Math 3321

Matrices and Vectors. Part I

University of Houston

Lecture 19

Outline

1. Matrices - Basic Definitions and Operations
2. Inverse Matrix
3. Determinants

Matrices

Definition

A $m \times n$ **matrix** is a $m \times n$ rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The elements a_{ij} are called the **entries** of the matrix.
$m \times n$ is called the **size** of the matrix, and the numbers m and n are its **dimensions**.
If $m = n$ the matrix is a **square matrix** of order n.

Shorter notation: $A = (a_{i,j})$ or $A = (a_{i,j})_{1 \le i \le m, 1 \le j \le n}$

Special Cases: Vectors

A $1 \times n$ matrix

$$v = (a_1 \; a_2 \; \dots \; a_n)$$

is called an **row vector**.

An $m \times 1$ matrix

$$v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

is called a **column vector**.

The entries of a row or column vector are called the **components** of the vector.

- **Equality**

Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be a $p \times q$ matrix. We say that $A = B$ if and only if
(i) $m = p$ and $n = q$;
(ii) $a_{ij} = b_{ij}$ for all i and j.
That is, $A = B$ if and only if A and B are identical.

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That is, $A = B$ if and only if A and B are identical.

Example:

$$\begin{pmatrix} a & b & 3 \\ 2 & c & 0 \end{pmatrix} = \begin{pmatrix} 7 & -1 & x \\ 2 & 4 & 0 \end{pmatrix}$$

if and only if

$$a = 7,\ b = -1,\ c = 4,\ x = 3.$$

- **Addition**

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices.
$A + B$ is the $m \times n$ matrix $C = (c_{ij})$ where

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i \text{ and } j.$$

That is,

$$A + B = (a_{ij} + b_{ij}).$$

Addition of matrices is only defined for matrices of same size

Examples:

(a)

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 & 6 \\ -1 & 2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 2+(-4) & 4+0 & -3+6 \\ 2+(-1) & 5+2 & 0+0 \end{pmatrix} = \begin{pmatrix} -2 & 4 & 3 \\ 1 & 7 & 0 \end{pmatrix}$$

Operation of Matrices

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(a)

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(b)

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & -3 \\ 0 & 6 \end{pmatrix} \quad \text{is not defined.}$$

Operation of Matrices

PROPERTIES:

Let A, B, and C be matrices of the same size. Then:

- $A + B = B + A$ (Commutative)
- $(A + B) + C = A + (B + C)$ (Associative)
- $A + \mathbf{O} = \mathbf{O} + A = A$ (Identity)

Here the symbol **O** will be used to denote the **zero matrix** of arbitrary size. The zero matrix is the **additive identity**.

A matrix with all entries equal to 0 is called a **zero matrix**. E.g.,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The **negative of a matrix** A, denoted by $-A$, is the matrix whose entries are the negatives of the entries of A.

$-A$ is also called the **additive inverse of** A.

Example:

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 2 & 0 & 6 \\ -4 & -1 & 5 \end{pmatrix}$$

$$-A = \begin{pmatrix} -1 & -7 & 2 \\ -2 & 0 & -6 \\ 4 & 1 & -5 \end{pmatrix}$$

Operation of Matrices

- **Subtraction**

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then

$$A - B = A + (-B).$$

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Example:

$$\begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} - \begin{pmatrix} 4 & 0 & -6 \\ 1 & -2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 4 & -3 \\ 2 & 5 & 0 \end{pmatrix} + \begin{pmatrix} -4 & 0 & 6 \\ -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 & 3 \\ 1 & 7 & 0 \end{pmatrix}$$

PROPERTIES:

Let A, B, and C be matrices of the same size. Then:

- $A + B = B + A$ (commutative)
- $(A + B) + C = A + (B + C)$ (associative)
- $A + \mathbf{O} = \mathbf{O} + A = A$ (additive identity)
- $A + (-A) = (-A) + A = \mathbf{O}$ (additive inverse)

Operation of Matrices

- **Multiplication of a Matrix by a Number**

The **product of a number** k **and a matrix** A, denoted kA, is given by

$$kA = (ka_{ij}).$$

This product is called **multiplication by a scalar**.

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Examples:

$2\,(1\ 2\ 3) = (2\ 4\ 6)$

$$-3\begin{pmatrix} 2 & -1 & 4 \\ 1 & 5 & -2 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 3 & -12 \\ -3 & -15 & 6 \\ -12 & 0 & -9 \end{pmatrix}$$

PROPERTIES:

Let A, B be $m \times n$ matrices and let α, β be real numbers. Then

- $1\,A = A$
- $0\,A = \mathbf{O}$
- $\alpha(A + B) = \alpha\,A + \alpha\,B$
- $(\alpha + \beta)A = \alpha\,A + \beta\,A$

The last 2 properties are called **distributive** laws.

• **The Product of Two Matrices**

Let A and B be given matrices.

The product AB, in that order, is defined if and only if the number of columns of A equals the number of rows of B.

If the product AB is defined, then the size of the product is a matrix

$$\underset{m\times p}{A}\ \underset{p\times n}{B} = \underset{m\times n}{C}$$

whose dimension is (no. of rows of A)$\times$(no. of columns of B):

Operation of Matrices

Matrix multiplication, when defined, is computed by row-column multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 10 + 2\times 20 + 3\times 30 & 1\times 11 + 2\times 21 + 3\times 31 \\ 4\times 10 + 5\times 20 + 6\times 30 & 4\times 11 + 5\times 21 + 6\times 31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

Operation of Matrices

Example:

Let $A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -2 \end{pmatrix}$

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$$\begin{aligned} AB &= \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} (1)(3)+(4)(-1)+(2)(1) & (1)(0)+(4)(2)+(2)(-2) \\ (3)(3)+(1)(-1)+(5)(1) & (3)(0)+(1)(2)+(5)(-2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ 13 & -8 \end{pmatrix} \end{aligned}$$

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Note:

$$\underset{2\times 3}{A}\ \underset{3\times 2}{B} = \underset{2\times 2}{C}$$

Operation of Matrices

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Using the same matrices as the last example, we obtain:

$$\begin{aligned} BA &= \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 3+0 & 12+0 & 6+0 \\ -1+6 & -4+2 & -2+10 \\ 1-6 & 4-2 & 2-10 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 12 & 6 \\ 5 & -2 & 8 \\ -5 & 2 & -8 \end{pmatrix} \end{aligned}$$

Note that $AB \neq BA$, they even have different dimensions.
In general, if AB is defined, BA is not even guaranteed to exist.

Operation of Matrices

PROPERTIES:

Let A, B, and C be matrices.

- $AB \neq BA$ in general;
- $(AB)C = A(BC)$ matrix multiplication is associative (when the multiplications are defined).
- If A is an $m \times n$ matrix, then

$$I_m A = A \text{ and } A I_n = A.$$

where I_n is the $n \times n$ **identity matrix** defined by

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

PROPERTIES:

Let A, B, and C be matrices. Assuming the following operations are well defined, the following properties hold.

- $A(B+C) = AB + AC$. This is called the *left distributive law.*
- $(A+B)C = AC + BC$. This is called the *right distributive law.*
- $k(AB) = (kA)B = A(kB)$

Operation of Matrices

The **product between two vectors** is a special case of the matrix multiplication.

The product of a $1 \times n$ row vector and an $n \times 1$ column vector is the **number** given by

$$(a_1,\ a_2,\ a_3,\ \ldots,\ a_n)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$= a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_nb_n.$$

This is called the **dot product** or **inner product**. It maps two vectors of the same length to a number.

Operation of Matrices

The product of $n \times 1$ column vector and a $1 \times m$ row vector is a $n \times m$ **matrix**.

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} (b_1,\ b_2,\ b_3,\ \ldots,\ b_m)$$

$$= \begin{pmatrix} a_1b_1 & a_1b_2 & \ldots & a_1b_m \\ a_2b_1 & a_2b_2 & \ldots & a_2b_m \\ \vdots & \vdots & \vdots & \vdots \\ a_nb_1 & a_nb_2 & \ldots & a_nb_m \end{pmatrix}$$

Operation of Matrices

Examples:

$$(2, 1, -1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2 + 1 = 3$$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}(1, 0, -1) = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Inverse of a Matrix

Given a matrix A, a matrix B is a *multiplicative inverse* of A if

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- AB and BA might not both exist.
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- If AB and BA both exist and have the same size, $AB \neq BA$, in general.

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Fact:
- If AB and BA both exist and have the same size, then A and B must be square.

Inverse of a Matrix

Let A be an $n \times n$ matrix. An $n \times n$ matrix B with the property that

$$AB = BA = I_n$$

is called the **multiplicative inverse of** A or, more simply, **the inverse of** A.

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Uniqueness

If A has an inverse, then it is unique. That is, there is one and only one matrix B such that

$$AB = BA = I.$$

B is denoted by A^{-1}.

Inverse of a Matrix

Because of the way we defined multiplication, a system of linear equations can be written as:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$

or in the vector-matrix form

$$Ax = b$$

This suggest that we can formally write the solution as $x = A^{-1}b$

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• **Procedure for finding the inverse of a matrix**

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We want ro solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

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We split the problem into two parts

1) Solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & -2 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3/2 \end{array}\right)$$

This shows that $\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3/2 \end{pmatrix}$

Inverse of a Matrix

Inverse of a Matrix

2) Now solve $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 1 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{array}\right)$$

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$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1/2 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1/2 \end{array} \right)$$

This shows that $\begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

Hence: $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

Inverse of a Matrix

Example.

Compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

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We can solve the two systems described above simultaneously applying the method of Gaussian elimination:

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$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

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This method gives that: $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

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Example.

Compute the inverse of $A = \begin{pmatrix} 2 & -8 \\ -1 & 6 \end{pmatrix}$

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$$\left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ -4 & 2 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

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In this problem, it is not possible to find the inverse matrix.

A does not have an inverse.

Inverse of a Matrix

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$$\left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ -4 & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

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A does not have an inverse.

NOTE: Not every nonzero $n \times n$ matrix A has an inverse!

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$$\left(\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right) \to$$
$$\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right)$$

Inverse of a Matrix

Example. Compute the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$

Inverse of a Matrix

We can summarize the method for **finding the inverse of a** $n \times n$ **matrix** A as follows.

Let A be an $n \times n$ matrix.

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1. Form the augmented matrix $(A|I_n)$.
2. Reduce $(A|I_n)$ to reduced row echelon form.
3. If the reduced row echelon form is

$$(I_n|B), \quad \text{then} \quad B = A^{-1}$$

   If the reduced row echelon form is **not** $(I_n|B)$, then A does not have an inverse. That is, if the reduced row echelon form of A is not the identity, then A does not have an inverse.

Application:

Solve the system

$$\begin{array}{rrrcr} x_1 & & +2x_3 & = & 3 \\ 2x_1 & -x_2 & +3x_3 & = & 5 \\ 4x_1 & +x_2 & +8x_3 & = & -2 \end{array}$$

We can write the system in matrix form:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$$

Inverse of a Matrix

Existence of solution

The $n \times n$ system of equations

$$Ax = b$$

has a unique solution if and only if the matrix of coefficients, A, has an inverse.

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If the matrix A is invertible, the linear system $Ax = b$ has solution $x = A^{-1}\, b$. That is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} -29 \\ -14 \\ 15 \end{pmatrix}$$

Determinants

The **determinant** is a special number that can be calculated from a matrix.

- 1×1 matrix $A = (a)$
$$\det A = a$$
- 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Determinants

- 3×3 matrix $\quad A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

  $$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

  $$a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1$$

Determinants

- 3×3 matrix $\quad A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

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  $$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1$$

  Re-write as:

  $$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

  $$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

  This is called the **expansion across the first row**.

Determinants

There are alternative ways to compute the determinants of a 3×3 matrix.

$$\det A = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

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Re-write as:

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1) =$$

$$-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

This is called the **expansion down the second column**.

You can expand across any row, or down any column, as long as each position is associated with an algebraic sign according to table below:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

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For example, across the second row:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$$

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or down the first column:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

Example: $A = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{pmatrix}$

We compute the determinant by expansion across the first row

$$\begin{aligned} \begin{vmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{vmatrix} &= 2\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 2 \\ -2 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ -2 & 1 \end{vmatrix} \\ &= 2(12-2) - (-3)(0+4) + 3(0+8) \\ &= 2(10) - (-3)(4) + 3(8) = 56 \end{aligned}$$

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Alternatively, we compute the determinant by expansion down the first column (where there are only 2 non-zero entries)

$$\begin{aligned} \begin{vmatrix} 2 & -3 & 3 \\ 0 & 4 & 2 \\ -2 & 1 & 3 \end{vmatrix} &= 2\begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} -3 & 3 \\ 4 & 2 \end{vmatrix} \\ &= 2(12-2) - 2(-6-12) \\ &= 2(10) - 2(-18) = 56 \end{aligned}$$

Determinants

Same idea applies for the determinants of larger matrices

- 4×4 matrix $\quad A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$

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  The determinant $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$ is computed by expansion into 3×3 determinants using the sign chart

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

Properties of Determinants

Properties of Determinants

- Let A be an $n \times n$ matrix.
  If A has a row or column of zeros, then $\det A = 0$

Example:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & 3 \\ 4 & 0 & 8 \end{pmatrix}$$

Expand down second column:

$$-0\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 0$$

Properties of Determinants

- If A is a diagonal matrix,

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix},$$

$$\det A = a_1 \begin{vmatrix} b_2 & 0 \\ 0 & c_3 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 0 & c_3 \end{vmatrix} + 0 \begin{vmatrix} 0 & b_2 \\ 0 & 0 \end{vmatrix} = a_1 \cdot b_2 \cdot c_3.$$

In particular, $\det I_n = 1$
For example, for I_3 we have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Properties of Determinants

- If A is a triangular matrix, e.g.,

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & b_2 & b_3 & b_4 \\ 0 & 0 & c_3 & c_4 \\ 0 & 0 & 0 & d_4 \end{pmatrix} \quad \text{(upper triangular)}$$

  then $\det A = a_1 \cdot b_2 \cdot c_3 \cdot d_4$.

This is easily shown by expanding down first column.

Properties of Determinants

- Let A and B be $n \times n$ matrices. Then

$$\det[AB] = \det A \det B.$$

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- If a row (column) of A is multiplied by a nonzero number k to obtain a matrix B. Then
$$\det B = k \det A.$$
- If a row (column) of A is multiplied by a number k and added to another row (column) to obtain a matrix B. Then
$$\det B = \det A.$$

An application of determinants. Find the solution of the linear system:

$$\begin{aligned} ax_1 + bx_2 &= \alpha \\ cx_1 + dx_2 &= \beta \end{aligned}$$

Linear systems of equations

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\end{aligned}
$$

In matrix notation:

$$
Ax = b, \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, b = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
$$

If A has an inverse then direct calculation shows

$$
A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
$$

Also

$$
x = \frac{\begin{vmatrix} \alpha & b \\ \beta & d \end{vmatrix}}{\det A}, \quad y = \frac{\begin{vmatrix} a & \alpha \\ c & \beta \end{vmatrix}}{\det A}
$$

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- If $\det A = 0$, then the system either has infinitely many solutions or no solutions.

Definition

- A is **nonsingular** if $\det A \neq 0$;
- A is **singular** if $\det A = 0$.

Theorem

The following statements are equivalent:

1. The system of equations: $Ax = b$ has a unique solution.
2. The reduced row echelon form of A is I_n.
3. The rank of A is n.
4. A has an inverse.
5. $\det A \neq 0$.

Cramer's Rule & systems of equations.

Given a system of n linear equations in n unknowns:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n & = & b_3 \\ \ldots \quad \ldots & = & \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array}$$

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The solutions can be written as

$$x_i = \frac{\det A_i}{\det A}, \qquad \text{provided} \quad \det A \neq 0,$$

where A_i is the matrix A with the i^{th} column replaced by the vector b.

Example

Find the x_2 solution system using Cramer's rule, if the method applies

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & = & 3 \\ 2x_1 & - & x_2 & + & x_3 & = & 5 \\ -4x_1 & + & x_2 & - & 2x_3 & = & -2 \end{array}$$

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We have

$$\begin{aligned} \det A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & -1 & 1 \\ -4 & 1 & -2 \end{vmatrix} &= 1\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} \\ &= (2-1) - 2(-4+4) + 2(2-4) = -2 \end{aligned}$$

Since $\det A \neq 0$, Cramer's method applies.

Linear systems of equations

$$\det A_2 = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ -4 & -2 & -2 \end{vmatrix} \begin{aligned} &= 1\begin{vmatrix} 5 & 1 \\ -2 & -2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ -4 & -2 \end{vmatrix} \\ &= (-10+2) - 3(-4+4) + 2(-4+20) = 24 \end{aligned}$$

Linear systems of equations

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Hence

$$x_2 = \frac{\det A_2}{\det A} = \frac{24}{-2} = -12$$

Linear systems of equations

Example

Find the x_2 solution system using Cramer's rule, if the method applies

$$\begin{array}{rcrcrcr} -2x_1 & + & 7x_2 & + & 6x_3 & = & -1 \\ 5x_1 & + & x_2 & - & 2x_3 & = & 7 \\ 3x_1 & + & 8x_2 & + & 4x_3 & = & -1 \end{array}$$

$$\det A = \begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix} = 0$$

Hence Cramer's method does not apply.