisfies the following two properties.

- $L[y_1 + y_2] = L[y_1] + L[y_2]$
- $L[cy] = cL[y]$, c is a constant

What are some linear operations we have seen in calculus?

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What are some linear operations we have seen in calculus?

Differentiation, Integration

Linearity

Consider the linear differential equation

$$y' + p(x)y = q(x).$$

Show that $L[y] = y' + p(x)y$ can be viewed as a linear operation on y.

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$$y' + p(x)y = q(x).$$

Show that $L[y] = y' + p(x)y$ can be viewed as a linear operation on y.

We need to show that
(1) $L[y_1 + y_2] = L[y_1] + L[y_2]$ *and (2)* $L[cy] = cL[y]$

Part (1):

$$\begin{aligned} L[y_1 + y_2] &= (y_1 + y_2)' + p(x)(y_1 + y_2) \\ &= y_1' + y_2' + p(x)y_1 + p(x)y_1 \\ &= (y_1' + p(x)y_1) + (y_2' + p(x)y_2) \\ &= L[y_1] + L[y_2] \end{aligned}$$

Part (2): $L[cy] = cy' + p(x)cy = c(y' + p(x)y) = cL[y]$

Linearity

Definition

The fact that the operation $L[y] = y' + p(x)y$ is a linear operation is the reason for calling $y' + p(x)y = q(x)$ a linear differential equation. Also, in this context, L is called a *linear differential operator*.

Linearity

Example: Given the ODE $y' + 2xy = 4x$. Set $L[y] = y' + 2xy$. Find:

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$$L[\cos(2x)] = (\cos(2x))' + 2x\cos(2x) = -2\sin(2x) + 2x\cos(2x)$$

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2. $L[e^{-x^2}]$

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2. $L[e^{-x^2}]$

$$L[e^{-x^2}] = (e^{-x^2})' + 2xe^{-x^2} = -2xe^{-x^2} + 2xe^{-x^2} = 0$$