Math 3321

Applications of First Order Equations

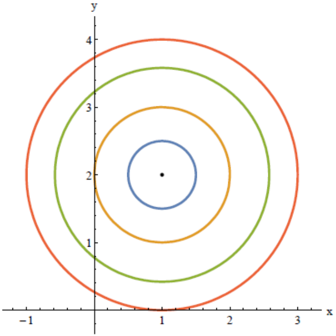
University of Houston

Lecture 06

Outline

Orthogonal Trajectories

The family of circles $(x-1)^2 + (y-2)^2 = C$ is the general solution for an ODE. Find this ODE.



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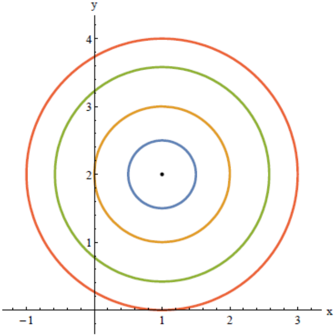

We can differentiate this equation with respect to x...

Orthogonal Trajectories

Differentiating $(x-1)^2 + (y-2)^2 = C$
we get

$$2(x-1) + 2(y-2)y' = 0$$

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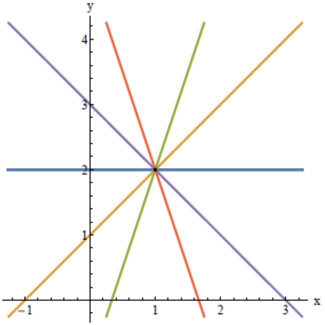
$$2(x-1) + 2(y-2)y' = 0$$

Hence

$$y' = -\frac{2(x-1)}{2(y-2)} = -\frac{x-1}{y-2}$$

Orthogonal Trajectories

The family of lines $y - 2 = K(x - 1)$ is the general solution for an ODE. Find this ODE.



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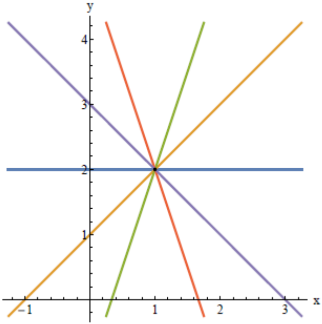


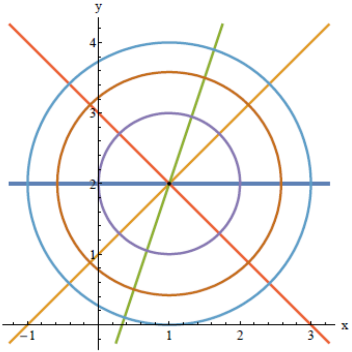
Claim: $y' = \frac{y-2}{x-1}$

In fact, the lines are the solution of the separable ODE:

$$\frac{y'}{y-2} = \frac{1}{x-1}$$

Orthogonal Trajectories

Show that the circles and lines are orthogonal (perpendicular).



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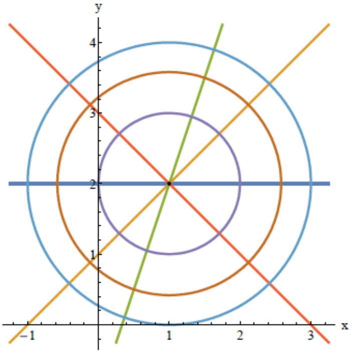

If $P(x_0, y_0)$ is a point of intersection of one of the circles and one of the lines, their slopes are the **negative reciprocal** *of each other.*

Orthogonal Trajectories

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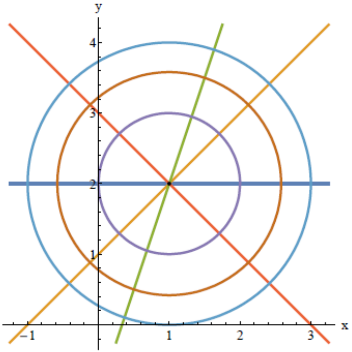

If $P(x_0, y_0)$ is a point of intersection of one of the circles and one of the lines, their slopes are the **negative reciprocal** *of each other. This means the tangent lines are perpendicular:*
$\tan(\theta + \pi/2) = -\cot(\theta)$.

Orthogonal Trajectories

Definitions

A curve which intersects each member of a given family of curves at right angles (orthogonally) is called an *orthogonal trajectory* of the family.

In general, when we have two one-parameter families of curves

$$F(x, y, C) = 0 \text{ and } G(x, y, K) = 0$$

such that each member of one family is an orthogonal trajectory of the other family, then the two families are said to be *orthogonal trajectories*.

Orthogonal Trajectories

Procedure for finding orthogonal trajectories:

Our steps are as follows:

1. Starting with the family $F(x, y, C) = 0$, find the differential equation for this family.
2. Replace y' in this equation with $-\frac{1}{y'}$. Now solve for y' to find the differential equation for the family of orthogonal trajectories.
3. Find the general solution for this new differential equation. This is the family of orthogonal trajectories.

Orthogonal Trajectories

Example:

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We first calculate the differential equation for the family:

$$y' = 2K(x-1)$$

Hence:

$$K = \frac{y'}{2(x-1)}$$

We will next substitute this expression of K into the family of parabolas.

Orthogonal Trajectories

By substituting the expression of K into the family of parabolas:

$$(y-3) = \frac{y'}{2(x-1)}(x+1)^2$$

which simplifies to

$$2(y-3) = y'(x+1)$$

Therefore, the differential equation for the family of parabolas is

$$y' = \frac{2(y-3)}{(x+1)}$$

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To obtain the differential equation for the family of orthogonal trajectories, we will take the **negative reciprocal** *of this equation.*

Orthogonal Trajectories

By taking the **negative reciprocal** *of the last equation, we obtain (I apologize for the abuse of notation, I use the same symbol to denote the reciprocal)*

$$y' = -\frac{(x+1)}{2(y-3)}$$

We solve the separable ODE

$$2(y-3)y' = -(x-1) \;\Rightarrow\; \int 2(y-3)dy = -\int (x+1)dx$$

The solution is

$$(y-3)^2 = -\frac{1}{2}(x+1)^2 + C$$

or

$$\frac{1}{2}(x+1)^2 + (y-3)^2 = C$$

The orthogonal trajectories are ellipses with center at the point $(-1, 3)$.

Radioactive Decay

It is well known that the rate of decay of a radioactive material at time t is proportional to the amount of material present at time t. Letting $A = A(t)$ be the amount at time t, we can express this relationship mathematically as

$$\frac{dA}{dt} = kA$$

where k, the proportionality constant, is negative.

This differential equation can be viewed as either separable or linear. Solving this equation gives

$$A(t) = Ce^{kt}.$$

If $A_0 = A(0)$ is the amount at time 0, then $C = A_0$ and our solution is

$$A(t) = A_0 e^{kt}.$$

Radioactive Decay

Example: A certain radioactive material is decaying at a rate proportional to the amount present. If a sample of 50 grams of the material was present initially and after 2 hours the sample lost 10% of its mass, find:

1. An expression for the mass of the material remaining at any time t.

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Let $A(t)$ denote the amount of material at time t. As we have $A(0) = 50$ so radioactive decay equation is

$$A(t) = A(0)e^{-rt} = 50e^{-rt}$$

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Next we use the information that the material lost 10% of its mass (= 5 grams) in 2 hours.
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$$A(2) = 50e^{-2r} = 45 \Rightarrow e^{-2r} = \frac{45}{50} = 0.9$$

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Alternatively

$$-2r = \ln(0.9) = \ln(9/10) \Rightarrow r = -\tfrac{1}{2}\ln(9/10)$$

Thus we have

$$A(t) = 50e^{-rt} = 50e^{-\frac{t}{2}\ln(9/10)} = 50\left(\tfrac{9}{10}\right)^{t/2}$$

Radioactive Decay

2. The mass of the material after 4 hours.

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We use the expression

$$A(t) = 50e^{-rt} = 50e^{-\frac{t}{2}\ln(9/10)} = 50\left(\frac{9}{10}\right)^{t/2}$$

with $t = 4$*:*

$$A(4) = 50\left(\frac{9}{10}\right)^{2} = 40.5$$

Radioactive Decay

3. How long will it take for 75% of the material to decay?

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If the material has lost 75% of its mass, then 25% (=12.5 grams) remains.

Need to solve the following equation for t

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Hence

$$t = \frac{2\ln(1/4)}{\ln(9/10)} = 26.3153$$

4. The half-life of the material.

The half-life T is given by equation

$$A(0)e^{-rT} = \frac{A(0)}{2}$$

Hence

$$e^{-rT} = \frac{1}{2} \Rightarrow T = \frac{\ln 2}{r}$$

Radioactive Decay

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The half-life of the material is

$$T = \frac{\ln 2}{r} = \frac{\ln 2}{0.0527} = 13.1527 \ hours$$

Exponential Growth

Under ideal conditions, a population increases at a rate proportional to the current size of the population. Letting $P = P(t)$ be the population at time t, we can express this relationship mathematically as

$$\frac{dP}{dt} = kP$$

where k, the proportionality constant, is positive.

As in the case of radioactive decay, the solution can be expressed

$$P(t) = P_0 e^{kt}.$$

Note that continuously compounded interest can be modeled in the same way.

Exponential Growth

Example: In 1980 the world population was approximately 4.5 billion and in the year 2000 it was approximately 6 billion. Assume that the population increases at a rate proportional to the size of population.

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Let P(t) denote the world population at time t. Since $P(1980) = 4.5$ *billion and* $P(2000) = 6$ *we have*

$$P(t) = P(1980)e^{k(t-1980)} = 4.5e^{k(t-1980)}$$

$$P(2000) = 4.5e^{k20} = 6$$

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Thus

$$e^{k20} = \tfrac{4}{3} \Rightarrow k = \tfrac{\ln(4/3)}{20} = 0.0144$$

and

$$P(t) = 4.5\, e^{\frac{\ln(4/3)}{20}(t-1980)} = 4.5\, \left(\tfrac{4}{3}\right)^{\frac{t-1980}{20}}$$

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The doubling time is

$$T = \frac{\ln 2}{k} = \frac{\ln 2}{0.0144} = 48.135$$

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$$P(2002) = 4.5\ \left(\frac{4}{3}\right)^{\frac{2002-1980}{20}} = 4.5\ \left(\frac{4}{3}\right)^{\frac{22}{20}} = 6.175$$

Exponential Growth

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We want to find the time t_E such that $P(t_E) = 30$.

$$P(t_E) = 4.5\ \left(\tfrac{4}{3}\right)^{\frac{t_E - 1980}{20}} = 30$$

Hence

$$\left(\tfrac{4}{3}\right)^{\frac{t_E - 1980}{20}} = \frac{30}{4.5} = \frac{20}{3}$$

Hence

$$t_E = 1980 + 20\frac{\ln(\frac{20}{3})}{\ln(\frac{4}{3})} = 2111.9$$

Newton's Law of Cooling/Heating

The rate of change of the temperature of an object at time t is proportional to the difference between the temperature of the object $u = u(t)$ and the (constant) temperature σ of the surrounding medium (e.g., air or water), called the *ambient temperature*.

$$\frac{du}{dt} = -k(u - \sigma), \; k > 0 \text{ constant.}$$

Newton's Law of Cooling/Heating

Mathematical Model:

The differential equation for the law of cooling or heating is given by the differential equation

$$\frac{du}{dt} = -k(u - \sigma),\ k > 0 \text{ constant}.$$

Letting $u(0) = u_0$ be the initial temperature we get the solution

$$u(t) = \sigma + [u_0 - \sigma]e^{-kt}.$$

Newton's Law of Cooling/Heating

Example: Suppose that a corpse is discovered at 10 p.m. and its temperature is determined to be 85° F. Two hours later, its temperature is 74° F. If the ambient temperature is 68° F, estimate the time of death.

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By Newton's Law of Cooling,

$$u(t) = 68 + [u(t_0) - 68]e^{-k(t-t_0)}$$

The two conditions imply that

$$u(10) = 68 + [u(t_0) - 68]e^{-k(10-t_0)} = 85$$

$$u(12) = 68 + [u(t_0) - 68]e^{-k(12-t_0)} = 74$$

Newton's Law of Cooling/Heating

Equivalently, we can write the two conditions as

$$[u(t_0) - 68]e^{-k(10-t_0)} = 85 - 68 = 17$$

$$[u(t_0) - 68]e^{-k(12-t_0)} = 74 - 68 = 6$$

By taking the ratio

$$e^{k(12-10)} = \tfrac{17}{6} \Rightarrow e^{2k} = \tfrac{17}{6}$$

Hence

$$k = \tfrac{1}{2}\ln(\tfrac{17}{6}) = 0.521$$

Now we can use the fact that $u(t_0) = 98.6$ *to find the time of death* t_0.

Using $k = 0.521$*, we use the equation*

$$[98.6 - 68]e^{-k(10-t_0)} = 17$$

to find an expression for t_0

$$t_0 - 10 = \frac{1}{k}\ln\left(\tfrac{17}{98.6-68}\right) = -1.128$$

The time of death was $t_0 = 10 - 1.128 = 8.872$

Example: A disease is infecting a colony of 1000 penguins living on a remote island. Let $P(t)$ be the number of sick penguins t days after the outbreak. Suppose that 50 penguins had the disease initially, 200 are sick after two days, and suppose that the disease is spreading at a rate proportional to the product of the time elapsed and the number of penguins who do not have the disease.

1. Give the mathematical model (IVP) for P.

Other Models

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The rate of change in population of sick penguins, denoted as $\frac{dP}{dt}$ *is proportional to the time* t *and the the number of penguins who do not have the disease, that is* $(1000 - P)$.

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1. Give the mathematical model (IVP) for P.

The rate of change in population of sick penguins, denoted as $\frac{dP}{dt}$ is proportional to the time t and the the number of penguins who do not have the disease, that is $(1000 - P)$.

Thus we can model the population as

$$\frac{dP}{dt} = kt(1000 - P)$$

We have an initial condition: $P(0) = 50$. Thus the IVP is

$$\frac{dP}{dt} = kt(1000 - P), \quad P(0) = 50$$

Other Models

2. Find the general solution of the differential equation in part (1).

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To find the general solution of $P' = kt(1000 - P)$*, we separate the equation*

$$\begin{aligned}
\tfrac{1}{1000-P}\, dP &= kt\, dt \\
-\ln(1000-P) &= \tfrac{1}{2}kt^2 + C \\
\ln(1000-P) &= -\tfrac{1}{2}kt^2 + C \\
|1000-P| &= e^C\, e^{-\frac{1}{2}kt^2} \\
1000 - P &= Ke^{-\frac{1}{2}kt^2} \\
P(t) &= 1000 - Ke^{-\frac{1}{2}kt^2}
\end{aligned}$$

Other Models

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Using the IVP, we observe that

$$P(0) = 1000 - K = 50$$

3. Find the particular solution that satisfies the initial condition.

We found the general solution

$$P(t) = 1000 - Ke^{-\frac{1}{2}kt^2}$$

Using the IVP, we observe that

$$P(0) = 1000 - K = 50$$

Thus $K = 950$ *and the IVP solution is*

$$P(t) = 1000 - 950\,e^{-\frac{1}{2}kt^2}$$

Other Models

We can use the condition $P(2) = 200$ to find the value of k. Using the expression into the IVP solution

$$P(t) = 1000 - 950e^{-\frac{1}{2}kt^2}$$

Hence at $t = 2$ we get

$$P(2) = 1000 - 950e^{-\frac{1}{2}k2^2} = 1000 - 950e^{-2k} = 200$$

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Hence at $t = 2$ we get

$$P(2) = 1000 - 950e^{-\frac{1}{2}k2^2} = 1000 - 950e^{-2k} = 200$$

Thus

$$950e^{-2k} = 800 \Rightarrow e^{-2k} = \frac{80}{95}$$

Thus

$$k = -\tfrac{1}{2}\ln(\tfrac{80}{95}) = 0.086$$