Math 3321

Inverse Laplace Transforms and Initial-Value Problems

University of Houston

Lecture 14

Outline

1. Introduction
2. The Inverse Laplace Transform
3. Solving Initial-Value Problems

Introduction

In the last lecture, we learned the definition and basic properties of the Laplace transform.

We applied the Laplace transform to the inital-value problem

$$y'' + ay' + by = f(x);\; y(0) = \alpha,\, y'(0) = \beta.$$

Introduction

In the last lecture, we learned the definition and basic properties of the Laplace transform.

We applied the Laplace transform to the inital-value problem

$$y'' + ay' + by = f(x);\ y(0) = \alpha,\, y'(0) = \beta.$$

This enabled us to solve for the Laplace transform of the solution to the IVP:

$$\mathcal{L}[y(x)] = Y(s) = \frac{F(s)}{s^2 + as + b} + \frac{\alpha s + \beta + a\alpha}{s^2 + as + b}$$

where $F(s) = \mathcal{L}[f(x)]$ is the Laplace transform of f.

Introduction

Key Observation

We found the Laplace transform can be applied to y' and y'', allowing us to express these in terms of $\mathcal{L}[y(x)] = Y(s)$ and given intial values:

$$\mathcal{L}[y'(x)] = sY(s) - y(0)$$

and

$$\mathcal{L}[y''(x)] = s^2Y(s) - sy(0) - y'(0).$$

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The general problem of finding a function with a given Laplace transform is called the *inversion problem.*
This inversion problem and its applications to solving inital-value problems is the topic of this lecture.

The Inverse Laplace Transform

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Theorem 1

Let f and g be continuous functions on $[0, \infty)$. Then $\mathcal{L}[f(x)] = \mathcal{L}[g(x)]$ if and only if $f(x) = g(x)$ for all $x \in [0, \infty)$.

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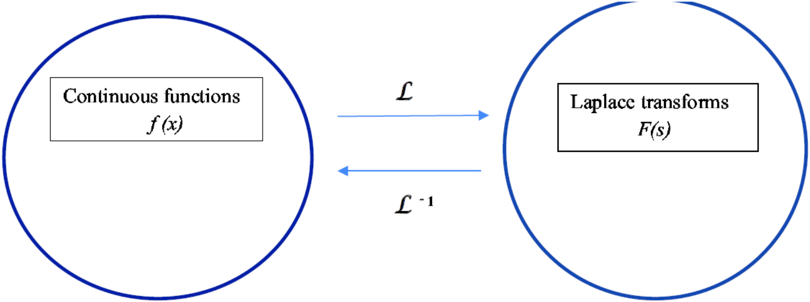
Definition

If $F(s)$ is a given transform and if the function f, continuous on $[0, \infty)$, has the property that $\mathcal{L}[f(x)] = F(s)$, then f is called the *inverse transform of* F, and is denoted by

$$f(x) = \mathcal{L}^{-1}[F(s)].$$

The operator $\mathcal{L}^{-1}$ is called the *inverse operator of* $\mathcal{L}$.

The Inverse Laplace Transform



The Inverse Laplace Transform

Theorem 2

The operator $\mathcal{L}^{-1}$ is linear:

$$\mathcal{L}^{-1}[F(s) + G(s)] = \mathcal{L}^{-1}[F(s)] + \mathcal{L}^{-1}[G(s)], \text{ and}$$

$$\mathcal{L}^{-1}[cF(s)] = c\mathcal{L}^{-1}[F(s)], \, c \text{ any constant.}$$

The Inverse Laplace Transform

Table of Laplace Transforms

$f(x)$	$F(s) = \mathcal{L}[f(x)]$
k (constant)	$\frac{k}{s}, \quad s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}, \quad s > \alpha$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}, \quad s > 0$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}, \quad s > 0$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s > \alpha$
$x^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$x^n e^{rx}, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-r)^{n+1}}, \quad s > r$
$x \cos \beta x$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}, \quad s > 0$
$x \sin \beta x$	$\frac{2\beta s}{(s^2+\beta^2)^2}, \quad s > 0$

The Inverse Laplace Transform

Examples:

1. Find $\mathcal{L}^{-1}[F(s)]$ if $F(s) = \dfrac{4}{s-2} + \dfrac{3s+2}{s^2+9}$.

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1. Find $\mathcal{L}^{-1}[F(s)]$ if $F(s) = \dfrac{4}{s-2} + \dfrac{3s+2}{s^2+9}$.

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{4}{s-2} + \frac{3s+2}{s^2+9}\right] &= 4\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 3\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] \\ &= 4\,e^{2x} + 3\,\cos(3x) + \frac{2}{3}\,\sin(3x) \end{aligned}$$

The Inverse Laplace Transform

2. Find $\mathcal{L}^{-1}[F(s)]$ if $F(s) = \dfrac{4}{(s-1)^3} + \dfrac{s}{s^2+2s+10}$.

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Observe that

$$\frac{s}{s^2+2s+10} = \frac{s}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+9} - \frac{1}{(s+1)^2+9}$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}[\frac{4}{(s-1)^3}] + \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+9}] - \mathcal{L}^{-1}[\frac{1}{(s+1)^2+9}] \\ &= 2\mathcal{L}^{-1}[\frac{2}{(s-1)^3}] + \mathcal{L}^{-1}[\frac{s+1}{(s+1)^2+9}] - \frac{1}{3}\mathcal{L}^{-1}[\frac{3}{(s+1)^2+9}] \\ &= 2x^2e^x + e^{-x}\cos(3x) - \tfrac{1}{3}e^{-x}\sin(3x) \end{aligned}$$

The Inverse Laplace Transform

3. Find $\mathcal{L}^{-1}[F(s)]$ if $F(s) = \dfrac{s^2 - 7s + 9}{(s-1)^2(s+2)}$.

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Observe that

$$\begin{aligned}\frac{s^2 - 7s + 9}{(s-1)^2(s+2)} &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2} \\ &= \frac{(A+C)s^2 + (A+B-2C)s - 2A + 2B + C}{(s-1)^2(s+2)}\end{aligned}$$

To find the coefficients of the partial fractions, we need to solve the system

$$\begin{cases} A + C = 1 \\ A + B - 2C = -7 \\ -2A + 2B + C = 9 \end{cases}$$

whose solution is $A = -2, B = 1, C = 3.$

Hence we can write

$$F(s) = \frac{-2}{s-1} + \frac{1}{(s-1)^2} + \frac{3}{s+2}$$

It follows that

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= -2\mathcal{L}^{-1}[\frac{1}{s-1}] + \mathcal{L}^{-1}[\frac{1}{(s-1)^2}] + 3\mathcal{L}^{-1}[\frac{1}{s+2}] \\ &= -2e^x + xe^x + 3e^{-2x}\end{aligned}$$

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Observe that

$$\begin{aligned}\frac{2}{(s+1)(s^2+1)} &= \frac{A}{s+1} + \frac{Bs+C}{s^2+1} \\ &= \frac{(A+B)s^2 + (B+C)s + A + C}{(s+1)(s^2+1)}\end{aligned}$$

To find the coefficients of the partial fractions, we need to solve the system

$$\begin{cases} A + B = 0 \\ B + C = 0 \\ A + C = 2 \end{cases}$$

whose solution is $A = 1, B = -1, C = 1$.

The Inverse Laplace Transform

Hence we can write

$$F(s) = \frac{1}{s+1} + \frac{1-s}{s^2+1}$$

It follows that

$$\begin{aligned}
\mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}[\frac{1}{s+1}] + \mathcal{L}^{-1}[\frac{1-s}{s^2+1}] \\
&= \mathcal{L}^{-1}[\frac{1}{s+1}] + \mathcal{L}^{-1}[\frac{1}{s^2+1}] - \mathcal{L}^{-1}[\frac{s}{s^2+1}] \\
&= e^{-x} + \sin(x) - \cos(x)
\end{aligned}$$

Solving Initial-Value Problems

Examples:

1. Find the solution to the IVP:

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We apply the Laplace transform

$$\begin{aligned}
\mathcal{L}[y' - 3y] &= \mathcal{L}[4\cos(2x)] \\
\mathcal{L}[y'] - 3\mathcal{L}[y] &= 4\mathcal{L}[\cos(2x)] \\
s\mathcal{L}[y] - y(0) - 3\mathcal{L}[y] &= 4\frac{s}{s^2+4} \\
(s-3)\mathcal{L}[y] &= 4\frac{s}{s^2+4} + 4
\end{aligned}$$

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Hence

$$\mathcal{L}[y] = Y(s) = \frac{4s}{(s^2+4)(s-3)} + \frac{4}{s-3} = \frac{4s^2+4s+16}{(s^2+4)(s-3)}$$

By partial fraction decomposition,

$$Y(s) = \frac{4s^2+4s+16}{(s^2+4)(s-3)} = \frac{A}{s-3} + \frac{Bs+C}{s^2+4}$$
$$= \frac{64/13}{s-3} + \frac{-12/13s+16/13}{s^2+4}$$

Hence, we have

$$y(x) = \tfrac{64}{13}\mathcal{L}^{-1}[\frac{1}{s-3}] + \tfrac{8}{13}\mathcal{L}^{-1}[\frac{2}{s^2+4}] - \tfrac{12}{13}\mathcal{L}^{-1}[\frac{s}{s^2+4}]$$

Computing the inverse Laplace transform we obtain

$$y(x) = \tfrac{64}{13}e^{3x} + \tfrac{8}{13}\sin(2x) - \tfrac{12}{13}\cos(2x)$$

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We apply the Laplace transform

$$\begin{aligned}
\mathcal{L}[y'' - y] &= \mathcal{L}[4e^x] \\
\mathcal{L}[y''] - \mathcal{L}[y] &= 4\mathcal{L}[e^x] \\
s^2\mathcal{L}[y] - sy(0) - y'(0) - \mathcal{L}[y] &= 4\tfrac{1}{s-1} \\
(s^2 - 1)\mathcal{L}[y] &= 4\tfrac{1}{s-1} + 2s
\end{aligned}$$

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\mathcal{L}[y''] - \mathcal{L}[y] &= 4\mathcal{L}[e^x] \\
s^2\mathcal{L}[y] - sy(0) - y'(0) - \mathcal{L}[y] &= 4\tfrac{1}{s-1} \\
(s^2 - 1)\mathcal{L}[y] &= 4\tfrac{1}{s-1} + 2s
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}[y] = Y(s) &= \frac{4}{(s^2-1)(s-1)} + \frac{2s}{s^2-1} \\
&= \frac{4}{(s+1)(s-1)^2} + \frac{2s}{(s-1)(s+1)} \\
&= \frac{2s^2 + 2s + 4}{(s+1)(s-1)^2}
\end{aligned}$$

Solving Initial-Value Problems

By partial fraction decomposition,

$$\begin{aligned}\frac{2s^2+2s+4}{(s+1)(s-1)^2} &= \frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s+1}\\ &= \frac{1}{s-1}+\frac{4}{(s-1)^2}+\frac{1}{s+1}\end{aligned}$$

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Hence, we can write

$$Y(s)=\frac{1}{s-1}+4\frac{1}{(s-1)^2}+\frac{1}{s+1}$$

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Hence, we can write

$$Y(s)=\frac{1}{s-1}+4\frac{1}{(s-1)^2}+\frac{1}{s+1}$$

Computing the inverse Laplace transform we get

$$y(x)=e^x+4xe^x+e^{-x}$$