Math 3321

Higher Order Linear Differential Equations

University of Houston

Lecture 11

Outline

Introduction

So far, we have studied first order linear equations

$$y' + p(x)y = q(x)$$

and second order linear equations

$$y'' + p(x)y' + q(x)y = f(x).$$

Here we will continue with higher order linear differential equations.

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Definitions

An n^{th}-*order linear differential equation* is an equation which can be written in the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(N)}$$

where p_0, $p_1, \ldots,$ p_{n-1}, and f are continuous functions on some interval I. Once again, the functions p_0, $p_1, \ldots,$ p_{n-1}, are called the *coefficients* and f is the *forcing function* or the *nonhomogeneous term*.

Definitions

Equation (N) is *homogeneous* if the function f on the right side is 0 for all $x \in I$. In this case, (N) becomes

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \qquad \text{(H)}$$

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Remark: Linearity

Our intuitive understanding is that an n^{th}-order differential equation is **linear** when y and its derivatives appear in the equation with exponent 1 only, and there are no "cross-product" terms such as yy'.

Introduction

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Letting

$$L[y] = y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y,$$

we can write (N) as

$$L[y] = f(x)$$

and (H) becomes

$$L[y] = 0.$$

Introduction

Theorem 1: Existence and Uniqueness Theorem

Given the n^{th}-order linear equation (N). Let a be any point on the interval I, and let a_0, $a_1, \ldots, a_{n-1}$ be any n real numbers. Then the initial-value problem

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x);$$

$$y(a) = a_0,\ y'(a) = a_1, \ldots, y^{(n-1)}(a) = a_{n-1}$$

has a unique solution.

As stated when we introduced second order linear differential equations, *there is no general method for solving second or higher order linear differential equations*. However, there are methods for certain cases which we will discuss in this lecture.

Homogeneous Equations

Terminology

As in the second order case, we emphasize that (H)

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

is solved by the zero function $y \equiv 0$. We call the zero function the *trivial solution.* Our interest will be in finding *nontrivial solutions.* Unless otherwise stated, when we use the term solution we will assume this means nontrivial solution.

Homogeneous Equations

Theorem 2

Given any solution $y = y(x)$ for (H) and C a real number, then $u(x) = Cy(x)$ is also a solution for (H).

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Theorem 4

If $y = y_1(x)$, $y = y_2(x), \ldots, y = y_k(x)$ are solutions of (H) and C_1, $C_2, \ldots, C_k$ real numbers, then

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_k y_k(x)$$

is also a solution for (H).

Homogeneous Equations

When $k = n$ above, we get

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) \tag{1}$$

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We are interested in knowing when (1) will be the general solution for (H).
As in the second order case, this will depend on the relationship between the solutions $y_1,\, y_2, \ldots,\, y_n$.

Homogeneous Equations

Definition

Let $y = y_1(x)$, $y = y_2(x), \ldots, y = y_n(x)$ be solutions for (H). Then

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ y_1''(x) & y_2''(x) & \ldots & y_n''(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}$$

is the *Wronskian* of the solutions y_1, $y_2, \ldots, y_n$.

Homogeneous Equations

Theorem 5

Let $y = y_1(x)$, $y = y_2(x), \ldots,$ $y = y_n(x)$ be solutions of (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$ and y_1, $y_2, \ldots,$ y_n are linearly dependent.

(ii) $W(x) \neq 0$ for all $x \in I$ which implies y_1, $y_2, \ldots,$ y_n are linearly independent and $y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$ is the general solution of (H).

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Definition

A set $\{y = y_1(x),\ y = y_2(x), \ldots,\ y = y_n(x)\}$ of n linearly independent solutions of (H) is called a *fundamental set of solutions.*

A set of solutions $\{y_1,\ y_2, \ldots,\ y_n\}$ of (H) is a fundamental set if and only if

$$W[y_1, y_2, \ldots, y_n](x) \neq 0 \text{ for all } x \in I.$$

Homogeneous Equations with Constant Coefficients

Definition

An n^{th}*-order linear homogeneous differential equation with constant coefficients* is an equation which can be written in the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{2}$$

where a_0, $a_1, \ldots,$ a_{n-1} are real numbers.

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Definitions

Given the differential equation (2), the corresponding polynomial equation

$$p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0 \tag{3}$$

is called the *characteristic equation* of (2). The n^{th}-degree polynomial $p(r)$ is the *characteristic polynomial.* Finally, the roots of the equation/polynomial are known as the *characteristic roots.*

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(1) If r_1, $r_2, \ldots,$ r_k are distinct numbers (real or complex), then the distinct exponential functions $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}, \ldots,$ $y_k = e^{r_k x}$ are linearly independent.

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(2) For any real number r, the functions $y_1 = e^{rx}$, $y_2 = xe^{rx}, \ldots,$ $y_k = x^{k-1}e^{rx}$ are linearly independent.

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(2) For any real number r, the functions $y_1 = e^{rx}$, $y_2 = xe^{rx}, \ldots$, $y_k = x^{k-1}e^{rx}$ are linearly independent.

(3) For any real numbers α and β, the functions $y_1(x) = e^{\alpha x}\cos(\beta x)$, $y_2(x) = e^{\alpha x}\sin(\beta x)$, $y_3(x) = xe^{\alpha x}\cos(\beta x)$, $y_4(x) = xe^{\alpha x}\sin(\beta x), \ldots$ are linearly independent.

Moreover, the functions in one of the groups are independent of the functions in the other groups.

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Examples:

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$$p(r) = r^3 + 3r^2 - 6r - 8$$

we factor out the term $(r - 2)$

$$p(r) = r^3 + 3r^2 - 6r - 8 = (r - 2)(r^2 + 5r + 4)$$

Hence the roots are $r = 2, r = -1, r = -4$ *and the general solution is*

$$y = C_1 e^{-4x} + C_2 e^{-x} + C_3 e^{2x}$$

Homogeneous Equations with Constant Coefficients

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Since $r = -1 + 3i$ *is a root, then* $r = -1 - 3i$ *is also a root, hence* $(r + 1 - 3i)(r + 1 + 3i) = r^2 + 2r + 10$ *is a factor of the characteristic polynomial.*

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$$p(r) = (r^2 + 2r + 10)(r^2 - 1)$$

and the roots of the characteristic polynomial are $r = -1 \pm 3i, r = 1, r = -1$. *The general solution is*

$$y = C_1 e^{-x} \cos(3x) + C_2 e^{-x} \sin(3x) + C_3 e^{-x} + C_4 e^{x}$$

3. Find the general solution of $y^{(4)} - y''' - 7y'' + 13y' - 6y = 0$ if $r = 1$ is a root of the characteristic polynomial with multiplicity 2.

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We write the characteristic polynomial

$$p(r) = r^4 - r^3 - 7r^2 + 13r - 6$$

we factor out the term $(r-1)^2$

$$p(r) = (r-1)^2(r^2 + r - 6)$$

Hence the roots are $r = 1, r = 1, r = -3, r = 2$ *and the general solution is*

$$y = C_1e^x + C_2xe^x + C_3e^{2x} + C_4e^{-3x}$$

Homogeneous Equations with Constant Coefficients

4. Find the homogeneous linear ODE with constant coefficients of least order for which $y = 2xe^{2x} - 3e^{-2x} + 4\cos(2x) + 10$ is a solution.

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Since $10 = 10e^{0x}$ is a solution, $r = 0$ must be a root

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Hence we write the characteristic polynomial

$$p(r) = (r-2)^2(r+2)(r^2+4)r = r^6 - 2r^5 - 16r^2 + 32r$$

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Hence we have the homogeneous differential equation

$$y^{(6)} - 2y^{(5)} - 16y'' + 32y' = 0$$

Nonhomogeneous Equations

Theorem 6

Given any two solutions $z = z_1(x)$ and $z = z_2(x)$ for (N),

$$y(x) = z_1(x) - z_2(x)$$

is a solution of the reduced equation (H).

Nonhomogeneous Equations

Theorem 7

Let $\{y = y_1(x),\ y = y_2(x), \ldots,\ y = y_n(x)\}$ be a fundamental set of solutions of the reduced equation (H) and let $z = z(x)$ be a particular solution of (N). If $u = u(x)$ is *any* solution of (N), then there exist constants $C_1,\ C_2, \ldots,\ C_n$ such that

$$u(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + z(x).$$

Theorem 7 tells us that when we have $y = y_1(x),\ y = y_2(x), \ldots,\ y = y_n(x)$ are linearly independent solutions for (H) and $z = z(x)$ is a particular solution of (N), then all solutions of (N) can be expressed as

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + z(x). \tag{4}$$

That is, (4) is the general solution of equation (N).

Nonhomogeneous Equations

The *superposition principle* also holds in the n^{th}-order equation setting.

Theorem 8

Suppose $z = z_f(x)$ and $z = z_g(x)$ are particular solutions of

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

and

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x),$$

respectively, then $z = z_f(x) + z_g(x)$ is a particular solution of

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x) + g(x).$$

Finding a Particular Solution

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The method of variation of parameters can be extended to n^{th}-order linear differential equations.
However, the calculations become quite difficult.
We will instead focus on cases where we can use the method of undetermined coefficients. That is, we will restrict our focus to equations of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = f(x)$$

where a_0, $a_1, \ldots,$ a_{n-1} are real numbers and the nonhomogeneous term f is a polynomial, an exponential function, a sine, a cosine, or a suitable combination of such functions.

Finding a Particular Solution

Updating the basic table from our study of undetermined coefficients in the second order case we find the following for an n^{th}-order ODE.

A particular solution of $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$

If $f(x) =$	try $z(x) =$*
ce^{rx}	Ae^{rx}
$c \cos \beta x + d \sin \beta x$	$z(x) = A \cos \beta x + B \sin \beta x$
$ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x$	$z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$

*Note: If z satisfies the reduced equation, then $x^k z$, where k is the least integer such that $x^k z$ does not satisfy the reduced equation, will give a particular solution

Finding a Particular Solution

Examples:

1. Give the form of a particular solution of

$$y^{(4)} + 4y''' + 13y'' + 36y' + 36y = 5x^2e^{2x} + x\sin(3x) + 6.$$

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We first examine the homogeneous equation. The characteristic polynomial can be factored as

$$p(r) = r^4 + 4r^3 + 13r^2 + 36r + 36 = (r+2)^2(r^2+9)$$

This implies that the functions A_1e^{-2x}*,* A_2xe^{-2x}*,* $B_1\cos(3x)$*,* $B_2\sin(3x)$ *are fundamental solutions of the homogeneous equation and, thus, cannot be particular solutions.*

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This implies that the functions $A_1 e^{-2x}$, $A_2 x e^{-2x}$, $B_1 \cos(3x)$, $B_2 \sin(3x)$ *are fundamental solutions of the homogeneous equation and, thus, cannot be particular solutions.*

It follows that the particular solution has the form

$$z = (C_0 + C_1 x + C_2 x^2)e^{2x} + D_1 x\cos(3x) + D_2 x\sin(3x) + E$$

Finding a Particular Solution

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We first examine the homogeneous equation.
By substitution we see that $r = -1$ is a root of the characteristic polynomial which can be factored as

$$p(r) = r^3 + 3r^2 + 3r + 1 = (r + 1)^3$$

This implies that the homogeneous equation has general solution

$$y_h = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x}$$

Finding a Particular Solution

We look for a particular solution has the form $z = De^{2x}$. *We have*

$$z' = 2De^{2x},\ z'' = 4De^{2x},\ z''' = 8De^{2x}$$

By substitution into the differential equation we obtain

$$8De^{2x} + 12De^{2x} + 6De^{2x} + De^{2x} = 3e^{2x}$$

which simplifies to

$$27D = 3 \quad \Rightarrow \quad D = 1/9$$

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Thus the general solution is

$$y = y_h + z = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x} + \tfrac{1}{9}e^{2x}$$