Math 3321
Nonhomogeneous Equations with Constant Coefficients
(Undetermined Coefficients)

University of Houston

Lecture 10
Outline

1. Introduction

2. Nonhomogeneous Term: Exponential

3. Nonhomogeneous Term: Sine or Cosine

4. Nonhomogeneous Term: Product of Exponential and Sine/Cosine

5. Complications

6. Undetermined Coefficients in General
Introduction

As we have seen, solving a linear nonhomogeneous equation depends, in part, on finding a particular solution of the equation. In the last lecture, we leaned the method of variation of parameters, which allowed us to create a particular solution for the nonhomogeneous equation using two linearly independent solutions for the corresponding reduced equation. In this lecture we will learn a second method.
Introduction

As we have seen, solving a linear nonhomogeneous equation depends, in part, on finding a particular solution of the equation. In the last lecture, we learned the method of variation of parameters, which allowed us to create a particular solution for the nonhomogeneous equation using two linearly independent solutions for the corresponding reduced equation. In this lecture we will learn a second method.

Remark: Limitations of the method

Unlike variation of parameters, which can be applied to any nonhomogeneous equation, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form

\[ y'' + ay' + by = f(x) \]  (1)

where \( a \) and \( b \) are constants and the nonhomogeneous term \( f \) is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions.
Introduction

Throughout this lecture, we will emphasize the fact that the left-hand side of equation (1) can be viewed as a linear operator $L$ applied to the function $y$. That is

$$L[y] = y'' + ay' + by$$

so that the differential equation is

$$L[y] = f(x).$$
Nonhomogeneous Term: Exponential

We will first look at equations of the form

$$y'' + ay' + by = ce^{rx}.$$ 

We will **guess that a solution** function might take the form $z = Ae^{rx}$ where $A$ is some constant.
Nonhomogeneous Term: Exponential

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\[ y'' + ay' + by = ce^{rx} . \]

We will guess that a solution function might take the form \( z = Ae^{rx} \) where \( A \) is some constant.

Observe:
\[ z = Ae^{rx} \implies z' = Ar e^{rx} \text{ and } z'' = Ar^2 e^{rx}. \]

Then we find
\[
L[z] = z'' + az' + bz \\
= Ar^2 e^{rx} + aAr e^{rx} + bA e^{rx} \\
= (Ar^2 + aAr + bA)e^{rx}.
\]

This means \( L[z] = Ke^{rz} \) where \( K = Ar^2 + aAr + b \).
Examples: Find a particular solution for the given equation.

1. \( y'' - 5y' + 6y = 7e^{-4x} \)
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We will guess that a solution might take the form \( z = Ae^{-4x} \).

Observe:

\[
    z = Ae^{-4x}, \quad z' = -4Ae^{-4x}, \quad z'' = 16Ae^{-4x}
\]

Hence, by substitution,

\[
    16Ae^{-4x} - 5(-4Ae^{-4x}) + 6Ae^{-4x} = 7e^{-4x}
\]
Examples: Find a particular solution for the given equation.

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**We will guess that a solution might take the form** \( z = Ae^{-4x} \).

**Observe:**

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\begin{align*}
z &= Ae^{-4x}, & z' &= -4Ae^{-4x}, & z'' &= 16Ae^{-4x}
\end{align*}
\]

**Hence, by substitution,**

\[
16Ae^{-4x} - 5(-4Ae^{-4x}) + 6Ae^{-4x} = 7e^{-4x}
\]

**We obtain**

\[
(16A + 20A + 6A)e^{-4x} = 7e^{-4x}
\]

**Thus**

\[
42A = 7 \quad \rightarrow \quad A = \frac{1}{6}
\]

A particular solution is \( z = \frac{1}{6}e^{-4x} \)
2. \( y'' - 2y' + y = 5e^{-x} + 3e^{2x} + 2 \)
2. $y'' - 2y' + y = 5e^{-x} + 3e^{2x} + 2$

According to the Superposition Principle, we can treat each term individually. For the $4e^{-x}$ term we set $z_1 = Ae^{-x}$, for the $3e^{2x}$ term, we set $z_2 = Be^{2x}$, and for $2 = 2e^{0x}$, we set $z_3 = Ce^{0x} = C$. 
2. $y'' - 2y' + y = 5e^{-x} + 3e^{2x} + 2$

According to the Superposition Principle, we can treat each term individually. For the $4e^{-x}$ term we set $z_1 = Ae^{-x}$, for the $3e^{2x}$ term, we set $z_2 = Be^{2x}$, and for $2 = 2e^{0x}$, we set $z_3 = Ce^{0x} = C$.

Thus, we look for a particular solution of the equation with the form

$$z = Ae^{-x} + Be^{2x} + C$$

We compute

$$z' = -Ae^{-x} + 2Be^{2x}, \quad z'' = Ae^{-x} + 4Be^{2x}$$
Hence, substituting into the differential equation we obtain
\[ Ae^{-x} + 4Be^{2x} - 2(-Ae^{-x} + 2Be^{2x}) + Ae^{-x} + Be^{2x} + C = 5e^{-x} + 3e^{2x} + 2 \]

which simplifies to
\[ 4Ae^{-x} + Be^{2x} + C = 5e^{-x} + 3e^{2x} + 2 \]

This implies
\[ A = \frac{5}{4}, \quad B = 3, \quad C = 2 \]

Thus
\[ z = \frac{5}{4}e^{-x} + 3e^{2x} + 2 \]

is a particular solution of the differential equation.
Nonhomogeneous Term: Sine or Cosine

We will next look at equations of the form
\[ y'' + ay' + by = c \cos(\beta x) + d \sin(\beta x). \]

We will guess that a solution function might take the form
\[ z = A \cos(\beta x) \text{ where } A \text{ is some constant.} \]

Observe:
\[ z = A \cos(\beta x) \implies z' = -\beta A \sin(\beta x) \text{ and } z'' = -\beta^2 A \cos(\beta x). \]

Then we find
\[ L[z] = z'' + az' + bz = (-\beta^2 A + bA) \cos(\beta x) + (-a\beta A) \sin(\beta x). \]
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\[ z = A \cos(\beta x) \implies z' = -\beta A \sin(\beta x) \text{ and } z'' = -\beta^2 A \cos(\beta x). \]

Then we find

\[ L[z] = z'' + az' + bz = (-\beta^2 A + bA) \cos(\beta x) + (-a\beta A) \sin(\beta x). \]

This means \( L[z] = K \cos(\beta x) + M \sin(\beta x) \) where \( K \) and \( M \) depend on \( a, b, \beta, \) and \( A \). If we were to use \( z = B \sin(\beta x) \) instead, we would find a similar result.
Nonhomogeneous Term: Sine or Cosine

We will next look at equations of the form

$$y'' + ay' + by = c \cos(\beta x) + d \sin(\beta x).$$

We will guess that a solution function might take the form

$$z = A \cos(\beta x) \text{ where } A \text{ is some constant.}$$

Observe:

$$z = A \cos(\beta x) \implies z' = -\beta A \sin(\beta x) \text{ and } z'' = -\beta^2 A \cos(\beta x).$$

Then we find

$$L[z] = z'' + az' + bz = (-\beta^2 A + bA) \cos(\beta x) + (-a\beta A) \sin(\beta x).$$

This means

$$L[z] = K \cos(\beta x) + M \sin(\beta x) \text{ where } K \text{ and } M \text{ depend on } a, b, \beta, \text{ and } A.$$ If we were to use

$$z = B \sin(\beta x) \text{ instead, we would find a similar result.}$$

This leads us to the use

$$z = A \cos(\beta x) + B \sin(\beta x)$$

as our trial solution which will also give

$$L[z] = K \cos(\beta x) + M \sin(\beta x)$$

for some $K$ and $M$ which will depend on $a, b, \beta, \text{ and } A.$
Nonhomogeneous Term: Sine or Cosine

Example: Find a particular solution for the given equation.

1. $y'' - 2y' + y = 3 \cos(2x)$
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1. $y'' - 2y' + y = 3 \cos(2x)$

*We look for a particular solution of the equation with the form*

$$z = A \cos(2x) + B \sin(2x)$$

*We compute*

$$z' = -2A \sin(2x) + 2B \cos(2x),$$

$$z'' = -4A \cos(2x) - 4B \sin(2x)$$
Substituting into the differential equation we obtain

$$-4A \cos(2x) - 4B \sin(2x) - 2(-2A \sin(2x) + 2B \cos(2x)) + A \cos(2x) + B \sin(2x) = 3 \cos(2x)$$

This simplifies to

$$(−3A − 4B) \cos(2x) + (4A − 3B) \sin(2x) = 3 \cos(2x)$$
Substituting into the differential equation we obtain

\[-4A \cos(2x) - 4B \sin(2x) - 2(-2A \sin(2x) + 2B \cos(2x)) + A \cos(2x) + B \sin(2x) = 3 \cos(2x)\]

This simplifies to

\[-3A - 4B) \cos(2x) + (4A - 3B) \sin(2x) = 3 \cos(2x)\]

Hence we find

\[-3A - 4B = 3, \ 4A - 3B = 0, \ \Rightarrow \ A = -\frac{36}{100}, \ B = -\frac{12}{25}\]

Thus

\[z = -\frac{36}{100} \cos(2x) - \frac{12}{25} \sin(2x)\]

is a particular solution of the differential equation.
2. \( y'' - 2y' + 5y = 2 \cos(3x) - 4 \sin(3x) + e^{2x} \)
2. \( y'' - 2y' + 5y = 2 \cos(3x) - 4 \sin(3x) + e^{2x} \)

According to the Superposition Principle, we can treat each term individually. For the \(2 \cos(3x) - 4 \sin(3x)\) term we set \(z_1 = A \cos(3x) + B \sin(3x)\), for the \(e^{2x}\) term, we set \(z_2 = C e^{2x}\).

Thus, we look for a particular solution of the equation with the form

\[
z = A \cos(3x) + B \sin(3x) + C e^{2x}
\]

We compute

\[
z' = -3A \sin(3x) + 3B \cos(3x) + 2C e^{2x},
\]

\[
z'' = -9A \cos(3x) - 9B \sin(3x) + 4C e^{2x},
\]
Nonhomogeneous Term: Sine or Cosine

Hence substituting into the differential equation we obtain

\[-9A \cos(3x) - 9B \sin(3x) + 4Ce^{2x} - 2(-3A \sin(3x) + 3B \cos(3x) + 2Ce^{2x})
+ 5(A \cos(3x) + B \sin(3x) + Ce^{2x}) = 2 \cos(3x) - 4 \sin(3x) + e^{2x}\]

This simplifies to

\[2A \cos(3x) - 10B \sin(3x) + 5Ce^{2x} = 2 \cos(3x) - 4 \sin(3x) + e^{2x}\]
Nonhomogeneous Term: Sine or Cosine

\[\text{Hence substituting into the differential equation we obtain}\]

\[-9A \cos(3x) - 9B \sin(3x) + 4Ce^{2x} - 2(-3A \sin(3x) + 3B \cos(3x) + 2Ce^{2x}) + 5(A \cos(3x) + B \sin(3x) + Ce^{2x}) = 2 \cos(3x) - 4 \sin(3x) + e^{2x}\]

\[\text{This simplifies to}\]

\[2A \cos(3x) - 10B \sin(3x) + 5Ce^{2x} = 2 \cos(3x) - 4 \sin(3x) + e^{2x}\]

\[\text{Hence we find}\]

\[A = 1, \quad B = 2/5, \quad C = 1/5\]

\[\text{Thus}\]

\[z = \cos(3x) + \frac{2}{5} \sin(3x) + \frac{1}{5}e^{2x}\]

\[\text{is a particular solution of the differential equation.}\]
The next case is equations of the form

\[ y'' + ay' + by = ce^{rx} \cos(\beta x) + de^{rx} \sin(\beta x). \]

We will guess that a solution function might take the form

\[ z = Ae^{rx} \cos(\beta x) + Be^{rx} \sin(\beta x) \]

where \( A \) and \( B \) are some constants.
Nonhomogeneous Term: Product of Exponential and Sine/Cosine

Example: Find a particular solution for the given equation.

1. \(y'' + 9y = 4e^x \sin(2x)\)
Nonhomogeneous Term: Product of Exponential and Sine/Cosine

Example: Find a particular solution for the given equation.

1. \( y'' + 9y = 4e^x \sin(2x) \)

We look for a particular solution of the equation with the form

\[
z = Ae^x \cos(2x) + Be^x \sin(2x)
\]

We compute

\[
z' = (A \cos(2x) - 2A \sin(2x) + B \sin(2x) + 2B \cos(2x))e^x,
\]

\[
= (A + 2B) \cos(2x)e^x + (B - 2A) \sin(2x)e^x
\]

\[
z'' = (A + 2B) \cos(2x)e^x - 2(A + 2B) \sin(2x)e^x + (B - 2A) \sin(2x)e^x
\]

\[
+ 2(B - 2A) \cos(2x)e^x
\]

\[
= (4B - 3A) \cos(2x)e^x + (-3B - 4A) \sin(2x)e^x
\]
Nonhomogeneous Term: Product of Exponential and Sine/Cosine

Hence substituting into the differential equation we obtain

\[(4B - 3A) \cos(2x)e^x + (-3B - 4A) \sin(2x)e^x\]
\[+ 9(Ae^x \cos(2x) + Be^x \sin(2x))\]
\[= 4e^x \sin(2x)\]

which simplifies to

\[(4B + 6A) \cos(2x)e^x + (6B - 4A) \sin(2x)e^x = 4e^x \sin(2x)\]
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Hence substituting into the differential equation we obtain

\[(4B - 3A) \cos(2x)e^x + (-3B - 4A) \sin(2x)e^x\]
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\[= 4e^x \sin(2x)\]

which simplifies to

\[(4B + 6A) \cos(2x)e^x + (6B - 4A) \sin(2x)e^x = 4e^x \sin(2x)\]

This implies

\[(4B + 6A) = 0, (6B - 4A) = 4 \quad \Rightarrow \quad A = -4/13, B = 6/13\]

Thus

\[z = -\frac{4}{13}e^x \cos(2x) + \frac{6}{13}e^x \sin(2x)\]

is a particular solution of the differential equation.
Complications

The following table summarizes our findings thus far.

<table>
<thead>
<tr>
<th>If $f(x) =$</th>
<th>set $z(x) =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ce^{rx}$</td>
<td>$Ae^{rx}$</td>
</tr>
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<td>$c \cos \beta x + d \sin \beta x$</td>
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<td>$ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x$</td>
<td>$z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$</td>
</tr>
</tbody>
</table>

*Note: The first line includes the case $r = 0$; if $f(x) = c = ce^{0x}$, then $z = Ae^{0x} = A$. |

We do have a complicating factor. This happens when our guess at $z$, formed by looking at the nonhomogeneous term $f$, is a solution to the reduced equation.
Complications

The next example will illustrate the issue.

1. Find a particular solution for $y'' + y' - 6y = 3e^{2x}$.

We solve the homogeneous equation first. The characteristic equation is

$$r^2 + r - 6 = 0$$

with characteristic roots $r = 2, -3$.

Hence the solution of the homogeneous problem is

$$y_h(x) = c_1 e^{-3x} + c_2 e^{2x}.$$

This shows that $z = Ae^{2x}$ cannot be a particular solution of the non-homogeneous problem.
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Hence, we look for a particular solution of the equation with the form

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We have
\[ z' = Ae^{2x} + 2Axe^{2x} \]
\[ z'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} = 4Ae^{2x} + 4Axe^{2x} \]

Substitution into the equation gives
\[ 4Ae^{2x} + 4Axe^{2x} + Ae^{2x} + 2Axe^{2x} - 6Axe^{2x} = 3e^{2x} \]

which simplifies to
\[ 5Ae^{2x} = 3e^{2x} \implies A = 3/5 \]

This shows that \( z = \frac{3}{5}xe^{2x} \) is a particular solution of the non-homogeneous problem
2. Find a particular solution for $y'' - 6y' + 9y = 4e^{3x}$. 

We solve the homogeneous equation first. The characteristic equation is $r^2 - 6r + 9 = (r - 3)^2 = 0$ with repeated characteristic root $r = 3$. This shows that $z = (A + Bx)e^{3x}$ cannot be a particular solution of the differential equation. So we will look for a solution of the form $z = Ax^2e^{3x}$. We have $z' = 2Axe^{3x} + 3Ax^2e^{3x}$, $z'' = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}$. 


2. Find a particular solution for \( y'' - 6y' + 9y = 4e^{3x} \).

*We solve the homogeneous equation first. The characteristic equation is*

\[
    r^2 - 6r + 9 = (r - 3)^2 = 0
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*with repeated characteristic root \( r = 3 \).*

*This shows that \( z = (A + Bx)e^{3x} \) cannot be a particular solution of the differential equation*

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So we will look for a solution of the form \( z = Ax^2e^{3x} \)

We have

\[
z' = 2Axe^{3x} + 3Ax^2e^{3x}, \quad z'' = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x}
\]
Substitution into the equation gives

\[2Ae^{3x} + 12Ax e^{3x} + 9Ax^2 e^{3x} - 6(2Ax e^{3x} + 3Ax^2 e^{3x}) + 9Ax^2 e^{3x} = 4e^{3x}\]

This simplifies to

\[2Ae^{3x} = 4e^{3x} \Rightarrow A = 2\]

This shows that

\[z = 2x^2 e^{3x}\]

is a particular solution of the non-homogeneous problem
Complications

When we include this information, our summary becomes:

<table>
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* If $z$ satisfies the reduced equation, use $xz$; if $xz$ also satisfies the reduced equation, then $x^2z$ will give a particular solution

For this reason, it is good practice to first solve the reduced equation before searching for a function $z$ which solves the nonhomogeneous equation.
3. Give the form of a particular solution of

\[ y'' - 4y' + 4y = 4e^{2x} + \cos(3x) - 5. \]
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\[ y'' - 4y' + 4y = 4e^{2x} + \cos(3x) - 5. \]

*The first step is to solve the reduced equation*

\[ y'' - 4y' + 4y = 0 \]

*The characteristic equation is* \( r^2 - 4r + 4 = (r - 2)^2 = 0 \), *which has repeated roots* \( r = 2 \).
3. Give the form of a particular solution of

\[ y'' - 4y' + 4y = 4e^{2x} + \cos(3x) - 5. \]

The first step is to solve the reduced equation

\[ y'' - 4y' + 4y = 0 \]

The characteristic equation is \( r^2 - 4r + 4 = (r - 2)^2 = 0 \), which has repeated roots \( r = 2 \).

This implies that \( z = Ae^{2x} \) and \( z = Axe^{2x} \) are solution of the reduced equation.
3. Give the form of a particular solution of

\[ y'' - 4y' + 4y = 4e^{2x} + \cos(3x) - 5. \]

The first step is to solve the reduced equation

\[ y'' - 4y' + 4y = 0 \]

The characteristic equation is \( r^2 - 4r + 4 = (r - 2)^2 = 0 \), which has repeated roots \( r = 2 \).

This implies that \( z = Ae^{2x} \) and \( z = Axe^{2x} \) are solutions of the reduced equation. Thus, to find a particular solution of

\[ y'' - 4y' + 4y = 4e^{2x} \]

we need to choose a solution of the form \( z_1 = Ax^2e^{2x} \)
To find a particular solution of

\[ y'' - 4y' + 4y = \cos(3x) \]

we need to choose a solution of the form \( z_2 = B \cos(3x) + C \sin(3x) \)

To find a particular solution of

\[ y'' - 4y' + 4y = -5 \]

we need to choose a solution of the form \( z_3 = D \).
Complications

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\[ y'' - 4y' + 4y = \cos(3x) \]

we need to choose a solution of the form \( z_2 = B \cos(3x) + C \sin(3x) \)

To find a particular solution of

\[ y'' - 4y' + 4y = -5 \]

we need to choose a solution of the form \( z_3 = D \).

Hence, combining these observations, we have that the particular solution has the form

\[ z = z_1 + z_2 + z_3 = Ax^2 e^{2x} + B \cos(3x) + C \sin(3x) + D \]
Complications

4. Give the form of a particular solution of

\[ y'' - 4y' = 4e^{4x} \sin(x) + \cos(2x) + 3. \]
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\[ y'' - 4y' = 4e^{4x} \sin(x) + \cos(2x) + 3. \]

The first step is to solve the reduced equation

\[ y'' - 4y' = 0 \]

The characteristic equation is \( r^2 - 4r = r(r - 4) = 0 \), which has roots \( r = 0, r = 4 \). This implies that \( z = C_1 \) and \( z = C_2 e^{4x} \) are solution of the reduced equation.
4. Give the form of a particular solution of

\[ y'' - 4y' = 4e^{4x} \sin(x) + \cos(2x) + 3. \]

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The characteristic equation is \( r^2 - 4r = r(r - 4) = 0 \), which has roots \( r = 0, r = 4 \). This implies that \( z = C_1 \) and \( z = C_2 e^{4x} \) are solution of the reduced equation.

This implies that \( z = C_1 \) is not a particular solution and this should be amended to a solution of the form \( z = C_1 x \).
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\[ y'' - 4y' = 4e^{4x} \sin(x) + \cos(2x) + 3. \]

The first step is to solve the reduced equation

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The characteristic equation is \( r^2 - 4r = r(r - 4) = 0 \), which has roots \( r = 0, r = 4 \). This implies that \( z = C_1 \) and \( z = C_2e^{4x} \) are solution of the reduced equation.

This implies that \( z = C_1 \) is not a particular solution and this should be amended to a solution of the form \( z = C_1x \).

Thus the particular solution has the form

\[ z = A \cos(x)e^{4x} + B \sin(x)e^{4x} + C \cos(2x) + D \sin(2x) + Ex \]
Undetermined Coefficients in General

Thus far we have found solutions for the nonhomogeneous differential equation (1) in cases where the nonhomogeneous term $f$ is a constant multiple of one of the functions $e^{rx}$, $\cos(\beta x)$, $\sin(\beta x)$, $e^{rx} \cos(\beta x)$, $e^{rx} \sin(\beta x)$, or is a sum of such functions. In general, this method can be applied in cases of the form

$$f(x) = p(x)e^{rx}$$

$$f(x) = p(x)\cos(\beta x) \text{ or } p(x)\sin(\beta x)$$

$$f(x) = p(x)e^{rx} \cos(\beta x) \text{ or } p(x)e^{rx} \sin(\beta x)$$

where $p$ is a polynomial or where $f$ is the sum of such functions.
Undetermined Coefficients in General

Method of Undetermined Coefficients

We can summarize the general method as follows:

(1) If $f(x) = p(x)e^{rx}$ where $p$ is a polynomial of degree $n$, then

$$z = (A_0 + A_1x + A_2x^2 + \cdots + A_nx^n)e^{rx}.$$ 

(2) If $f(x) = p_1(x)\cos(\beta x) + p_2(x)\sin(\beta x)$ where $p_1$ and $p_2$ are polynomials of degree $k$ and $m$, respectively, then

$$z = (A_0 + A_1x + \cdots + A_nx^n)\cos(\beta x)$$

$$+ (B_0 + B_1x + \cdots + B_nx^n)\sin(\beta x)$$

where $n = \max\{k, m\}$. 

Undetermined Coefficients in General

Method of Undetermined Coefficients

(3) If \( f(x) = p_1(x)e^{rx} \cos(\beta x) + p_2(x)e^{rx} \sin(\beta x) \) where \( p_1 \) and \( p_2 \) are polynomials of degree \( k \) and \( m \), respectively, then

\[
z = (A_0 + A_1 x + \cdots + A_n x^n)e^{rx} \cos(\beta x) + (B_0 + B_1 x + \cdots + B_n x^n)e^{rx} \sin(\beta x)
\]

where \( n = \max\{k, m\} \).

If any term in \( z \) satisfies the reduced equation \( y'' + ay' + by = 0 \), then use \( xz \) as the trial solution. If \( xz \) satisfies the reduced equation, then use \( x^2z \).
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1. Find the general solution to $y'' - y' - 6y = (2x^2 - 1)e^{2x}$. 
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1. Find the general solution to $y'' - y' - 6y = (2x^2 - 1)e^{2x}$.

   Based on the form of the function on the right hand side, the particular solution should have the form

   $$z_1 = (A_0 + A_1x + A_2x^2)e^{2x}$$

   The reduced equation $y'' - y' - 6y = 0$ has characteristic equation is $r^2 - r - 6 = 0$, which has roots $r = -2, r = 3$. Thus the solution of the homogeneous problem is

   $$z = C_1e^{-2x} + C_2e^{3x}$$
Undetermined Coefficients in General

It remains to find the particular solution. We have

\[ z'_1 = (A_1 + 2A_2 x)e^{2x} + 2(A_0 + A_1 x + A_2 x^2)e^{2x} \]
\[ = ((2A_0 + A_1) + (A_1 - 2A_2)x + A_2 x^2) e^{2x} \]

\[ z''_1 = 2A_2 e^{2x} + 2(A_1 + 2A_2 x)e^{2x} + 2(A_1 + 2A_2 x)e^{2x} + 4(A_0 + A_1 x + A_2 x^2)e^{2x} \]
\[ = ((4A_0 + 4A_1 + 2A_2) + (4A_1 + 8A_2)x + 4A_2 x^2) e^{2x} \]
It remains to find the particular solution. We have

\[ z' = (A_1 + 2A_2x)e^{2x} + 2(A_0 + A_1x + A_2x^2)e^{2x} \]
\[ = ((2A_0 + A_1) + (A_1 - 2A_2)x + A_2x^2) e^{2x} \]

\[ z'' = 2A_2e^{2x} + 2(A_1 + 2A_2x)e^{2x} + 2(A_1 + 2A_2x)e^{2x} + 4(A_0 + A_1x + A_2x^2)e^{2x} \]
\[ = ((4A_0 + 4A_1 + 2A_2) + (4A_1 + 8A_2)x + 4A_2x^2) e^{2x} \]

Substitution into the differential equation (ignoring the exponential on both sides) gives

\[ ((4A_0 + 4A_1 + 2A_2) + (4A_1 + 8A_2)x + 4A_2x^2) \]
\[ - ((2A_0 + A_1) + (A_1 - 2A_2)x + A_2x^2) - 6(A_0 + A_1x + A_2x^2) \]
\[ = (2x^2 - 1) \]

Hence we have

\[ -4A_0 + 3A_1 + 2A_2 = -1, \quad -3A_1 + 10A_2 = 0, \quad -3A_2 = 2 \]

giving \( A_0 = -21/12, \ A_1 = -20/9, \ A_2 = -2/3 \)
Thus the general solution is

\[ y = C_1 e^{-2x} + C_2 e^{3x} + \left( -\frac{21}{12} - \frac{20}{9} x - \frac{2}{3} x^2 \right) e^{2x} \]
2. Give the form of a particular solution for

\[ y'' + 8y' + 16y = 2x - 1 + 7x^2 e^{-4x}. \]
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Based on the form of the function on the right hand side, the particular solution should have the form

\[ z = A + Bx + (C + Dx + Ex^2)e^{-4x} \]

However, we need to check the solution of the reduced equation

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\[ y'' + 8y' + 16y = 0 \]

The characteristic equation is \( r^2 + 8r + 16 = (r + 4)^2 = 0 \), which has repeated roots \( r = -4 \). This implies that \( z = C_1 e^{-4x} \) and \( z = C_2 xe^{-4x} \) are solution of the reduced equation and they are not particular solutions.

Thus, a particular solution has the form

\[ z = A + Bx + Ex^2 e^{-4x} \]
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\[ y'' + y = 4x \sin(x) - 2x \cos(2x) + x^2 \sin(2x) + xe^{2x}. \]
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Based on the form of the function on the right hand side, the particular solution should include the terms

\[ z_1 = (A_0 + A_1x) \cos(x) + (B_0 + B_1x) \sin(x) \]
\[ z_2 = (C_0 + C_1x + C_2x^2) \cos(2x) + (D_0 + D_1x + D_2x^2) \sin(2x) \]
\[ z_3 = (E_0 + E_1x) e^{2x} \]

We also need to check the solution of the reduced equation

\[ y'' + y = 0 \]

The characteristic equation is \( r^2 + 1 = 0 \), which has roots \( r = \pm i \). This implies that \( z = \alpha \cos x + \beta \sin x \) is the solution of the reduced equation and it is not a particular solution.
Thus, a particular solution has the form

\[
z = A_1 x \cos(x) + B_1 x \sin(x) + (C_0 + C_1 x + C_2 x^2) \cos(2x) \\
+ (D_0 + D_1 x + D_2 x^2) \sin(2x) + (E_0 + E_1 x)e^{2x}
\]