Math 3321

Eigenvalues and Eigenvectors

University of Houston

Lecture 21

Outline

1. Matrices and Linear Transformations

2. Eigenvalues/Eigenvectors

Linear Transformations

A **linear transformation** (or a **linear map**) is a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

that satisfies the following properties. For any $x, y \in \mathbb{R}^n$ and any $\alpha \in \mathbb{R}$, we have

$$T(x + y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

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$$T(x + y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

Example. The function

$$f(x, y, z) = (x + y, y - z, -2z), \quad x, y, z \in \mathbb{R}$$

is a linear map.

Linear Transformations

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Proof: We clearly have that for any vectors $v, w \in \mathbb{R}^n$

$$A[v + w] = Av + Aw \text{ and } A[\alpha\, v] = \alpha\, Av$$

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Example:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ that maps vectors from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Linear Transformations

Fact

Every linear transformation is associated with a matrix and vice versa.

That is, every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented as a matrix.

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Example. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(x, y, z) = (x + y, y - z, -2z)$$

We can represent it via the 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

Linear Transformations

In fact

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y-z \\ -2z \end{pmatrix}$$

Linear Transformations

A linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

maps a vector $x \in \mathbb{R}^n$ into another vector $T(x) \in \mathbb{R}^n$.

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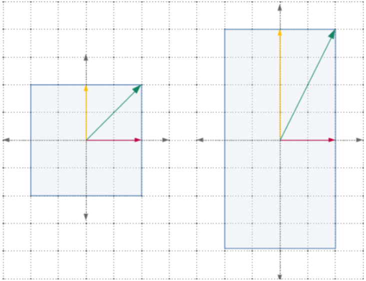
maps a vector $x \in \mathbb{R}^n$ into another vector $T(x) \in \mathbb{R}^n$.

This can be interpreted geometrically as a combination of rotation and stretching.

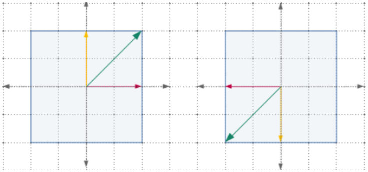
Linear Transformations

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2y \end{pmatrix}$$

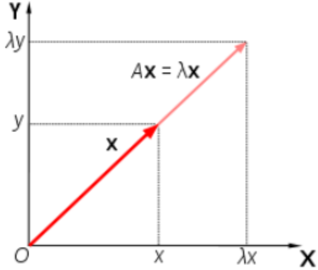


Linear Transformations

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$



For a given matrix, there are special vectors on which the matrix produces only stretching but no rotation . That is, the matrix does not change the direction of the vector.



Linear Transformations

Example: Set

$$A = \begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix}.$$

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$$\begin{pmatrix} 1 & -3 & 1 \\ -1 & 1 & 1 \\ 3 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ (same direction)}$$

Definition

Let A be an $n \times n$ matrix.
A number λ is an **eigenvalue** of A if there is a **non-zero vector** v such that

$$A\,v = \lambda\,v$$

Eigenvalues/Eigenvectors

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Note: Eigenvectors are not unique; an eigenvalue has infinitely many eigenvectors.

Eigenvalues/Eigenvectors

To find the eigenvalues of A, we find the values of λ that satisfy

$$\det(A - \lambda I) = 0.$$

By arguments explained above, finding such λ implies that equation $A v = \lambda v$ has nontrivial solutions.

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Hence we find the **eigenvalues**: $\lambda_1 = 5, \quad \lambda_2 = 1$

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Hence we find the **eigenvalues**: $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$

Terminology:

- $\det(A - \lambda I)$ is a polynomial of degree n, called the **characteristic polynomial** of A.
- The zeros of the characteristic polynomial are the eigenvalues of A
- The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

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Characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} \\ &= (2-\lambda)(-1-\lambda) - 4 \\ &= \lambda^2 - \lambda - 6 \end{aligned}$$

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Eigenvalues: $\lambda_1 = 3, \quad \lambda_2 = -2$

To find eigenvectors, we solve

$$(A - \lambda\, I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

Eigenvalues/Eigenvectors

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For $\lambda_1 = 3$, solve

$(A - 3\,I)x = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Note that the two rows are linearly dependent, hence the system reduces to the equation

$$-x_1 + 2x_2 = 0 \quad \Rightarrow x_1 = 2x_2$$

The eigenvectors corresponding to $\lambda_1 = 3$ are the vectors

$$v_1 = \alpha(2,1), \qquad \alpha \in \mathbb{R}$$

This is a family of vectors with having same direction, that is, they form a linearly dependent set.

Eigenvalues/Eigenvectors

Simlarly, for $\lambda_2 = -2$, we solve
$(A - (-2)\, I)x = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Again the system reduces to a single equation

$$2x_1 + x_2 = 0 \quad \Rightarrow x_2 = -2x_1$$

The eigenvectors corresponding to $\lambda_2 = -2$ are the vectors

$$v_2 = \beta(1, -2), \qquad \beta \in \mathbb{R}$$

This is also a family of vectors with having same direction. Note they are independent from v_1.

Example 2: Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 4 & -1 \\ 4 & 0 \end{pmatrix}$

Eigenvalues/Eigenvectors

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$$= \lambda^2 - 4\lambda + 4$$

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Characteristic equation:

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Eigenvalues: $\lambda_1 = \lambda_2 = 2$

To find the eigenvectors, we solve

$$(A - \lambda\, I)x = \begin{pmatrix} 4-\lambda & -1 \\ 4 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

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Eigenvalues/Eigenvectors

To find the eigenvectors, we solve

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for $\lambda_1 = \lambda_2 = 2$

Hence we solve:

$$(A - 2\, I)x = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This reduces to $2x_1 - x_2 = 0$, that is, $x_2 = 2x_1$.
Hence, we have the family of eigenvectors

$$v = \alpha(1,2), \qquad \alpha \in \mathbb{R}$$

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Characteristic equation:

$$\lambda^2 - 4\lambda + 13 = 0$$

Eigenvalues:

$$\lambda_1 = 2 + 3i, \quad \lambda_2 = 2 - 3i$$

Eigenvalues/Eigenvectors

To find eigenvectors, need to solve

$$(A - \lambda I)x = \begin{pmatrix} 5-\lambda & 3 \\ -6 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

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$(A - \lambda I)x = \begin{pmatrix} 5-\lambda & 3 \\ -6 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$
for $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$.

For $\lambda_1 = 2 + 3i$, we solve

$$(A - (2+3i)\,I)x = \begin{pmatrix} 3-3i & 3 \\ -6 & -3-3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies the equation $(1-i)x_1 + x_2 = 0$, hence $x_2 = (i-1)x_1$

We find the family of eigenvectors $v_1 = \alpha(1, i-1)$, $\alpha \in \mathbb{R}$.

Eigenvalues/Eigenvectors

For $\lambda_2 = 2 - 3i$, we solve

$$(A - (2-3i)\,I)x = \begin{pmatrix} 3+3i & 3 \\ -6 & -3+3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This implies the equation $(1+i)x_1 + x_2 = 0$, hence $x_2 = -(1+i)x_1$

We find the family of eigenvectors $v_2 = \beta(1, -1-i)$, $\beta \in \mathbb{R}$

Eigenvalues/Eigenvectors

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Characteristic equation:

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = (\lambda - 3)(\lambda - 2)(\lambda + 1) = 0.$$

Eigenvalues: $\lambda_1 = 3, \; \lambda_2 = 2, \; \lambda_3 = -1$

Eigenvalues/Eigenvectors

To find eigenvectors, we solve

$$(A - \lambda I)x = \begin{pmatrix} 4-\lambda & -3 & 5 \\ -1 & 2-\lambda & -1 \\ -1 & 3 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

for λ_1, λ_2, λ_3.

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for λ_1, λ_2, λ_3.

For $\lambda_1 = 3$, we have $(A - 3I)x = \begin{pmatrix} 1 & -3 & 5 \\ -1 & -1 & -1 \\ -1 & 3 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

It reduces to $\begin{pmatrix} 1 & -3 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

We find $x_2 = x_3$ and $x_1 = 3x_2 - 5x_3$.

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Hence, we obtain the family of eigenvectors $v_1 = \alpha(-2, 1, 1)$, $\alpha \in \mathbb{R}$.

Eigenvalues/Eigenvectors

For $\lambda_2 = 2$, we have $(A - 2\,I)x = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 0 & -1 \\ -1 & 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

It reduces to $\begin{pmatrix} 1 & -3 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

We find $x_2 = x_3$ and $x_1 = 3x_2 - 4x_3$.

Eigenvalues/Eigenvectors

For $\lambda_2 = 2$, we have $(A - 2\,I)x = \begin{pmatrix} 2 & -3 & 5 \\ -1 & 0 & -1 \\ -1 & 3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

It reduces to $\begin{pmatrix} 1 & -3 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

We find $x_2 = x_3$ and $x_1 = 3x_2 - 4x_3$.

Hence, we obtain the family of eigenvectors $v_2 = \beta(-1, 1, 1,)\ \beta \in \mathbb{R}$.

Eigenvalues/Eigenvectors

For $\lambda_3 = -1$, we have $(A + 1\,I)x = \begin{pmatrix} 5 & -3 & 5 \\ -1 & 1 & -1 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

It reduces to $\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

We find $x_2 = 0$ and $x_1 = -x_3$.

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We find $x_2 = 0$ and $x_1 = -x_3$.

Hence, we obtain the family of eigenvectors $v_3 = \gamma(1, 0, -1)$, $\gamma \in \mathbb{R}$.

Theorem

If $v_1,\ v_2,\ \ldots,\ v_k$ are eigenvectors of a matrix A corresponding to distinct eigenvalues $\lambda_1,\ \lambda_2, \ldots,\ \lambda_k$, then $v_1,\ v_2,\ \ldots,\ v_k$ are linearly independent.

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Theorem

Let A be a (real) $n \times n$ matrix. If the complex number $\lambda = a + bi$ is an eigenvalue of A with corresponding (complex) eigenvector $u + iv$, then $\lambda = a - bi$, the *conjugate* of $a + bi$, is also an eigenvalue of A and $u - iv$ is a corresponding eigenvector.

Theorem

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$\det A = (-1)^n \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \cdots \lambda_n.$$

That is, $\det A$ is the ± 1 times the product of the eigenvalues of A. (The λ's are not necessarily distinct, the multiplicity of an eigenvalue may be greater than 1, and they are not necessarily real.)

Theorem

Let A be an $n \times n$ matrix. The following are equivalent

1. The system $Ax = b$ has a unique solution.
2. The reduced row echelon form of A is I_n.
3. The rank of A is n.
4. A has an inverse.
5. $\det A \neq 0$.
6. The rows of A are linearly independent.
7. 0 is not an eigenvalue of A.

The last theorem can be equivalently re-formulated as follows.

Theorem

Let A be an $n \times n$ matrix. The following are equivalent

1. The system $Ax = b$ does not have a unique solution.
2. The reduced row echelon form of A is not I_n.
3. The rank of A is less than n.
4. A does not have an inverse.
5. $\det A = 0$.
6. The rows of A are linearly dependent.
7. 0 is an eigenvalue of A.