Math 3321

Separable Differential Equations

University of Houston

Lecture 04

Outline

Definition

A first order differential equation $y' = f(x, y)$ is a *separable equation* if the function f can be seen as the product of a function of x and a function of y. This means we can factor f to write

$$f(x, y) = p(x)h(y),$$

where p and h are continuous on some domain in the xy-plane.

Separable Equations

The solution method will be based on writing

$$y' = p(x)h(y)$$

as

$$\frac{1}{h(y)}y' = p(x).$$

Letting $q(y) = \dfrac{1}{h(y)}$, we can write our equation as

$$q(y)y' = p(x). \tag{1}$$

Separable Equations

The solution method will be based on writing

$$y' = p(x)h(y)$$

as

$$\frac{1}{h(y)}y' = p(x).$$

Letting $q(y) = \dfrac{1}{h(y)}$, we can write our equation as

$$q(y)y' = p(x). \tag{1}$$

Possible Singular Solutions

Anytime we divide by a function, we must be careful about the possiblilty of division by 0. In this case, we must be sure that $h(y) \neq 0$. If r is a real number such that $h(r) = 0$, we **may** have $y \equiv r$ as a singular solution to the differential equation.

Separable Equations

When we write

$$y' = \frac{dy}{dx},$$

which we are interpretting as “differential y” divided by “differential x”, we can write

$$q(y)y' = p(x)$$

as

$$q(y)\frac{dy}{dx} = p(x).$$

Multiplying both sides by dx gives us the equation

$$q(y)dy = p(x)dx.$$

This is the inspiration for calling these *separable* differential equations.

Solution Method for Separable Equations

Solving $y' = p(x)h(y)$:

Our steps are as follows:

1. Identify: Can we write the given equation in the form (1) If yes, do so. Emphasizing that $y = y(x)$, equation (1) is

$$q(y(x))y'(x) = p(x).$$

Solution Method for Separable Equations

Solving $y' = p(x)h(y)$:

2. Integrate with respect to x:

$$\int q(y(x))y'(x)dx = \int p(x)dx + C$$

which can also be written

$$\int q(y)dy = \int p(x)dx + C$$

by setting $y = y(x)$ and $dy = y'(x)dx$. Now, assume P to be an antiderivative for p and Q an antiderivative of q, then we have

$$Q(y) = P(x) + C. \tag{2}$$

Solution Method for Separable Equations

Definition

Equation (2) defines a one-parameter family of curves called the *integral curves* of equation (1). In general, equation (2) defines y implicitly as a function of x. Such a family gives the general solution of (1). However, when Q is invertible, it is preferable to solve for y as a function of x explicitly.

Examples of Solving Separable Equations

1. $y' = \dfrac{x-5}{y^2}$

1. $y' = \dfrac{x-5}{y^2}$

$$\begin{aligned} y^2 y' &= (x-5) \\ \int y^2\,dy &= \int (x-5)\,dx \\ \frac{1}{3}y^3 &= \frac{1}{2}x^2 - 5x + C \\ y &= \sqrt[3]{\tfrac{3}{2}x^2 - 15x + C} \end{aligned}$$

Examples of Solving Separable Equations

2. $y' = \dfrac{y-1}{x+3}$

Examples of Solving Separable Equations

2. $y' = \dfrac{y-1}{x+3}$

We set aside the solution $y = 1$. This may or may not be a singular solution. Under the assumption that $y \neq 1$, we have

$$
\begin{aligned}
\tfrac{y'}{y-1} &= \tfrac{1}{x+3} \\
\int \tfrac{1}{y-1} dy &= \int \tfrac{1}{x+3}\, dx \\
\ln|y-1| &= \ln|x+3| + C \\
|y-1| &= e^{C}(x+3) \\
y-1 &= = \tilde{C}(x+3) \\
y &= \tilde{C}(x+3) + 1
\end{aligned}
$$

Note that $y = 1$ is a particular solution not a singular one.

Examples of Solving Separable Equations

3. $y' = \dfrac{x+3}{y-1}, \; y(-1) = 4$

3. $y' = \dfrac{x+3}{y-1}$, $y(-1) = 4$

We first find the general solution of $y' = \frac{x+3}{y-1}$

$$\begin{aligned} (y-1)y' &= x+3 \\ \int (y-1)dy &= \int (x+3)\,dx \\ \tfrac{1}{2}y^2 - y &= \tfrac{1}{2}x^2 + 3x + C \end{aligned}$$

Note that y *is only defined implicitly.*
To find the particular solution, we set $x = -1, y = 4$ *into the equation.*

$$\tfrac{1}{2}16 - 4 = \tfrac{1}{2} - 3 + C, \quad \Rightarrow C = \tfrac{13}{2}$$

IVP Solution: $\frac{1}{2}y^2 - y = \frac{1}{2}x^2 + 3x + \frac{13}{2}$

Examples of Solving Separable Equations

4. $y' = xy - ye^x$

Examples of Solving Separable Equations

4. $y' = xy - ye^x$

We first do variable separation on

$$y' = xy - ye^x = y(x - e^x)$$

Under the assumption that $y \neq 0$, *we write*

$$\begin{aligned}
\frac{y'}{y} &= x - e^x \\
\int \frac{1}{y} dy &= \int (x - e^x)\, dx \\
\ln |y| &= \frac{x}{2} - e^x + C \\
|y| &= e^{(\frac{x}{2} - e^x + C)} \\
y &= \tilde{C}\, e^{(\frac{x}{2} - e^x)}
\end{aligned}$$

Examples of Solving Separable Equations

5. $y' = \dfrac{xy^2 - 4x}{x^2 + 4}$

Examples of Solving Separable Equations

5. $y' = \dfrac{xy^2 - 4x}{x^2 + 4}$

We first do variable separation on

$$y' = \frac{xy^2 - 4x}{x^2 + 4} = \frac{x(y^2 - 4)}{x^2 + 4}$$

Under the assumption that $y \neq \pm 2$, we write

$$\begin{aligned}
\tfrac{y'}{y^2-4} &= \tfrac{x}{x^2+4} \\
\int \tfrac{1}{y^2-4} dy &= \int \tfrac{x}{x^2+4}\, dx \\
\int (\tfrac{1/4}{y-2} - \tfrac{1/4}{y+2}) dy &= \tfrac{1}{2} \ln(x^2 + 4) + C \\
\tfrac{1}{4} \ln |\tfrac{y-2}{y+2}| &= \tfrac{1}{2} \ln(x^2 + 4) + C
\end{aligned}$$

We next exponentiate both sides

Examples of Solving Separable Equations

$$
\begin{aligned}
\tfrac{1}{4}\ln\left|\tfrac{y-2}{y+2}\right| &= \tfrac{1}{2}\ln(x^2+4)+C \\
\left|\tfrac{y-2}{y+2}\right|^{1/4} &= e^C(x^2+4)^{1/2} \\
\left|\tfrac{y-2}{y+2}\right| &= e^{4C}\,(x^2+4)^2 \\
\tfrac{y-2}{y+2} &= K\,(x^2+4)^2 \\
y-2 &= (y+2)K\,(x^2+4)^2 = yK(x^2+4)^2+2K(x^2+4)^2 \\
y-yK(x^2+4)^2 &= 2+2K(x^2+4)^2 \\
y &= \tfrac{2+2K(x^2+4)^2}{1-K(x^2+4)^2}
\end{aligned}
$$

Note that $y=2$ is a particular solution, not a singular one; $y=-2$ is a singular solution.