Math 3321

Nonhomogeneous Equations with Constant Coefficients (Undetermined Coefficients)

University of Houston

Lecture 10

Outline

Introduction

As we have seen, solving a linear nonhomogeneous equation depends, in part, on finding a particular solution of the equation. In the last lecture, we leaned the method of variation of parameters, which allowed us to create a particular solution for the nonhomogeneous equation using two linearly independent solutions for the corresponding reduced equation. In this lecture we will learn a second method.

Introduction

As we have seen, solving a linear nonhomogeneous equation depends, in part, on finding a particular solution of the equation. In the last lecture, we leaned the method of variation of parameters, which allowed us to create a particular solution for the nonhomogeneous equation using two linearly independent solutions for the corresponding reduced equation. In this lecture we will learn a second method.

Remark: Limitations of the method

Unlike variation of parameters, which can be applied to any nonhomogeneous equation, the *method of undetermined coefficients* can be applied only to nonhomogeneous equations of the form

$$y'' + ay' + by = f(x) \tag{1}$$

where a and b are constants and the nonhomogeneous term f is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions.

Introduction

Throughout this lecture, we will emphasize the fact that the left-hand side of equation (1) can be viewed as a linear operator L applied to the function y. That is

$$L[y] = y'' + ay' + by$$

so that the differential equation is

$$L[y] = f(x).$$

Nonhomogeneous Term: Exponential

We will first look at equations of the form

$$y'' + ay' + by = ce^{rx}.$$

We will **guess that a solution** function might take the form $z = Ae^{rx}$ where A is some constant.

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We will **guess that a solution** function might take the form $z = Ae^{rx}$ where A is some constant.
Observe:

$$z = Ae^{rx} \implies z' = Are^{rx} \text{ and } z'' = Ar^2e^{rx}.$$

Then we find

$$\begin{aligned} L[z] =& z'' + az' + bz \\ =& Ar^2e^{rx} + aAre^{rx} + bAe^{rx} \\ =& (Ar^2 + aAr + bA)e^{rx}. \end{aligned}$$

This means $L[z] = Ke^{rz}$ where $K = Ar^2 + aAr + b$.

Nonhomogeneous Term: Exponential

Examples: Find a particular solution for the given equation.

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We will guess that a solution might take the form $z = Ae^{-4x}$.
Observe:

$$z = Ae^{-4x}, \quad z' = -4Ae^{-4x}, \quad z'' = 16Ae^{-4x}$$

Hence, by substitution,

$$16Ae^{-4x} - 5(-4Ae^{-4x}) + 6Ae^{-4x} = 7e^{-4x}$$

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Hence, by substitution,

$$16Ae^{-4x} - 5(-4Ae^{-4x}) + 6Ae^{-4x} = 7e^{-4x}$$

We obtain

$$(16A + 20A + 6A)e^{-4x} = 7e^{-4x}$$

Thus

$$42A = 7 \quad \rightarrow \quad A = \frac{1}{6}$$

A particular solution is $z = \frac{1}{6}e^{-4x}$

Nonhomogeneous Term: Exponential

2. $y'' - 2y' + y = 5e^{-x} + 3e^{2x} + 2$

Nonhomogeneous Term: Exponential

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According to the Superposition Principle, we can treat each term individually. For the $4e^{-x}$ term we set $z_1 = Ae^{-x}$, for the $3e^{2x}$ term, we set $z_2 = Be^{2x}$, and for $2 = 2e^{0x}$, we set $z_3 = Ce^{0x} = C$.

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Thus, we look for a particular solution of the equation with the form

$$z = Ae^{-x} + Be^{2x} + C$$

We compute

$$z' = -Ae^{-x} + 2Be^{2x}, \qquad z'' = Ae^{-x} + 4Be^{2x}$$

Nonhomogeneous Term: Exponential

Hence, substituting into the differential equation we obtain

$Ae^{-x} + 4Be^{2x} - 2(-Ae^{-x} + 2Be^{2x}) + Ae^{-x} + Be^{2x} + C = 5e^{-x} + 3e^{2x} + 2$

which simplifies to

$$4Ae^{-x} + Be^{2x} + C = 5e^{-x} + 3e^{2x} + 2$$

This implies

$$A = 5/4, \quad B = 3, \quad C = 2$$

Thus

$$z = \frac{5}{4}e^{-x} + 3e^{2x} + 2$$

is a particular solution of the differential equation.

Nonhomogeneous Term: Sine or Cosine

We will next look at equations of the form

$$y'' + ay' + by = c\cos(\beta x) + d\sin(\beta x).$$

We will guess that a solution function might take the form $z = A\cos(\beta x)$ where A is some constant. Observe:

$$z = A\cos(\beta x) \implies z' = -\beta A\sin(\beta x) \text{ and } z'' = -\beta^2 A\cos(\beta x).$$

Then we find

$$L[z] = z'' + az' + bz = (-\beta^2 A + bA)\cos(\beta x) + (-a\beta A)\sin(\beta x).$$

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Then we find

$$L[z] = z'' + az' + bz = (-\beta^2 A + bA)\cos(\beta x) + (-a\beta A)\sin(\beta x).$$

This means $L[z] = K\cos(\beta x) + M\sin(\beta x)$ where K and M depend on a, b, β, and A. If we were to use $z = B\sin(\beta x)$ instead, we would find a similar result.

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This means $L[z] = K\cos(\beta x) + M\sin(\beta x)$ where K and M depend on a, b, β, and A. If we were to use $z = B\sin(\beta x)$ instead, we would find a similar result.

This leads us to the use

$$z = A\cos(\beta x) + B\sin(\beta x)$$

as our trial solution which will also give $L[z] = K\cos(\beta x) + M\sin(\beta x)$ for some K and M which will depend on a, b, β, and A.

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We look for a particular solution of the equation with the form

$$z = A\cos(2x) + B\sin(2x)$$

We compute

$$z' = -2A\sin(2x) + 2B\cos(2x),$$

$$z'' = -4A\cos(2x) - 4B\sin(2x)$$

Nonhomogeneous Term: Sine or Cosine

Substituting into the differential equation we obtain

$$-4A\cos(2x) - 4B\sin(2x) - 2(-2A\sin(2x) + 2B\cos(2x))$$
$$+A\cos(2x) + B\sin(2x) = 3\cos(2x)$$

This simplifies to

$$(-3A - 4B)\cos(2x) + (4A - 3B)\sin(2x) = 3\cos(2x)$$

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$$+A\cos(2x) + B\sin(2x) = 3\cos(2x)$$

This simplifies to

$$(-3A - 4B)\cos(2x) + (4A - 3B)\sin(2x) = 3\cos(2x)$$

Hence we find

$$-3A - 4B = 3, 4A - 3B = 0, \quad \Rightarrow \quad A = -36/100, B = -12/25$$

Thus

$$z = -\frac{36}{100}\cos(2x) - \frac{12}{25}\sin(2x)$$

is a particular solution of the differential equation.

Nonhomogeneous Term: Sine or Cosine

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According to the Superposition Principle, we can treat each term individually. For the $2\cos(3x) - 4\sin(3x)$ *term we set* $z_1 = A\cos(3x) + B\sin(3x)$, *for the* e^{2x} *term, we set* $z_2 = Ce^{2x}$.

Thus, we look for a particular solution of the equation with the form

$$z = A\cos(3x) + B\sin(3x) + Ce^{2x}$$

We compute

$$z' = -3A\sin(3x) + 3B\cos(3x) + 2Ce^{2x},$$

$$z'' = -9A\cos(3x) - 9B\sin(3x) + 4Ce^{2x},$$

Nonhomogeneous Term: Sine or Cosine

Hence substituting into the differential equation we obtain

$$-9A\cos(3x) - 9B\sin(3x) + 4Ce^{2x} - 2(-3A\sin(3x) + 3B\cos(3x) + 2Ce^{2x})$$
$$+5(A\cos(3x) + B\sin(3x) + Ce^{2x}) = 2\cos(3x) - 4\sin(3x) + e^{2x}$$

This simplifies to

$$2A\cos(3x) - 10B\sin(3x) + 5Ce^{2x} = 2\cos(3x) - 4\sin(3x) + e^{2x}$$

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This simplifies to

$$2A\cos(3x) - 10B\sin(3x) + 5Ce^{2x} = 2\cos(3x) - 4\sin(3x) + e^{2x}$$

Hence we find

$$A = 1, \quad B = 2/5, \quad C = 1/5$$

Thus

$$z = \cos(3x) + \tfrac{2}{5}\sin(3x) + \tfrac{1}{5}e^{2x}$$

is a particular solution of the differential equation.

Nonhomogeneous Term: Product of Exponential and Sine/Cosine

The next case is equations of the form

$$y'' + ay' + by = ce^{rx}\cos(\beta x) + de^{rx}\sin(\beta x).$$

We will guess that a solution function might take the form

$$z = Ae^{rx}\cos(\beta x) + Be^{rx}\sin(\beta x)$$

where A and B are some constants.

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Example: Find a particular solution for the given equation.

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We look for a particular solution of the equation with the form

$$z = Ae^x \cos(2x) + Be^x \sin(2x)$$

We compute

$$\begin{aligned} z' &= (A\cos(2x) - 2A\sin(2x) + B\sin(2x) + 2B\cos(2x))e^x, \\ &= (A + 2B)\cos(2x)e^x + (B - 2A)\sin(2x)e^x \end{aligned}$$

$$\begin{aligned} z'' &= (A + 2B)\cos(2x)e^x - 2(A + 2B)\sin(2x)e^x + (B - 2A)\sin(2x)e^x \\ &+ 2(B - 2A)\cos(2x)e^x \\ &= (4B - 3A)\cos(2x)e^x + (-3B - 4A)\sin(2x)e^x \end{aligned}$$

Nonhomogeneous Term: Product of Exponential and Sine/Cosine

Hence substituting into the differential equation we obtain

$$\begin{aligned}&(4B-3A)\cos(2x)e^x+(-3B-4A)\sin(2x)e^x\\&+9(Ae^x\cos(2x)+Be^x\sin(2x))\\&=4e^x\sin(2x)\end{aligned}$$

which simplifies to

$$(4B+6A)\cos(2x)e^x+(6B-4A)\sin(2x)e^x=4e^x\sin(2x)$$

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which simplifies to

$$(4B+6A)\cos(2x)e^x+(6B-4A)\sin(2x)e^x=4e^x\sin(2x)$$

This implies

$$(4B+6A)=0,(6B-4A)=4 \quad \Rightarrow \quad A=-4/13, B=6/13$$

Thus

$$z=-\tfrac{4}{13}e^x\cos(2x)+\tfrac{6}{13}e^x\sin(2x)$$

is a particular solution of the differential equation.

Complications

The following table summarizes our findings thus far.

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	set $z(x) =$
ce^{rx}	Ae^{rx}
$c \cos \beta x + d \sin \beta x$	$z(x) = A \cos \beta x + B \sin \beta x$
$ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x$	$z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$

Note: The first line includes the case $r = 0$;
if $f(x) = c = ce^{0x}$, then $z = Ae^{0x} = A$.

We do have a complicating factor. This happens when our guess at z, formed by looking at the nonhomogeneous term f, is a solution to the reduced equation.

Complications

The next example will illustrate the issue.

1. Find a particular solution for $y'' + y' - 6y = 3e^{2x}$.

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We solve the homogeneous equation first. The characteristic equation is

$$r^2 + r - 6 = 0$$

with characteristic roots $r = 2, -3$.

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We solve the homogeneous equation first. The characteristic equation is

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with characteristic roots $r = 2, -3$. *Hence the solution of the homogeneous problem is*

$$y_h(x) = c_1 e^{-3x} + c_2 e^{2x}$$

This shows that $z = Ae^{2x}$ *cannot be a particular solution of the non-homogeneous problem.*

Complications

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We have

$$z' = Ae^{2x} + 2Axe^{2x}$$

$$z'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x}$$

Substitution into the equation gives

$$4Ae^{2x} + 4Axe^{2x} + Ae^{2x} + 2Axe^{2x} - 6Axe^{2x} = 3e^{2x}$$

which simplifies to

$$5Ae^{2x} = 3e^{2x} \quad \Rightarrow \quad A = 3/5$$

This shows that $z = \frac{3}{5}xe^{2x}$ *is a particular solution of the non-homogeneous problem*

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We solve the homogeneous equation first. The characteristic equation is

$$r^2 - 6r + 9 = (r-3)^2 = 0$$

with repeated characteristic root $r = 3$.

This shows that $z = (A + Bx)e^{3x}$ cannot be a particular solution of the differential equation

So we will look for a solution of the form $z = Ax^2e^{3x}$

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This shows that $z = (A + Bx)e^{3x}$ *cannot be a particular solution of the differential equation*

So we will look for a solution of the form $z = Ax^2e^{3x}$

We have

$$z' = 2Axe^{3x} + 3Ax^2e^{3x}, \quad z'' = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}$$

Complications

Substitution into the equation gives

$$2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x} - 6(2Axe^{3x} + 3Ax^2e^{3x}) + 9Ax^2e^{3x} = 4e^{3x}$$

This simplifies to

$$2Ae^{3x} = 4e^{3x} \quad \Rightarrow \quad A = 2$$

This shows that

$$z = 2x^2e^{3x}$$

is a particular solution of the non-homogeneous problem

Complications

When we inlcude this information, our summary becomes:

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	set $z(x) =$*
ce^{rx}	Ae^{rx}
$c \cos \beta x + d \sin \beta x$	$z(x) = A \cos \beta x + B \sin \beta x$
$ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x$	$z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$

* If z satisfies the reduced equation, use xz; if xz also satisfies the reduced equation, then x^2z will give a particular solution

For this reason, it is good practice to first solve the reduced equation before searching for a function z which solves the nonhomogeneous equation.

Complications

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$$y'' - 4y' + 4y = 0$$

The characteristic equation is $r^2 - 4r + 4 = (r-2)^2 = 0$, *which has repeated roots* $r = 2$.

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This implies that $z = Ae^{2x}$ *and* $z = Axe^{2x}$ *are solution of the reduced equation.*

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The first step is to solve the reduced equation

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The characteristic equation is $r^2 - 4r + 4 = (r-2)^2 = 0$*, which has repeated roots* $r = 2$*.*
This implies that $z = Ae^{2x}$ *and* $z = Axe^{2x}$ *are solution of the reduced equation. Thus, to find a particular solution of*

$$y'' - 4y' + 4y = 4e^{2x}$$

we need to choose a solution of the form $z_1 = Ax^2e^{2x}$

Complications

To find a particular solution of

$$y'' - 4y' + 4y = \cos(3x)$$

we need to choose a solution of the form $z_2 = B\cos(3x) + C\sin(3x)$
To find a particular solution of

$$y'' - 4y' + 4y = -5$$

we need to choose a solution of the form $z_3 = D$.

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To find a particular solution of

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we need to choose a solution of the form $z_3 = D$*.*

Hence, combining these observations, we have that the particular solution has the form

$$z = z_1 + z_2 + z_3 = Ax^2e^{2x} + B\cos(3x) + C\sin(3x) + D$$

Complications

4. Give the form of a particular solution of

$$y'' - 4y' = 4e^{4x}\sin(x) + \cos(2x) + 3.$$

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The first step is to solve the reduced equation

$$y'' - 4y' = 0$$

The characteristic equation is $r^2 - 4r = r(r-4) = 0$*, which has roots* $r = 0, r = 4$*. This implies that* $z = C_1$ *and* $z = C_2e^{4x}$ *are solution of the reduced equation.*

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$$y'' - 4y' = 4e^{4x}\sin(x) + \cos(2x) + 3.$$

The first step is to solve the reduced equation

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The characteristic equation is $r^2 - 4r = r(r-4) = 0$, *which has roots* $r = 0, r = 4$. *This implies that* $z = C_1$ *and* $z = C_2e^{4x}$ *are solution of the reduced equation.*
This implies that $z = C_1$ *is not a particular solution and this should be amended to a solution of the form* $z = C_1x$.

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The first step is to solve the reduced equation

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The characteristic equation is $r^2 - 4r = r(r-4) = 0$, which has roots $r = 0, r = 4$. This implies that $z = C_1$ and $z = C_2e^{4x}$ are solution of the reduced equation.
This implies that $z = C_1$ is not a particular solution and this should be amended to a solution of the form $z = C_1x$.

Thus the particular solution has the form

$$z = A\cos(x)e^{4x} + B\sin(x)e^{4x} + C\cos(2x) + D\sin(2x) + Ex$$

Undetermined Coefficients in General

Thus far we have found solutions for the nonhomogeneous differential equation (1) in cases where the nonhomogeneous term f is a constant multiple of one of the functions e^{rx}, $\cos(\beta x)$, $\sin(\beta x)$, $e^{rx}\cos(\beta x)$, $e^{rx}\sin(\beta x)$, or is a sum of such functions. In general, this method can be applied in cases of the form

$$f(x) = p(x)e^{rx}$$
$$f(x) = p(x)\cos(\beta x) \text{ or } p(x)\sin(\beta x)$$
$$f(x) = p(x)e^{rx}\cos(\beta x) \text{ or } p(x)e^{rx}\sin(\beta x)$$

where p is a polynomial or where f is the sum of such functions.

Undetermined Coefficients in General

Method of Undetermined Coefficients

We can summarize the general method as follows:

(1) If $f(x) = p(x)e^{rx}$ where p is a polynomial of degree n, then

$$z = (A_0 + A_1x + A_2x^2 + \cdots + A_nx^n)e^{rx}.$$

(2) If $f(x) = p_1(x)\cos(\beta x) + p_2(x)\sin(\beta x)$ where p_1 and p_2 are polynomials of degree k and m, respectively, then

$$\begin{aligned} z =& (A_0 + A_1x + \cdots + A_nx^n)\cos(\beta x) \\ &+ (B_0 + B_1x + \cdots + B_nx^n)\sin(\beta x) \end{aligned}$$

where $n = \max\{k, m\}$.

Undetermined Coefficients in General

Method of Undetermined Coefficients

(3) If $f(x) = p_1(x)e^{rx}\cos(\beta x) + p_2(x)e^{rx}\sin(\beta x)$ where p_1 and p_2 are polynomials of degree k and m, respectively, then

$$\begin{aligned} z =& (A_0 + A_1 x + \cdots + A_n x^n)e^{rx}\cos(\beta x) \\ &+ (B_0 + B_1 x + \cdots + B_n x^n)e^{rx}\sin(\beta x) \end{aligned}$$

where $n = \max\{k, m\}$.

If any term in z satisfies the reduced equation $y'' + ay' + by = 0$, then use xz as the trial solution. If xz satisfies the reduced equation, then use $x^2 z$.

Undetermined Coefficients in General

Method of Undetermined Coefficients

(3) If $f(x) = p_1(x)e^{rx}\cos(\beta x) + p_2(x)e^{rx}\sin(\beta x)$ where p_1 and p_2 are polynomials of degree k and m, respectively, then

$$\begin{aligned} z =& (A_0 + A_1 x + \cdots + A_n x^n)e^{rx}\cos(\beta x) \\ &+ (B_0 + B_1 x + \cdots + B_n x^n)e^{rx}\sin(\beta x) \end{aligned}$$

where $n = \max\{k, m\}$.

If any term in z satisfies the reduced equation $y'' + ay' + by = 0$, then use xz as the trial solution. If xz satisfies the reduced equation, then use $x^2 z$.

Undetermined Coefficients in General

Examples:

1. Find the general solution to $y'' - y' - 6y = (2x^2 - 1)e^{2x}$.

Examples:

1. Find the general solution to $y'' - y' - 6y = (2x^2 - 1)e^{2x}$.

Based on the form of the function on the right hand side, the particular solution should have the form

$$z_1 = (A_0 + A_1 x + A_2 x^2)e^{2x}$$

The reduced equation $y'' - y' - 6y = 0$ *has characteristic equation is* $r^2 - r - 6 = 0$*, which has roots* $r = -2$*,* $r = 3$*. Thus the solution of the homogeneous problem is*

$$z = C_1 e^{-2x} + C_2 e^{3x}$$

Undetermined Coefficients in General

It remains to find the particular solution. We have

$$\begin{aligned} z_1' &= (A_1 + 2A_2x)e^{2x} + 2(A_0 + A_1x + A_2x^2)e^{2x} \\ &= \left((2A_0 + A_1) + (A_1 - 2A_2)x + A_2x^2\right)e^{2x} \\ z_1'' &= 2A_2e^{2x} + 2(A_1 + 2A_2x)e^{2x} + 2(A_1 + 2A_2x)e^{2x} + 4(A_0 + A_1x + A_2x^2)e^{2x} \\ &= \left((4A_0 + 4A_1 + 2A_2) + (4A_1 + 8A_2)x + 4A_2x^2\right)e^{2x} \end{aligned}$$

Undetermined Coefficients in General

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$$\begin{aligned} z_1' &= (A_1 + 2A_2x)e^{2x} + 2(A_0 + A_1x + A_2x^2)e^{2x} \\ &= \left((2A_0 + A_1) + (A_1 - 2A_2)x + A_2x^2\right)e^{2x} \\ z_1'' &= 2A_2e^{2x} + 2(A_1 + 2A_2x)e^{2x} + 2(A_1 + 2A_2x)e^{2x} + 4(A_0 + A_1x + A_2x^2)e^{2x} \\ &= \left((4A_0 + 4A_1 + 2A_2) + (4A_1 + 8A_2)x + 4A_2x^2\right)e^{2x} \end{aligned}$$

Substitution into the differential equation (ignoring the exponential on both sides) gives

$$\begin{aligned} &\left((4A_0 + 4A_1 + 2A_2) + (4A_1 + 8A_2)x + 4A_2x^2\right) \\ &- \left((2A_0 + A_1) + (A_1 - 2A_2)x + A_2x^2\right) - 6(A_0 + A_1x + A_2x^2) \\ &= (2x^2 - 1) \end{aligned}$$

Hence we have

$$-4A_0 + 3A_1 + 2A_2 = -1, -3A_1 + 10A_2 = 0, -3A_2 = 2$$

giving $A_0 = -21/12, A_1 = -20/9, A_2 = -2/3$

Undetermined Coefficients in General

Thus the general solution is

$$y = C_1 e^{-2x} + C_2 e^{3x} + (-\tfrac{21}{12} - \tfrac{20}{9}x - \tfrac{2}{3}x^2)e^{2x}$$

Undetermined Coefficients in General

2. Give the form of a particular solution for

$$y'' + 8y' + 16y = 2x - 1 + 7x^2e^{-4x}.$$

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Based on the form of the function on the right hand side, the particular solution should have the form

$$z = A + Bx + (C + Dx + Ex^2)e^{-4x}$$

However, we need to check the solution of the reduced equation

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$$y'' + 8y' + 16y = 0$$

The characteristic equation is $r^2 + 8r + 16 = (r+4)^2 = 0$, *which has repeated roots* $r = -4$. *This implies that* $z = C_1e^{-4x}$ *and* $z = C_2xe^{-4x}$ *are solution of the reduced equation and they are* **not** *particular solutions.*

Thus, a particular solution has the form

$$z = A + Bx + Ex^2e^{-4x}$$

Undetermined Coefficients in General

3. Give the form of a particular solution for

$$y'' + y = 4x\sin(x) - 2x\cos(2x) + x^2\sin(2x) + xe^{2x}.$$

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Based on the form of the function on the right hand side, the particular solution should include the terms

$$z_1 = (A_0 + A_1x)\cos(x) + (B_0 + B_1x)\sin(x)$$

$$z_2 = (C_0 + C_1x + C_2x^2)\cos(2x) + (D_0 + D_1x + D_2x^2)\sin(2x)$$

$$z_3 = (E_0 + E_1x)e^{2x}$$

We also need to check the solution of the reduced equation

$$y'' + y = 0$$

The characteristic equation is $r^2 + 1 = 0$*, which has roots* $r = \pm i$*. This implies that* $z = \alpha\cos x + +\beta\sin x$ *is the solution of the reduced equation and it is* **not** *a particular solution.*

Undetermined Coefficients in General

Thus, a particular solution has the form

$$\begin{aligned} z &= A_1 x\cos(x) + B_1 x\sin(x) + (C_0 + C_1 x + C_2 x^2)\cos(2x) \\ &+ (D_0 + D_1 x + D_2 x^2)\sin(2x) + (E_0 + E_1 x)e^{2x} \end{aligned}$$