Math 3321

Matrices and Vectors. Part II

University of Houston

Lecture 20

Outline

1. Vector Spaces
2. Linear Dependence/Independence

Vectors

Definition

A **vector space** is a set whose elements, called vectors, may be added together and multiplied by numbers, called scalars.

Examples

$\mathbb{R}^2 = \{(a, b) : a, b \in \mathbb{R}\}$ – "the plane"

$\mathbb{R}^3 = \{(a, b, c) : a, b, c \in \mathbb{R}\}$ – "3-space"

$\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n)\}$ ordered n-tuples of real numbers

For all these spaces, vector addition is defined componentwise.

Vectors

For any two vectors $u = (a_1,\ a_2,\ \ldots, a_n)$ and $v = (b_1,\ b_2,\ \ldots, b_n)$ in $\mathbb{R}^n$, we have

$$\begin{aligned} u + v &= (a_1,\ a_2,\ \ldots, a_n) + (b_1,\ b_2,\ \ldots, b_n) \\ &= (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n) \end{aligned}$$

and for any real number λ,

$$\begin{aligned} \lambda\, v &= \lambda\,(a_1,\ a_2,\ \ldots, a_n) \\ &= (\lambda a_1, \lambda a_2,\ \ldots, \lambda a_n). \end{aligned}$$

Clearly, the sum of two vectors in $\mathbb{R}^n$ is another vector in $\mathbb{R}^n$ and a scalar multiple of a vector in $\mathbb{R}^n$ is a vector in $\mathbb{R}^n$.

Properties of vector addition

Let u, v and w be vectors in $\mathbb{R}^n$.

1. $u + v \in \mathbb{R}^n$ **closed**
2. $u + v = v + u$ **commutative**
3. $u + (v + w) = (u + v) + w$ **associative**

4. **Zero vector:** There is a vector $0 = (0,\ 0,\ 0, \ldots,\ 0)$,

$$v + 0 = 0 + v = v \quad \text{(additive identity)}.$$

5. **Additive Inverse:** For each vector $v \in \mathbb{R}^n$, there is a unique vector $-v$ such that

$$v + (-v) = v - v = 0$$

$-v$ is the additive inverse (or negative) of v.

Properties of scalar multiplication

Let α and β be numbers, and u, v and w be vectors.

1. $\alpha\, w \in \mathbb{R}^n$ (closed)
2. $1w = w$ (multiplicative identity)
3. $\alpha(\beta\, w) = (\alpha\beta)\, w$ (associative property)
4. $(\alpha + \beta)w = \alpha\, w + \beta\, w$ (distributive property)
5. $\alpha(u + w) = \alpha\, u + \alpha\, w$ (distributive property)

Vector spaces

While we have exemplified the properties of vectors using $\mathbb{R}^n$, the concept of vector space is much more general.

Any non-empty set V on which there are defined two operations, addition (+) and multiplication by a scalar, which satisfy the properties 1 - 5 for addition and 1 - 5 for multiplication by a scalar is a vector space.

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Other examples of vector spaces:

1. $C(0,1) = \{f : (0,1) \to \mathbb{R} \,|\, f \text{ is continuous}\}$
2. The set P_n of polynomials of degree n.
3. The set S of solutions of the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

Linear Dependence/Independence

Let V be a vector space and let

$$\{v_1,\ v_2,\ v_3, \cdots,\ v_k\}$$

be a set of vectors in V. Let

$$\{c_1,\ c_2,\ c_3, \cdots,\ c_k\}$$

be real numbers.
Then

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_k v_k$$

is a **linear combination** of $v_1, \ldots, v_k$.

Linear Dependence/Independence

Definition

Let $\mathcal{S} = \{v_1, v_2, \cdots, v_k\}$ be a set of vectors.
The set $\mathcal{S}$ is **linearly dependent** if there exist k numbers $c_1,\ c_2,\ \cdots, c_k$ **not all zero** such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}.$$

($c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$ is a **linear combination** of $v_1,\ v_2, \ldots,\ v_k$)
$\mathcal{S}$ is **linearly independent** if it is not linearly dependent. That is, $\mathcal{S}$ is linearly independent if

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}$$

implies $c_1 = c_2 = \cdots = c_k = 0.$

Linear Dependence/Independence

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Another way to say this:
The set $\mathcal{S}$ is **linearly dependent** if one of the vectors can be written as a linear combination of the other vectors.
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$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = \mathbf{O}.$$

Note: If there is one such set

$$\{c_1,\ c_2,\ c_3,\ \ldots,\ c_k\},$$

then there are **infinitely many** such sets.

Linear Dependence/Independence

Special case: 2 vectors $v_1, \ v_2$.

Linearly dependent iff one vector is a constant multiple of the other.

Linear Dependence/Independence

Special case: 2 vectors $v_1,\ v_2$.

Linearly dependent iff one vector is a constant multiple of the other.

Examples:

- $v_1 = (1, -2, 4), \quad v_2 = (-\frac{1}{2}, 1, -2)$
  linearly dependent: $v_1 = -2v_2$
- $v_1 = (2, -4, 5), \quad v_2 = (0, 0, 0)$
  linearly dependent: $v_2 = 0v_1$
- $v_1 = (5, -2, 0), \quad v_2 = (-3, 1, 9)$
  linearly independent

Linear Dependence/Independence

Problem:
Given the three vectors $v_1,\ v_2,\ v_3$ in $\mathbb{R}^2$:

$$v_1 = (1, -1),\ v_2 = (-2, 3),\ v_3 = (3, -5)$$

is the set $\{v_1,\ v_2,\ v_3\}$ dependent or independent?

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is the set $\{v_1,\ v_2,\ v_3\}$ dependent or independent?

Solution. Suppose they are dependent. Then there exist three numbers, c_1, c_2, c_3, not all zero such that

$$c_1v_1 + c_2v_2 + c_3v_3 = (0, 0)$$

This implies that

$$c_1(1, -1) + c_2(-2, 3) + c_3(3, -5) = 0$$

giving the system of equations

$$\begin{aligned} c_1 - 2c_2 + 3c_3 &= 0 \\ -c_1 + 3c_2 - 5c_3 &= 0 \end{aligned}$$

Linear Dependence/Independence

Hence, if we suppose that $\{v_1,\ v_2,\ v_3\}$ is a dependent set, the system of equations

$$\begin{aligned} c_1 - 2c_2 + 3c_3 &= 0 \\ -c_1 + 3c_2 - 5c_3 &= 0 \end{aligned}$$

must have nontrivial solutions.

Linear Dependence/Independence

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must have nontrivial solutions.

We have:
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 3 & -5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -2/5 & 0 \end{array}\right)$$

This shows that the system has non-trivial solution.
Hence $\{v_1,\ v_2,\ v_3\}$ is a **dependent** set.

Linear Dependence/Independence

Problem:
Given the four vectors: $v_1,\ v_2,\ v_3,\ v_4$ in $\mathbb{R}^3$
$v_1 = (1, -1, 2), \quad v_2 = (2, -3, 0), \quad v_3 = (-1, -2, 2), \quad v_4 = (0, 4, -3)$
is the set $\{v_1,\ v_2,\ v_3, ,\ v_4\}$ dependent or independent?

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is the set $\{v_1,\ v_2,\ v_3, ,\ v_4\}$ dependent or independent?

Solution.
As we saw above, solving this problem requires to check if there are 4 real numbers c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = (0, 0, 0)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ -1 & -3 & -2 & 4 & 0 \\ 2 & 0 & 2 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -3 & 4 & 0 \\ 0 & -4 & 4 & -3 & 0 \end{array}\right) \to$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -3 & 4 & 0 \\ 0 & 0 & -4 & -13/4 & 0 \end{array}\right)$$

Hence it is a **dependent** set.

Linear Dependence/Independence

The examples above show that a homogeneous system with more unknowns than equations always has infinitely many nontrivial solutions.

This implies the following useful observation

Proposition (Linear dependence)

Let $v_1, v_2, \cdots, v_k$ be a set of k vectors in $\mathbb{R}^n$. If $k > n$, then the set of vectors is (automatically) linearly dependent.

Linear Dependence/Independence

Example. Consider the following vector in $\mathbb{R}^3$:
$v_1 = (1, -1, 2), \quad v_2 = (2, -3, 0), \quad v_3 = (-1, -2, 2), \quad v_4 = (0, 4, -3)$
Are the following sets dependent or independent?

(a) $\{v_1,\ v_2,\ v_3,\ v_4\}$
(b) $\{v_1,\ v_2,\ v_3\}$
(c) $\{v_1,\ v_2\}$

Linear Dependence/Independence

Example. Consider the following vector in $\mathbb{R}^3$:
$v_1 = (1, -1, 2), \quad v_2 = (2, -3, 0), \quad v_3 = (-1, -2, 2), \quad v_4 = (0, 4, -3)$
Are the following sets dependent or independent?

(a) $\{v_1,\ v_2,\ v_3,\ v_4\}$
(b) $\{v_1,\ v_2,\ v_3\}$
(c) $\{v_1,\ v_2\}$

Solution.
(a) Since the set $\{v_1,\ v_2,\ v_3,\ v_4\}$ consists of 4 vectors in $\mathbb{R}^3$, it is necessarily **dependent**.

(c) Since v_1 is not a constant multiple of v_2, then the set $\{v_1,\ v_2\}$ is **independent**.

Linear Dependence/Independence

(b) In this case, we need to check directly if the equation $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ has non-trivial solutions. That is, does

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 0 \\ -c_1 - 3c_2 - 2c_3 &= 0 \\ 2c_1 + 2c_3 &= 0 \end{aligned}$$

have non-trivial solutions?

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have non-trivial solutions?

Augmented matrix and row reduce:

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ -1 & -3 & -2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

This shows that the system of equation has only the trivial solution. Thus, the vectors $\{v_1,\ v_2,\ v_3\}$ are **independent**.

Linear Dependence/Independence

Alternatively, to solve problem (b), we can calculate the determinant

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & -2 \\ 2 & 0 & 2 \end{vmatrix}$$

We have $\begin{vmatrix} 1 & 2 & -1 \\ -1 & -3 & -2 \\ 2 & 0 & 2 \end{vmatrix} = 2(-4-3) + 2(-3+2) = -16$

$\det \neq 0$ implies unique solution $c_1 = c_2 = c_3 = 0$ and **independent**.

Linear Dependence/Independence

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$\det \neq 0$ implies unique solution $c_1 = c_2 = c_3 = 0$ and **independent**.

In general,

- $\det \neq 0$ implies unique solution and **independent**
- $\det = 0$ implies infinitely many solutions and **dependent**.

Linear Dependence/Independence

Example. Consider the vectors in $\mathbb{R}^3$
$w_1 = (a, 1, -1), \quad w_2 = (-1, 2a, 3), \quad w_3 = (-2, a, 2), \quad w_4 = (3a, -2, a)$
For what values of a are the vectors linearly dependent?

Linear Dependence/Independence

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For what values of a are the vectors linearly dependent?

Solution.
Since the set $\{w_1,\ w_2,\ w_3,\ w_4\}$ consists of 4 vectors in $\mathbb{R}^3$, it is necessarily **dependent**.

Linear Dependence/Independence

Example. Consider the vectors in $\mathbb{R}^3$
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$w_1 = (a, 1, -1), \quad w_2 = (-1, 2a, 3), \quad w_3 = (-2, a, 2)$
For what values of a are the vectors linearly dependent?

Solution.
We can solve the problem by applying row reduction on the matrix of coefficients or by computing the determinant (we can write the vectors by row or column).

$$\begin{vmatrix} a & -1 & -2 \\ 1 & 2a & a \\ -1 & 3 & 2 \end{vmatrix} = a(4a - 3a) + (2 + a) - 2(3 + 2a) = a^2 - 3a - 4$$

We find that $\det = 0$ if $a = 4$ or $a = -1$, in which cases the set of vectors is **dependent**.

(Note: if $a = -1$, then $w_3 = w_1 + w_2$)

Tests for independence/dependence

Let $\mathcal{S} = v_1, v_2, \cdots, v_k$ be a set of vectors in $\mathbb{R}^n$.

- **Case 1:** $k > n$: $\mathcal{S}$ is linearly dependent.
- **Case 2:** $k = n$:
  Solution 1. Solve the system of equations

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}.$$

If a unique solution:

$$c_1 = c_2 = \cdots = c_n = 0,$$

the vectors are **independent**.
If infinitely many solutions, the vectors are **dependent**.

- **Case 2:** $k = n$:
  Solution 2. Form the $n \times n$ matrix A whose rows are $v_1,\ v_2, \cdots, v_n$ and row reduce A:
  if the reduced matrix has n nonzero rows, i.e., if the rank of A is n, then **independent**;
  if the reduced matrix has one or more zero rows, then **dependent**.

  Solution 3. Calculate $\det A$:
  If $\det A \neq 0$, the vectors are **independent**.
  If $\det A = 0$, the vectors are **dependent**.

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  Solution 2. Form the $n \times n$ matrix A whose rows are $v_1,\ v_2, \cdots, v_n$ and row reduce A:
  if the reduced matrix has n nonzero rows, i.e., if the rank of A is n, then **independent**;
  if the reduced matrix has one or more zero rows, then **dependent**.

  Solution 3. Calculate $\det A$:
  If $\det A \neq 0$, the vectors are **independent**.
  If $\det A = 0$, the vectors are **dependent**.

Note: If $v_1, v_2, \cdots, v_n$ is a linearly independent set of vectors in $\mathbb{R}^n$, then each vector in $\mathbb{R}^n$ has a unique representation as a linear combination of $v_1, v_2, \cdots, v_n$.

- **Case 3:** $k < n$:
  (i) Form the $k \times n$ matrix A whose rows are $v_1,\ v_2, \cdots, v_k$
  (ii) Row reduce A:
  if the reduced matrix has k nonzero rows, set is **independent**; if it has one or more zero rows, set is **dependent**.

  Equivalently, solve the system of equations

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \mathbf{0}.$$

  If unique solution: $c_1 = c_2 = \cdots = c_n = 0$, then **independent**; if infinitely many solutions, then **dependent**.

Linear Dependence/Independence

Example.
Consider the set $v_1 = (1, -2, 3), v_2 = (2, -3, 1), v_3 = (, 3, -4, -1)$.
Is it dependent or independent?

Linear Dependence/Independence

Example.

Consider the set $v_1 = (1, -2, 3), v_2 = (2, -3, 1), v_3 = (, 3, -4, -1)$.
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Row reduce:

$$\begin{pmatrix} 1 & -2 & 3 \\ 2 & -3 & 1 \\ 3 & -4 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 2 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

This shows that the set is **dependent**.

It also shows that row 3 is a linear combination of row 1 and and row 2; hence, the vector equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{O}$$

has infinitely many non-zero solutions.

Linear Dependence/Independence

Example. Consider the vectors in $\mathbb{R}^4$
$v_1 = (1, -1, 2, 1), \quad v_2 = (3, 2, 0, -1), \quad v_3 = (-1, -4, 4, 3),$
$v_4 = (2, 3, -2, -2)$

a. Is the set $\{v_1,\ v_2,\ v_3,\ v_4\}$ dependent or independent?

b. If dependent, what is the maximum number of independent vectors?

Linear Dependence/Independence

Example. Consider the vectors in $\mathbb{R}^4$

$v_1 = (1, -1, 2, 1), \quad v_2 = (3, 2, 0, -1), \quad v_3 = (-1, -4, 4, 3),$

$v_4 = (2, 3, -2, -2)$

a. Is the set $\{v_1,\ v_2,\ v_3,\ v_4\}$ dependent or independent?

b. If dependent, what is the maximum number of independent vectors?

Row reduce:

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ -1 & 4 & 4 & 3 \\ 2 & 3 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & -5 & 6 & 4 \\ 0 & 5 & -6 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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Row reduce:

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ -1 & 4 & 4 & 3 \\ 2 & 3 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & -5 & 6 & 4 \\ 0 & 5 & -6 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Since the rank of the matrix is less than 4, then the vectors are **dependent.**

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a. Is the set $\{v_1,\ v_2,\ v_3,\ v_4\}$ dependent or independent?

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Row reduce:

$$\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & -1 \\ -1 & 4 & 4 & 3 \\ 2 & 3 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & -5 & 6 & 4 \\ 0 & 5 & -6 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 5 & -6 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Since the rank of the matrix is less than 4, then the vectors are **dependent.**

(b) Since the rank of the matrix is 2, then there are **2 linearly independent vectors**.

Linear Dependence/Independence

General Result

Remark

Given a set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$. Form the matrix V with $v_1, v_2 \ldots$ as rows and row reduce.

If you get a row of 0's, the vectors are linearly dependent and at least one of the vectors is a linear combination of the other vectors.
If you get no rows of 0's, the vectors are linearly independent.