Math 3321

Vibrating Mechanical Systems

University of Houston

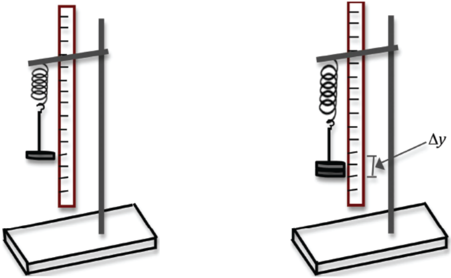
Lecture 12

Outline

1. Introduction
2. Free Vibrations
3. Undamped Free Vibrations
4. Damped Free Vibrations
5. Forced Vibrations
6. Undamped Forced Vibrations
7. Damped Forced Vibrations

Introduction

In this lecture, we will consider an application of second order linear differential equations.
Here we examine the **motion of a spring** of length l_0, from which an object of mass m is attached. When the mass is attached, the spring extends to a length l_1. If the object is puled down or pushed up an additional y_0 units at time $t = 0$ and then released, we are interested in the resulting motion of the object. We wish to find the position of the object at time $t > 0$ (t measured in seconds).



Introduction

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(A) *Free Vibrations* in which the forces acting on the spring-mass system are gravity and the restoring force of the spring. We will then include a damping force such as friction.

(B) *Forced Vibrations* in which an additional external force is applied to a freely vibrating system.

Free Vibrations

We consider the forces acting on the object at time $t > 0$. We begin with the downward force from gravity

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The second force is the restoring force from the spring. Hooke's Law tells us this force is proportional to the displacement $l_1 + y(t)$ and acts in the direction opposite the displacement. That is:

$$F_2 = -k[l_1 + y(t)] \text{ where } k > 0.$$

Here, k is known as the *spring constant*.

Undamped Free Vibrations

With undamped vibrations, we assume the forces F_1 and F_2 are the only forces. That is, we assume the spring is frictionless and there is no air resistence. Our force equation will be

$$F = mg - k[l_1 + y(t)] = (mg - kl_1) - ky(t)$$

Before the object was displaced, the system was in equilibrium, so it must be

$$mg - kl_1 = 0$$

Therefore, the total force F reduces to

$$F = -ky(t)$$

By Newton's Second Law of Motion, $F = ma$, hence

$$ma = -ky(t) \quad \Rightarrow \quad a = y''(t) = -\frac{k}{m}y(t)$$

Undamped Free Vibrations

Since $\frac{k}{m} > 0$, we introduce the notation $\omega^2 = \frac{k}{m}$ and we write the last expression as

$$y''(t) + \omega^2 y(t) = 0$$

This is the equation of **simple harmonic motion**.

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This is the equation of **simple harmonic motion**.

It is a second order linear homogeneous equation with constant coefficients whose characteristic equation is

$$r^2 + \omega^2 = 0.$$

The characteristic roots are $\pm i$, hence the general solution is

$$y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

Undamped Free Vibrations

Simple Harmonic Motion

Our equation of motion is

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where $\omega = \sqrt{k/m}$. We found the solution to be

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$$y = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

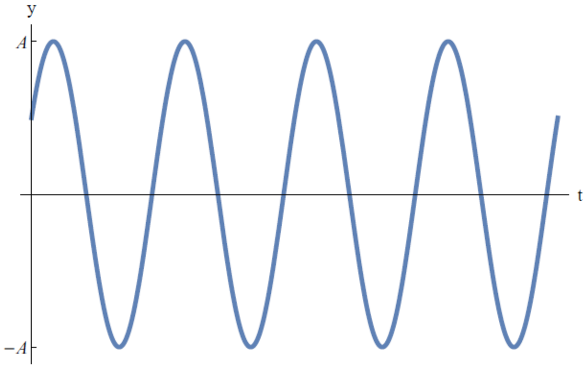
We can rewrite this as

$$y = A \sin(\omega t + \phi),$$

where $A > 0$ and $\phi \in [0, 2\pi)$. Here, the constant ω is the *natural frequency* of the system and A is the *amplitude* of the motion. The number ϕ is the *phase constant* or *phase shift*.

Undamped Free Vibrations

Below is a typical graph of a function $y = A\sin(\omega t + \phi)$



Undamped Free Vibrations

Examples:

1. An object is in simple harmonic motion. Find an equation for the motion given that the period is $\frac{2\pi}{3}$ and at time $t = 0$, $y = 1$, and $y' = 3$. What is the function which models this motion?

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Since the period is $\frac{2\pi}{\omega}$ *and this quantity is equal to* $\frac{2\pi}{3}$*, it follows that* $\omega = 3$ *Hence the equation of motion is of the form*

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To determine A and ϕ, we use the fact that

$$1 = y(0) = A\sin(\phi), \quad 3 = y'(0) = 3A\cos(3t + \phi)|_{t=0} = 3A\cos(\phi)$$

which simplifies to

$$A\sin(\phi) = 1, \quad A\cos(\phi) = 1$$

Undamped Free Vibrations

Thus we have that

$$A^2 \sin^2(\phi) + A^2 \cos^2(\phi) = 2 \quad \Rightarrow \quad A^2 = 2 \quad \Rightarrow \quad A = \sqrt{2}$$

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Finally, to find ϕ, we use the observation that

$$\sqrt{2}\,\sin(\phi) = 1, \sqrt{2}\,\cos(\phi) = 1 \quad \Rightarrow \quad \sin(\phi) = \cos(\phi) = \tfrac{\sqrt{2}}{2}$$

hence $\phi = \frac{\pi}{4}$. Thus, the equation of motion is

$$y(t) = 3\sin(3t + \pi/4)$$

Undamped Free Vibrations

2. An object is in simple harmonic motion. Find an equation for the motion given that the frequency is $\frac{5}{\pi}$ and at time $t = 0$, $y = 1$, and $y' = 0$. What is the function which models this motion?

Undamped Free Vibrations

2. An object is in simple harmonic motion. Find an equation for the motion given that the frequency is $\frac{5}{\pi}$ and at time $t = 0$, $y = 1$, and $y' = 0$. What is the function which models this motion?

Since the period is $\frac{2\pi}{\omega}$ and this quantity is equal to $\frac{\pi}{5} = \frac{2\pi}{10}$, it follows that $\omega = 10$ Hence the equation of motion is of the form

$$y(t) = A\sin(10t + \phi)$$

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To determine A and ϕ, we use the fact that

$$1 = y(0) = A\sin(\phi), \quad 0 = y'(0) = 10A\cos(10t + \phi)|_{t=0} = 10A\cos(\phi)$$

which simplifies to

$$A\sin(\phi) = 1, \quad \cos(\phi) = 0$$

Hence is must be $\phi = \pi/2$ and $A = 1$.

Damped Free Vibrations

We now introduce a damping force R, such as friction or air resistence. A damping force resists the movement and experiments have shown that R will be approximately proportional to the velocity $v = y'$ and acts in the opposite direction relative to the motion. That is:

$$R = -cy' \text{ with } c > 0.$$

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Our force equation is now

$$F = -ky - cy'$$

and our equation of motion is

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \tag{2}$$

This is the **equation of motion in the presence of a damping factor**.

The characteristic equation is

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

and has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

Damped Free Vibrations

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$$r^2 + \frac{c}{m} r + \frac{k}{m} = 0$$

and has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

There are three cases to consider:

$$c^2 - 4km < 0, \quad c^2 - 4km > 0, \quad c^2 - 4km = 0.$$

Damped Free Vibrations

Case 1: $c^2 - 4km < 0$

In this case, the characteristic equation has complex roots:

$$r_1 = -\frac{c}{2m} + i\omega,\ r_2 = -\frac{c}{2m} - i\omega,\ \text{where } \omega = \frac{\sqrt{4km - c^2}}{2m}.$$

Damped Free Vibrations

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Here the general solution is

$$y = e^{(-c/2m)t}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$$

which can also be written as

$$y = Ae^{(-c/2m)t} \sin(\omega t + \phi) \tag{3}$$

where $A > 0$ and $\phi \in [0, 2\pi)$. We call this the *underdamped case.*

Damped Free Vibrations

Below is a typical graph of a function (3).

Damped Free Vibrations

Case 2: $c^2 - 4km > 0$

In this case, the characteristic equation has two distinct real roots:

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m},\ r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

Damped Free Vibrations

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In this case, the characteristic equation has two distinct real roots:

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m}, \; r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

Here the general solution is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \tag{4}$$

We call this the *overdamped case.*

Case 3: $c^2 - 4km = 0$

In this case, the characteristic equation has a single repeated root:

$$r = -\frac{c}{2m}.$$

Damped Free Vibrations

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In this case, the characteristic equation has a single repeated root:

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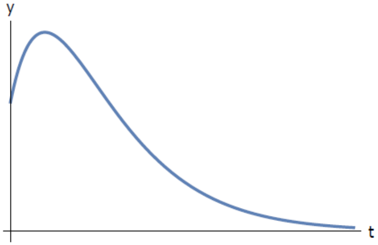
Here the general solution is

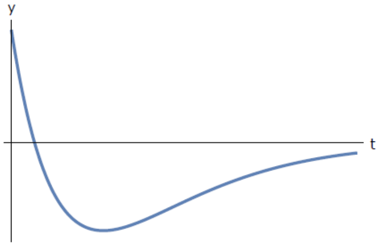
$$y = C_1 e^{-(c/2m)t} + C_2 t e^{-(c/2m)t}. \tag{5}$$

We call this the *critically damped case*.

Damped Free Vibrations

Below we see graphs which represent potential solutions in the overdamped and critically damped cases.





Forced Vibrations

The motion we have studied thus far is the result of the interplay between three forces: gravity, the restoring force of the spring, and the resistence force of friction or the surrounding medium. These were *free vibrations.*

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We now apply an external force to our freely vibrating system and study the resulting motion. We call these *forced vibrations.* Throughout, we will consider the application of a periodic external force of the form:

$$F_0 \cos(\gamma t),$$

where F_0 and γ are positive constants.

Undamped Forced Vibrations

In an undamped system with an external force $F_0 \cos(\gamma t)$, the behavior of our system will depend on the relationship between the *applied frequency* γ and the *natural frequency* ω. Here our force equation will be

$$F = -ky + F_0 \cos(\gamma t).$$

Hence the equation of motion takes the form

$$y''(t) + \frac{k}{m} y(t) = \frac{F_0}{m} \cos(\gamma t)$$

By introducing the notation $\omega^2 = \frac{k}{m}$ and above, we can write the last equation as

$$y''(t) + \omega^2 y(t) = \frac{F_0}{m} \cos(\gamma t)$$

Undamped Forced Vibrations

Let us examine the second order linear nonhomogeneous differential equation

$$y''(t) + \omega^2 y(t) = \frac{F_0}{m}\cos(\gamma t)$$

By the properties of the particular solution of the differential equation, it is clear that the nature of the motion depends on the relation between the applied frequency γ and the natural frequency ω of the system.

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By the properties of the particular solution of the differential equation, it is clear that the nature of the motion depends on the relation between the applied frequency γ and the natural frequency ω of the system.

Namely, if the $\omega \neq \gamma$ then the particular solution is of the form

$$y_p = A\cos(\gamma t) + B\sin(\gamma t)$$

However, if the $\omega = \gamma$ then the particular solution is of the form

$$y_p = At\cos(\gamma t) + Bt\sin(\gamma t)$$

Undamped Forced Vibrations

Case 1: $\gamma \neq \omega$

Here our equation of motion will be

$$y'' + \omega^2 y = F_0 \cos(\gamma t) \tag{6}$$

where $\omega = \sqrt{k/m}$. The method of undetermined coefficients will give a particular solution

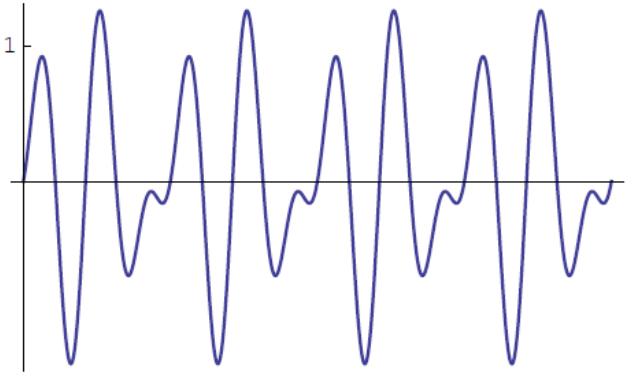
$$z = \frac{F_0}{m(\omega^2 - \gamma^2)} \cos(\gamma t)$$

and the general solution to our equation of motion is

$$y = A \sin(\omega t + \phi) + \frac{F_0}{m(\omega^2 - \gamma^2)} \cos(\gamma t). \tag{7}$$

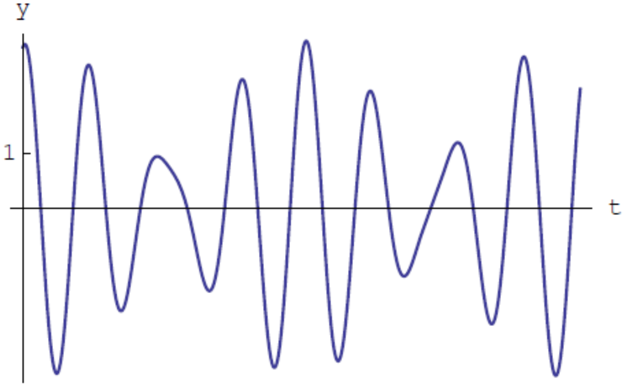
Undamped Forced Vibrations

Below we see a graph of a potential solution in the **case where $\frac{\omega}{\gamma}$ is rational**, in which case we see periodic motion.



Undamped Forced Vibrations

Below we see a graph of a potential solution in the **case where $\frac{\omega}{\gamma}$ is irrational**, in which case we see motion which is not periodic.



Undamped Forced Vibrations

Case 2: $\gamma = \omega$

Here our equation of motion will be

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The method of undetermined coefficients will give a particular solution

$$z = \frac{F_0}{2\omega m} t \sin(\omega t)$$

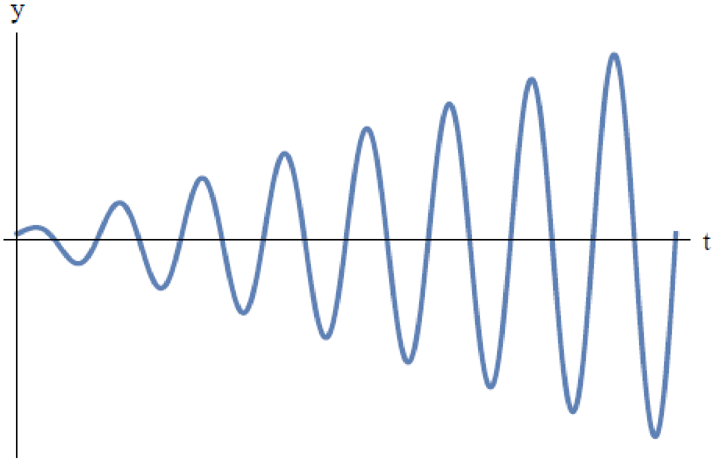
and the general solution to our equation of motion is

$$y = A\sin(\omega t + \phi) + \frac{F_0}{2\omega m} t \sin(\omega t). \tag{8}$$

The system is said to be in *resonance*. The motion will be oscillatory but the amplitude will grow linearly without bound.

Undamped Forced Vibrations

Below we see a graph of a potential solution (8). We see oscillatory motion with an amplitude which grows linearly without bound.



Damped Forced Vibrations

We introduce a damping force to our previous system of forced vibrations. Our force equation will be

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Hence the equation that governs the motion is

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$$F = -ky - cy' + F_0 \cos(\gamma t).$$

Hence the equation that governs the motion is

$$my'' = -cy' - ky + F_0 \cos(\gamma t)$$

Using again the notation $\omega^2 = k/m$, we can re-write it as

$$y'' + \frac{c}{m} y' + \omega^2 y = \frac{F_0}{m} \cos(\gamma t)$$

Damped Forced Vibrations

By the method of undetermined coefficients, we know that a particular solution of this equation will have the form

$$y_p(t) = A\cos(\gamma t) + B\sin(\gamma t),$$

which can be written as $y_p(t) = C\sin(\gamma t + \psi)$.

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$$y_p(t) = A\cos(\gamma t) + B\sin(\gamma t),$$

which can be written as $y_p(t) = C\sin(\gamma t + \psi)$.
Applying the method of undetermined coefficients, we obtain

$$y_p(t) = \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2\gamma^2}}\sin(\gamma t + \psi)$$

Damped Forced Vibrations

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Here our equation of motion will be

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_0}{m}\cos(\gamma t). \tag{9}$$

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$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_0}{m}\cos(\gamma t). \tag{9}$$

The method of undetermined coefficients will give a particular solution which we use to build the general solution to our equation of motion

$$y(t) = y_c(t) + \frac{F_0}{\sqrt{m^2(\omega^2 - \gamma^2)^2 + c^2\gamma^2}}\sin(\gamma t + \psi). \tag{10}$$

Here $y_c(t)$ is the general solution of the reduced equaton of (9) which we found in the damped free vibrations case. Recall, in all cases,

$$\lim_{t\to\infty} y_c(t) = 0.$$

Damped Forced Vibrations

Example:

1. Give the solution of the IVP below.

$$y'' + y' + \frac{101}{4}y = -200\cos(2t) - 4\sin(2t),\ y(0) = 4,\ y'(0) = \frac{3}{2}$$

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The characteristic equation of the homogeneous equation is

$$r^2 + r + \tfrac{101}{2} = 0$$

and the roots are

$$r = \frac{1 - \pm 1 - 101}{2} = \tfrac{1}{2} \pm 5$$

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and the roots are

$$r = \frac{1 - \pm 1 - 101}{2} = \tfrac{1}{2} \pm 5$$

Hence the general solution of the homogeneous equation is

$$y_h(t) = C_1 e^{-x/2}\cos(5t) + C_2 e^{-x/2}\sin(5t)$$

Damped Forced Vibrations

The particular solution is of the form

$$A\cos(2t) + B\sin(2t)$$

When we apply the method of undetermined coefficients we find $A = 2$ *and* $B = 0$. *Hence we get the general solution*

$$y = C_1 e^{-x/2}\cos(5t) + C_2 e^{-x/2}\sin(5t) + 2\cos(2t)$$

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$$y = C_1 e^{-x/2}\cos(5t) + C_2 e^{-x/2}\sin(5t) + 2\cos(2t)$$

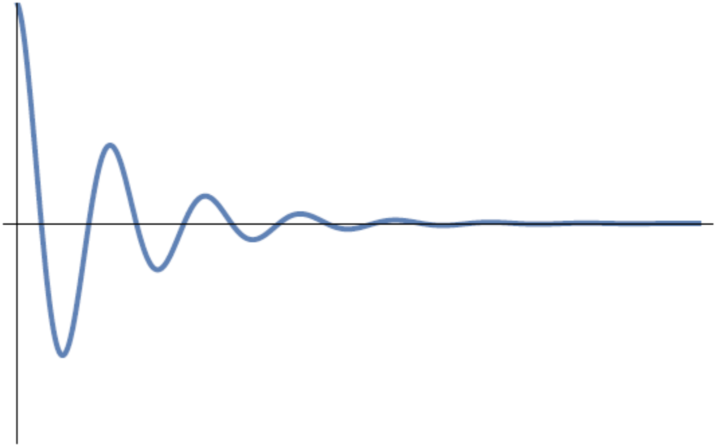
Imposing the initial conditions, we find $C_1 = 2$, $C_2 = 1/2$*. Hence we have the IVP solution*

$$y = 2e^{-x/2}\cos(5t) + \tfrac{1}{2}e^{-x/2}\sin(5t) + 2\cos(2t)$$

Damped Forced Vibrations

Below we see a graph of the transient solution

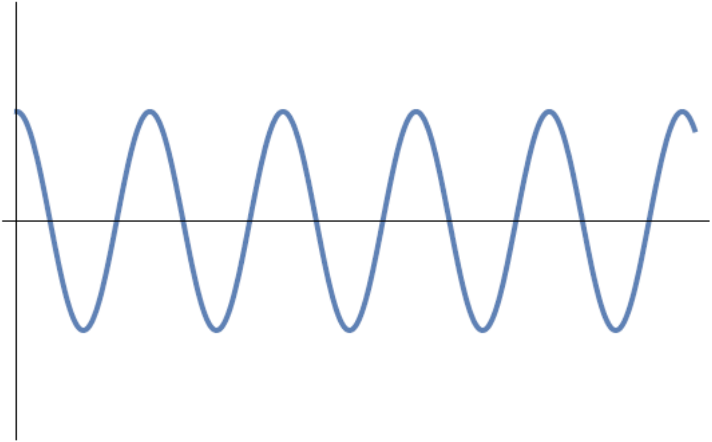
$$y_c(t) = 2e^{-t/2}\cos(5t) + \frac{1}{2}e^{-t/2}\sin(5t).$$



Damped Forced Vibrations

Below we see a graph of the steady state solution

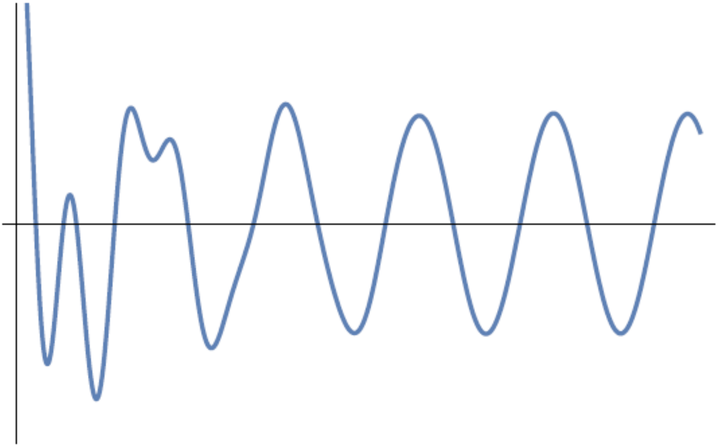
$$z(t) = 2\cos(2t).$$



Damped Forced Vibrations

Below we see a graph of the overall solution

$$y(t) = y_c(t) + z(t) = 2e^{-t/2}\cos(5t) + \frac{1}{2}e^{-t/2}\sin(5t) + 2\cos(2t).$$



Damped Forced Vibrations

Finally, we see the overall solution and the steady state solution graphed together.