Math 3321

Laplace Transforms of Piecewise Continuous Functions

University of Houston

Lecture 15

Outline

1. The Unit Step Function
2. Transforms of Piecewise Continuous Functions
3. Inverse Transforms and Piecewise Continuous Functions

The Unit Step Function

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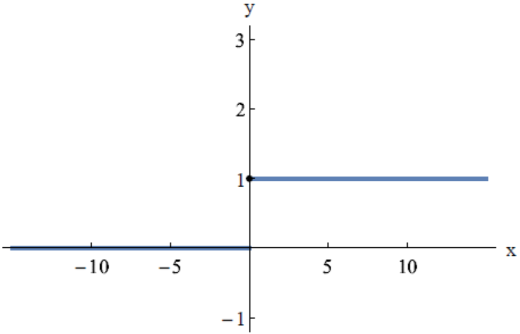
In this lecture we will consider the Laplace transform applied to certain typles of discontinuous functions.

The Unit Step Function

Our key tool in this lecture will be an important type of discontinuous function, that is the *unit step function*, also known as the *Heaviside function u*. This function is defined on $(-\infty, \infty)$ by

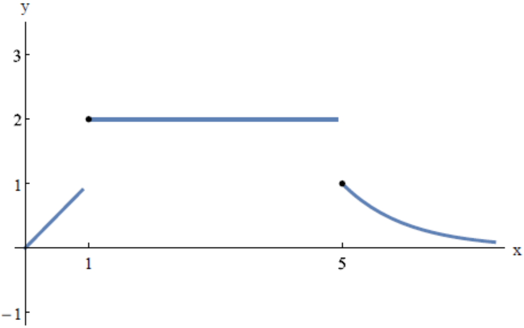
$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \tag{1}$$



Piecewise Continuous Functions

Definition

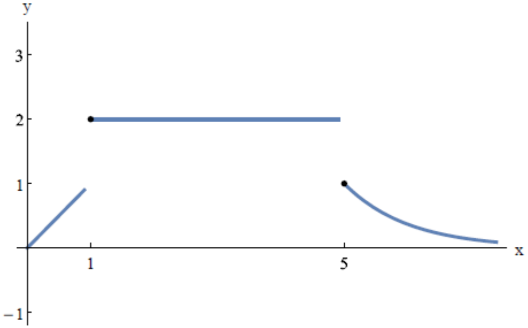
Let $f = f(x)$ be defined on an interval I and continuous except at a point $c \in I$, c not an endpoint of I. If the left-hand and right-hand limits of f at c both exist but are not equal, then f is said to have a *jump (or finite) discontinuity* at c.



Piecewise Continuous Functions

Definition

A function f defined on an interval I is *piecewise continuous on I* if it is continuous on I except for at most a finite number of points $c_1, c_2, \ldots, c_n \in I$ at which it has a jump discontinuity.



Piecewise Continuous Functions

Theorem 1

If the function f is piecewise continuous on $[0, \infty)$, and of exponential order λ, then the Laplace transform $\mathcal{L}[f(x)]$ exists for $s > \lambda$.

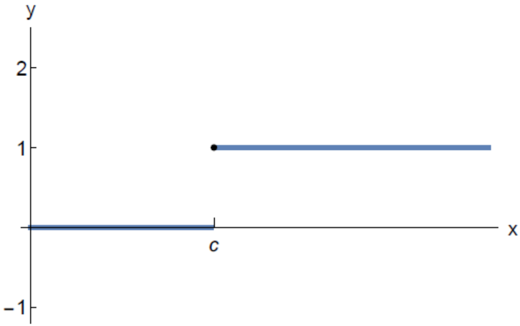
The calculation of the Laplace transform of a piecewise continuous can be carried out rather easily after learning how to compute the Laplace transform of the step function.

The Unit Step Function

Definition

Let $c > 0$ be a real number. The *translation of the unit step function* u by c is the function $u_c = u(x-c)$ defined on $[0,\infty)$ by

$$u_c(x) = u(x-c) = \begin{cases} 0, & 0 \le x < c \\ 1, & c \le x. \end{cases}$$



The Unit Step Function

Fact

The Laplace transform of $u_c(x) = u(x - c)$ is

$$\mathcal{L}[u(x-c)] = \frac{e^{-cs}}{s},\ s > 0. \tag{2}$$

Proof: By definition

$$\mathcal{L}[u(x-c)] = \int_0^\infty e^{-sx} u(x-c)\,dx = \int_0^c e^{-sx}\cdot 0\,dx + \int_c^\infty e^{-sx}\cdot 1\,dx$$

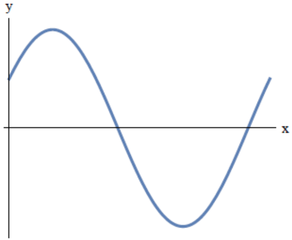
$$= \lim_{b\to\infty}\int_c^b e^{-sx}\,dx = \lim_{b\to\infty}\left[\frac{e^{-sx}}{-s}\right]_c^b = \lim_{b\to\infty}\frac{e^{-sb}}{-s} + \frac{e^{-sc}}{s} = \frac{e^{-cs}}{s},\ \ s > 0.$$

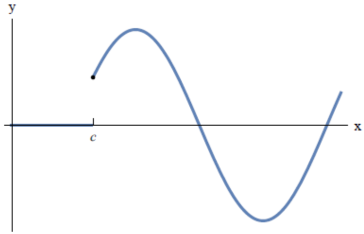
Note that if $c = 0$, then $u(x-0) = u(x) \equiv 1$ on $[0,\infty)$, and $\mathcal{L}[u(x)] = \mathcal{L}[1] = 1/s,\ s > 0$

The Unit Step Function

We can use the unit step function and its translations to build translations of a general function f. When f is defined on $[0, \infty)$ and $c > 0$, we can form the translation of f to the right c units below

$$f(x-c)u(x-c) = \begin{cases} 0, & 0 \le x < c \\ f(x-c), & c \le x. \end{cases}$$





Transforms of Piecewise Continuous Functions

Theorem 2

Let f be defined on $[0, \infty)$ and suppose $\mathcal{L}[f(x)] = F(s)$ exists for $s > \lambda$. Then $\mathcal{L}[f(x-c)u(x-c)]$ exists for $s > \lambda$ and is given by

$$\mathcal{L}[f(x-c)u(x-c)] = e^{-cs}F(s). \tag{3}$$

Proof: By the definition,

$$\mathcal{L}[f_c(x)] = \mathcal{L}[f(x-c)u(x-c)] = \int_0^\infty e^{-sx} f(x-c)u(x-c)\,dx$$

$$= \lim_{b\to\infty} \int_c^b e^{-sx} f(x-c)\,dx.$$

Now let $t = x - c$. Then

$$x = t + c, \qquad dx = dt, \qquad \text{and} \qquad t = 0 \quad \text{when} \quad x = c.$$

With this change of variable,

$$\lim_{b\to\infty}\int_c^b e^{-sx} f(x-c)\,dx = \lim_{b\to\infty}\int_0^{b-c} e^{-s(t+c)} f(t)\,dt$$

$$= e^{-cs}\lim_{b\to\infty}\left(\int_0^{b-c} e^{-st} f(t)\,dt\right) = e^{-cs}\int_0^{\infty} e^{-st} f(t)\,dt = e^{-cs}F(s)$$

since $b-c\to\infty$ as $b\to\infty$. ■

Transforms of Piecewise Continuous Functions

Examples:

1. Find the Laplace transform of $f(x) = \begin{cases} 2x, & 0 \leq x < 3 \\ 0, & 3 \leq x. \end{cases}$

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The key idea is to write f using the step function.
We have that (we always assume $x \geq 0$)

$$f(x) = 2x - 2x\,u(x-3)$$

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We have that

$$\mathcal{L}[f(x)] = 2\mathcal{L}[x] - 2\mathcal{L}[x\, u(x-3)]$$

To compute the second transform, it is convenient to write

$$x\, u(x-3) = (x-3)u(x-3) + 3u(x-3)$$

so that we can use Theorem 2.

Hence we have

$$\begin{aligned}\mathcal{L}[f(x)] &= 2\mathcal{L}[x] - 2\mathcal{L}[(x-3)u(x-3) + 3u(x-3)] \\ &= 2\mathcal{L}[x] - 2\mathcal{L}[(x-3)u(x-3)] - 6\mathcal{L}[u(x-3)]\end{aligned}$$

Since $\mathcal{L}[x] = \frac{1}{s^2}$, $\mathcal{L}[u(x-3)] = \frac{e^{-3s}}{s}$ *and* $\mathcal{L}[(x-3)u(x-3)] = e^{-3s}\frac{1}{s^2}$, *then*

$$\mathcal{L}[f(x)] = \frac{2}{s^2} - 2\frac{e^{-3s}}{s^2} - 6\frac{e^{-3s}}{s}$$

Transforms of Piecewise Continuous Functions

2. Find the Laplace transform of $f(x) = \begin{cases} 1, & 0 \leq x < 2 \\ x - 2, & 2 \leq x < 4 \\ x^2, & 4 \leq x. \end{cases}$

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Using the same idea as above, (assuming $x \ge 0$) we write f as

$$f(x) = f_1(x) + f_2(x) + f_3(x)$$

where each term is a continuous function on an interval, namely

$$\begin{aligned} f_1(x) &= 1 - 1\,u(x-2) \\ f_2(x) &= (x-2)u(x-2) - (x-2)u(x-4) \\ f_3(x) &= x^2 u(x-4) \end{aligned}$$

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To use Theorem 2 for the computation of f_2, it is convenient to write

$$(x-2)u(x-4) = (x-4)u(x-4) + 2u(x-4)$$

Hence

$$f_2(x) = (x-2)u(x-2) - (x-4)u(x-4) - 2u(x-4)$$

Transforms of Piecewise Continuous Functions

Similarly, to use Theorem 2 for the computation of f_3*, it is convenient to write* $x^2 = ((x-4)+4)^2 = (x-4)^2 + 8(x-4) + 16$*. Hence*

$$f_3(x) = x^2 u(x-4) = (x-4)^2 u(x-4) + 8(x-4)u(x-4) + 16u(x-4)$$

Now:

$$\mathcal{L}[f_1(x)] = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$\mathcal{L}[f_2(x)] = \frac{e^{-2s}}{s^2} - \frac{e^{-4s}}{s^2} - 2\frac{e^{-4s}}{s}$$

$$\mathcal{L}[f_3(x)] = 2\frac{e^{-4s}}{s^3} + 8\frac{e^{-4s}}{s^2} + 16\frac{e^{-4s}}{s}$$

Inverse Transforms and Piecewise Continuous Functions

Theorem 2 can be expressed equivalently in terms of the inverse Laplace transform.

Theorem 3

If $\mathcal{L}^{-1}[F(s)] = f(x)$ and $c > 0$, then

$$\mathcal{L}^{-1}[e^{-cs}F(s)] = f(x-c)u(x-c).$$

Examples:

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We can write

$$F(s) = \frac{e^{-2s}}{s(s+1)} = e^{-2s}\left(\frac{1}{s} - \frac{1}{s+1}\right) = e^{-2s}\left(F_1(s) + F_2(s)\right)$$

where $F_1(s) = \frac{1}{s}$ *and* $F_2(s) = -\frac{1}{s+1}$.

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where $F_1(s) = \frac{1}{s}$ *and* $F_2(s) = -\frac{1}{s+1}$.

Since $\mathcal{L}^{-1}[F_1(s)] = 1$ *and* $\mathcal{L}^{-1}[F_2(s)] = e^{-x}$, *then by Theorem 3*

$$\begin{aligned} f(x) = \mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}[e^{-2s}\frac{1}{s}] - \mathcal{L}^{-1}[e^{-2s}\frac{1}{s+1}] \\ &= u(x-2) - e^{-(x-2)}u(x-2) \end{aligned}$$

2. Find $f(x) = \mathcal{L}^{-1}[F(s)]$ where $F(s) = \dfrac{2}{s^3} + \dfrac{3e^{2-2s}}{(s-1)^2} + \dfrac{4e^{-\pi s}}{s^2+1}$.

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We write

$$F(s) = F_1(s) + F_2(s) + F_3(s)$$

where $F_1(s) = \frac{2}{s^3}$, $F_2(s) = 3e^2 e^{-2s}\frac{1}{(s-1)^2}$, $F_3(s) = 4e^{-\pi s}\frac{1}{s^2+1}$.

We have

$$\mathcal{L}^{-1}[F_1(s)] = \mathcal{L}^{-1}[\tfrac{2}{s^3}] = x^2$$

Since $\mathcal{L}^{-1}[\frac{1}{(s-1)^2}] = xe^x$, *using Theorem 3*

$$\mathcal{L}^{-1}[F_2(s)] = 3e^2\mathcal{L}^{-1}[e^{-2s}\frac{1}{(s-1)^2}] = 3e^2(x-2)e^{(x-2)}u(x-2)$$

using Theorem 3 and the observation that $\mathcal{L}^{-1}[\frac{1}{s^2+1}] = \sin x$,

$$\mathcal{L}^{-1}[F_3(s)] = 4\mathcal{L}^{-1}[e^{-\pi s}\frac{1}{s^2+1}] = 4\sin(x-\pi)u(x-\pi)$$

Hence, combining the 3 terms we have

$$f(x) = x^2 + 3e^x(x-2)u(x-2) + 4\sin(x-\pi)u(x-\pi)$$