Math 3321

Introduction to the Laplace Transform

University of Houston

Lecture 13

Outline

Background

In this lecture we will introduce an important method of solving initial-value problems for linear differential equations with constant coefficients. While this application is important for our purposes, it is far from the only use of the Laplace transform in the study of engineering and the sciences.

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In this lecture we will introduce an important method of solving initial-value problems for linear differential equations with constant coefficients. While this application is important for our purposes, it is far from the only use of the Laplace transform in the study of engineering and the sciences.

In order to understand the Laplace transform, we will need to make sure we understand **improper integrals** from calculus.

Background

In the definition of the definite integral, $\int_a^b f(x)dx$, it is assumed that $[a, b]$ is a finite closed interval and that f is defined and bounded on $[a, b]$. Even more, f is usually assumed to be continuous on $[a, b]$.

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Using limits, we are able to extend this concept to allow us to make sense of integrals of the form

$$\int_a^\infty f(x)dx$$

which we understand to mean

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

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$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx \\
&= \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Bigg|_1^b \\
&= \lim_{b\to\infty}\left(1-\frac{1}{b}\right) \\
&= 1
\end{aligned}$$

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$$\begin{aligned}
\int_0^\infty e^{-ax} dx &= \lim_{b\to\infty} \int_0^b e^{-ax} dx \\
&= \lim_{b\to\infty} \left(-\tfrac{1}{a} e^{-ax}\right)\Big|_0^b \\
&= \lim_{b\to\infty} \left(1 - \tfrac{1}{a} e^{-ab}\right) \\
&= 1
\end{aligned}$$

Introduction

The Laplace transform is defined in terms of an improper integral.

Definition

Let f be a continuous function on $[0, \infty)$. The Laplace transform of f, denoted by $\mathcal{L}[f(x)]$, or by $F(s)$, is the function given by

$$\mathcal{L}[f(x)] = F(s) = \int_0^\infty e^{-sx} f(x)dx. \tag{L}$$

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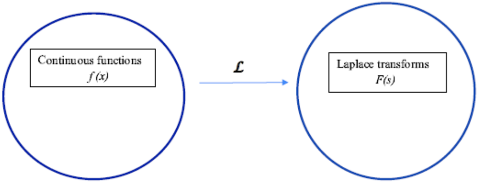
The Laplace transform $\mathcal{L}$ transforms a continuous function $f(x)$ into another function $F(s)$.



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Examples:

1. Let f be the constant function $f(x) \equiv 1$ for $x \in [0, \infty)$. Find the Laplace transform of $f(x)$.

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$$\begin{aligned}
\mathcal{L}[1] = F(s) &= \int_0^\infty e^{-sx} dx \\
&= \lim_{b \to \infty} \int_0^b e^{-sx} dx \\
&= \lim_{b \to \infty} \left(-\frac{1}{s} e^{-sx} \right) \Big|_0^b \\
&= \lim_{b \to \infty} \left(\frac{1}{s} - \frac{1}{s} e^{-sb} \right) \\
&= \frac{1}{s}
\end{aligned}$$

where, in the last step, we assume $s > 0$ to ensure convergence of the limit.

Introduction

2. Let $f(x) = e^{rx}$ for $x \in [0, \infty)$. Find the Laplace transform of $f(x)$.

$$\mathcal{L}[e^{rx}] = \int_0^\infty e^{-sx} \cdot e^{rx}\, dx = \lim_{b\to\infty} \int_0^b e^{-(s-r)x}\, dx = \lim_{b\to\infty} \left[\frac{e^{-(s-r)x}}{-(s-r)} \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{e^{-(s-r)b}}{-(s-r)} \right] + \frac{1}{s-r} = \lim_{b\to\infty} \left[\frac{-1}{e^{(s-r)b}(s-r)} \right] + \frac{1}{s-r}.$$

The limit exists (and has the value 0) if and only if $s - r > 0$. Therefore

$$\mathcal{L}[e^{rx}] = \frac{1}{s-r}, \quad s > r.$$

Note that if $r = 0$, then we have the result in Example 1 with $k = 1$. ■

3. Let $f(x) = \cos(\beta x)$ for $x \in [0, \infty)$. Find the Laplace transform of $f(x)$.

$$\mathcal{L}[\cos \beta x] = \int_0^\infty e^{-sx} \cos \beta x \, dx = \lim_{b \to \infty} \int_0^b e^{-sx} \cos \beta x \, dx$$

$$= \lim_{b \to \infty} \frac{e^{-sx}[-s \cos \beta x - \beta \sin \beta x]}{s^2 + \beta^2} \Bigg|_0^b .$$

(Note the integral was calculated using integration by parts. This integral is also a standard entry in a table of integrals.)

Now,

$$\mathcal{L}[\cos \beta x] = -\left[\lim_{b \to \infty} \frac{1}{e^{sb}} \cdot \frac{s \cos \beta b + \beta \sin \beta b}{s^2 + \beta^2} \right] + \frac{s}{s^2 + \beta^2}.$$

Since $(s \cos \beta b + \beta \sin \beta b)/(s^2 + \beta^2)$ is bounded and $1/e^{sb} \to 0$ when $s > 0$, the limit exists (and has the value 0). Therefore,

$$\mathcal{L}[\cos \beta x] = \frac{s}{s^2 + \beta^2}, \quad s > 0. \quad \blacksquare$$

Introduction

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
k (constant)	$\frac{k}{s}, \quad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

Basic Properties of the Laplace Transform

Our goal is to apply the Laplace transform to initial-value problems of the form:

$$y'' + ay' + by = f(x),\ y(0) = \alpha,\ y'(0) = \beta \tag{1}$$

where a, b, α, and β are constants and f is a continuous function on $[0, \infty)$.

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where a, b, α, and β are constants and f is a continuous function on $[0, \infty)$.
Our strategy will require us to find the Laplace transform of both sides of this differential equation. That is we wish to find

$$\mathcal{L}[y'' + ay' + by] = \mathcal{L}[f(x)]. \tag{2}$$

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Definition

A function f, continuous on $[0, \infty)$, is said to be of *exponential order* λ, λ a real number, if there exist numbers $M > 0$ and $A \geq 0$ such that for all $x \in [A, \infty)$ we have

$$|f(x)| \leq Me^{\lambda x}.$$

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(b) Let $f(x) = x$ for $x \in [0, \infty)$, then f is of exponential order λ for any positive number λ.

(c) Exponential functions are of exponential order. For example, let $f(x) = e^{2x}$, then f is of exponential order 2.

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(c) Exponential functions are of exponential order. For example, let $f(x) = e^{2x}$, then f is of exponential order 2.

(d) The function e^{x^2} is not of exponential order λ for any λ.

Basic Properties of the Laplace Transform

Theorem 1

Let f be a continuous function on $[0, \infty)$. If f is of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)]$ exists for $s > \lambda$.

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Theorem 2

The operator $\mathcal{L}$ is a linear operator. That is, if g and h are continuous functions on $[0, \infty)$, and if each of $\mathcal{L}[g(x)]$ and $\mathcal{L}[h(x)]$ exists for $s > \lambda$, then $\mathcal{L}[g(x) + h(x)]$ and $\mathcal{L}[cg(x)]$, c constant, each exist for $s > \lambda$, and

$$\mathcal{L}[g(x) + h(x)] = \mathcal{L}[g(x)] + \mathcal{L}[h(x)]$$

$$\mathcal{L}[cg(x)] = c\mathcal{L}[g(x)].$$

Basic Properties of the Laplace Transform

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$$\mathcal{L}[g(x) + h(x)] = \mathcal{L}[g(x)] + \mathcal{L}[h(x)]$$

$$\mathcal{L}[cg(x)] = c\mathcal{L}[g(x)].$$

The proof of Theorem 2 is a direct consequence of the linearity of integration.

Basic Properties of the Laplace Transform

Proof:

$$\begin{aligned}
\mathcal{L}[g(x)+h(x)] &= \int_0^\infty e^{-sx}[g(x)+h(x)]\,dx = \lim_{b\to\infty}\int_0^b e^{-sx}[g(x)+h(x)]\,dx \\
&= \lim_{b\to\infty}\left[\int_0^b e^{-sx}g(x)\,dx + \int_0^b e^{-sx}h(x)\,dx\right] \\
&= \lim_{b\to\infty}\int_0^b e^{-sx}g(x)\,dx + \lim_{b\to\infty}\int_0^b e^{-sx}h(x)\,dx \\
&= \int_0^\infty e^{-sx}g(x)\,dx + \int_0^\infty e^{-sx}h(x)\,dx = \mathcal{L}[g(x)] + \mathcal{L}[h(x)]
\end{aligned}$$

Corollary

Let $g_1(x)$, $g_2(x), \ldots,$ $g_n(x)$ be continuous functions on $[0, \infty)$. If $\mathcal{L}[g_1(x)]$, $\mathcal{L}[g_2(x)], \ldots,$ $\mathcal{L}[g_n(x)]$ all exist for $s > \lambda$, and if c_1, $c_2, \ldots,$ c_n are real numbers, then

$$\mathcal{L}[c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x)]$$

exists for $s > \lambda$ and

$$\mathcal{L}[c_1 g_1(x) + \cdots + c_n g_n(x)] = c_1 \mathcal{L}[g_1(x)] + \cdots + c_n \mathcal{L}[g_n(x)].$$

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$$\begin{aligned} \mathcal{L}[2\sin(x) - 3e^{-x} + 1] &= 2\mathcal{L}[\sin(x)] - 3\mathcal{L}[e^{-x}] + \mathcal{L}[1] \\ &= 2\frac{1}{s^2+1} - \frac{3}{s+1} + \frac{1}{s} \end{aligned}$$

2. Find the Laplace transform of

$$f(x) = 3e^{2x}\cos(3x) + x\sin(x) - 3x^2.$$

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$$\begin{aligned}\mathcal{L}[3e^{2x}\cos(3x) + x\sin(x) - 3x^2] &= 3\mathcal{L}[e^{2x}\cos(3x)] + \mathcal{L}[x\sin(x)] - 3\mathcal{L}[x^2] \\ &= 3\,\frac{s-3}{(s-2)^2+9} + \frac{2s}{(s^2+1)^2} - 3\,\frac{2}{s^3}\end{aligned}$$

Basic Properties of the Laplace Transform

Theorem 3

Let g be a continuously differentiable function on $[0, \infty)$. If g is of exponential order λ, then $\mathcal{L}[g'(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[g'(x)] = s\mathcal{L}[g(x)] - g(0).$$

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Remark. The fundamental implication of this property is that one can use the Laplace transform to map differential equations (in fact, IVPs) into algebraic equations with respect to the variable s.

Basic Properties of the Laplace Transform

Corollary

Let g be function which is n-times differentiable on $[0, \infty)$. If each of the functions g, $g', \ldots, g^{(n-1)}$ is of exponential order λ, then $\mathcal{L}[g^{(n)}(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[g^{(n)}(x)] = s^n \mathcal{L}[g(x)] - s^{n-1}g(0) - s^{n-2}g'(0) - \cdots - g^{(n-1)}(0).$$

Corollary

Let g be function which is n-times differentiable on $[0, \infty)$. If each of the functions g, $g', \ldots, g^{(n-1)}$ is of exponential order λ, then $\mathcal{L}[g^{(n)}(x)]$ exists for $s > \lambda$ and

$$\mathcal{L}[g^{(n)}(x)] = s^n \mathcal{L}[g(x)] - s^{n-1} g(0) - s^{n-2} g'(0) - \cdots - g^{(n-1)}(0).$$

It is worth mentioning the $n = 2$ case where we get

$$\mathcal{L}[g''(x)] = s^2 \mathcal{L}[g(x)] - s g(0) - g'(0).$$

Application to Initial-Value Problems

Examples:

1. Find the $\mathcal{L}[y(x)] = Y(s)$ for the solution of the IVP

$$y' - y = 3e^{2x},\ y(0) = 3.$$

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We apply the Laplace transform

$$\mathcal{L}[y' - y] = \mathcal{L}[3e^{2x}]$$

$$\mathcal{L}[y'] - \mathcal{L}[y] = 3\mathcal{L}[e^{2x}]$$

$$s\mathcal{L}[y] - y(0) - \mathcal{L}[y] = \frac{3}{s-2}$$

$$(s-1)\mathcal{L}[y] - 3 = \frac{3}{s-2}$$

$$Y(s) = \mathcal{L}[y] = \frac{3}{(s-2)(s-1)} + \frac{3}{s-1}$$

Application to Initial-Value Problems

2. Find the $\mathcal{L}[y(x)] = Y(s)$ for the solution of the IVP

$$y'' + 3y' - 4y = 2xe^{3x},\ y(0) = 3,\ y'(0) = -2.$$

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We apply the Laplace transform

$$\mathcal{L}[y'' + 3y' - 4y] = \mathcal{L}[2xe^{3x}]$$

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] - 4\mathcal{L}[y] = 2\mathcal{L}[xe^{3x}]$$

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3(s\mathcal{L}[y] - y(0)) - 4\mathcal{L}[y] = 2\tfrac{1}{(s-3)^2}$$

$$(s^2 + 3s - 4)\mathcal{L}[y] - (s+3)y(0) - y'(0) = \tfrac{2}{(s-3)^2}$$

$$(s-1)(s+4)\mathcal{L}[y] - 3(s+3) + 2 = \tfrac{2}{(s-3)^2}$$

$$\mathcal{L}[y] = \tfrac{2}{(s-3)^2(s-1)(s+4)} + \tfrac{3(s+3)}{(s-1)(s+4)} - \tfrac{2}{(s-1)(s+4)}$$