Math 3321

Systems of Linear Differential Equations

University of Houston

Lecture 22

Outline

Systems of Linear Differential Equations

Let $a_{11}(t),\ a_{12}(t),\ \ldots,\ a_{nn}(t)$, and $b_1(t),\ b_2(t),\ \ldots,\ b_n(t)$ be continuous functions on the interval I.

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The system of n first-order linear differential equations

$$\begin{array}{rcl} x_1' & = & a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ x_2' & = & a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots & & \vdots \\ x_n' & = & a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{array}$$

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is called a **first-order linear differential system**.

The system is **homogeneous** if

$$b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0 \quad \text{on} \quad I.$$

It is **nonhomogeneous** if the functions $b_i(t)$ are not all identically zero on I.

Systems of Linear Differential Equations

Set

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

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$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad b(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

The first-order linear differential system can be written in the **vector-matrix form**

$$x' = A(t)\,x + b(t). \tag{S}$$

Systems of Linear Differential Equations

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the matrix $A(t)$ is called the **matrix of coefficients** or the **coefficient matrix**.

The vector $b(t)$ is called the **non-homogeneous term**, or **forcing function**.

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the matrix $A(t)$ is called the **matrix of coefficients** or the **coefficient matrix**.

The vector $b(t)$ is called the **non-homogeneous term**, or **forcing function**.

If $b = 0$, we have a homogeneous first-order linear differential system

$$x' = A(t)\,x$$

A **solution** of the linear differential system

$$x' = A(t)\,x + b(t), \tag{S}$$

is a differentiable vector function

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

that satisfies (S) on the interval I.

Example 1:

$$x_1' = x_1 + 2x_2 - 5e^{2t}$$

$$x_2' = 3x_1 + 2x_2 + 3e^{2t}$$

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We can write the system in matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

or

$$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

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$$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

We will show that $x(t) = \begin{pmatrix} -e^{2t} \\ 2\,e^{2t} \end{pmatrix}$ is a solution.

By taking the derivative of $x(t)$, we obtain

$$x'(t) = \begin{pmatrix} -2e^{2t} \\ 4\,e^{2t} \end{pmatrix}$$

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Next, substituting $x(t)$ into the system we obtain

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This shows that $x(t)$ is a solution.

Example 2:

$$x_1' = 3x_1 - x_2 - x_3$$

$$x_2' = -2x_1 + 3x_2 + 2x_3$$

$$x_3' = 4x_1 - x_2 - 2x_3$$

Example 2:

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$$x_3' = 4x_1 - x_2 - 2x_3$$

We can write in matrix form as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

$$x' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} x$$

Systems of Linear Differential Equations

We will show by substitution into the systeem that

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$$x'(t) = \frac{d}{dt} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \\ 3e^{3t} \end{pmatrix} = 3\,x(t)$$

By substitution of x into the system of equations, we obtain

$$\begin{aligned} x' &= \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} (3+1-1)e^{3t} \\ (-2-3+2)e^{3t} \\ (4+1-2)e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \\ 3e^{3t} \end{pmatrix} = 3\,x(t) \end{aligned}$$

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Hence, x is a solution of the system of linear differential equations.

Systems of Linear Differential Equations

Systems of linear differential equations share most properties of regular linear differential equations.

Theorem

Let $x(t) = \begin{pmatrix} x_1(t) \\ \dots \\ x_n(t) \end{pmatrix}$ and $b(t) = \begin{pmatrix} b_1(t) \\ \dots \\ b_n(t) \end{pmatrix}$ be differentiable vector functions, $A(t)$ be a $n \times n$ matrix of continuous functions and c be a constant n-vector. The initial-value problem

$$x' = A(t)x + b(t), \quad x(t_0) = c$$

has a unique solution $x = x(t)$.

Higher-Order Linear Differential Equations

One major reason for studying systems of linear differential equations is their application to solve higher order linear differential equations.

Fact

An n^{th} order linear equation can be converted into a system of n first order linear equations.

Higher-Order Linear Differential Equations

Consider the homogeneous second order differential equation

$$y'' + p(t)y' + q(t)y = 0$$

To convert to a system of 2 first order linear equations, we proceed as follows

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2. Introduce new dependent variables x_1, x_2, as follows:

$$\begin{aligned} x_1 &= y \\ x_2 &= x_1' \; (= y') \end{aligned}$$

Higher-Order Linear Differential Equations

3. Now we can write the second order differential equation as the system :
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Note: this system is just a special case of the "general" homogeneous system of two, first-order differential equations:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or

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NOTE: We will learn in the next lecture how to find the solution of homogeneous systems first-order differential equation.

Example: $y'' - 5y' + 6y = 0$

Following the approach presented above, we introduce new dependent variables x_1, x_2, as follows:

$$x_1 = y$$
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We claim that

$z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

are solutions of the linear system

This is verified by substitution

$$z' = \frac{d}{dt}\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = 2z$$

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Also

$$\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -6e^{2t} + 10e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} = 2z$$

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This shows that $z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution.

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This shows that $z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a solution.

A very similar calculation shows that w is also a solution.

By computing the Wronskian of z and w, we find

$$W(z, w) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \neq 0,$$

so $z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $w = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$ are linearly independent solutions of the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

It follows that the general solution of the system is

$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$$

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$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$$

Since $y = x_1$, then $y(t) = C_1 e^{2t} + C_2 e^{3t}$ is the fundamental solution of the original second order linear equation.

Higher-Order Linear Differential Equations

Consider the homogeneous third order differential equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0$$

To convert to a system of 3 first order linear equations, we proceed as follows

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1. Solve for y''
$$y''' = -r(t)y - q(t)y' - p(t)y''$$

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1. Solve for y''
$$y''' = -r(t)y - q(t)y' - p(t)y''$$

2. Introduce new dependent variables $x_1,\ x_2,\ x_3,$ as follows:
$$\begin{aligned} x_1 &= y \\ x_2 &= x_1' \ (= y') \\ x_3 &= x_2' \ (= y'') \end{aligned}$$

Higher-Order Linear Differential Equations

3. We write the third order differential equation as the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -r(t)x_1 - q(t)x_2 - p(t)x_3 \end{aligned}$$

That is

$$\begin{aligned} x_1' &= 0x_1 + 1\,x_2 + 0x_3 \\ x_2' &= 0x_1 + 0x_2 + 1\,x_3 \\ x_3' &= -r(t)x_1 - q(t)x_2 - p(t)x_3 \end{aligned}$$

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In vector-matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Higher-Order Linear Differential Equations

As in the case of 2 equations and 2 unknowns, we note that the system in the last slide is a special case of the "general" system of three, first-order differential equations:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or in vector-matrix form:

$$x' = A(t)x$$

Example

$$y''' - 3y'' - 4y' + 12y = 0.$$

After introducing variables $x_1,\ x_2,\ x_3$, in can be written as

$$\begin{aligned} x_1 &= y \\ x_2 &= x_1' \ (= y') \\ x_3 &= x_2' \ (= y'') \end{aligned}$$

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After introducing variables $x_1,\ x_2,\ x_3$, in can be written as

$$\begin{aligned} x_1 &= y \\ x_2 &= x_1' \ (= y') \\ x_3 &= x_2' \ (= y'') \end{aligned}$$

In vector-matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or

$$x' = Ax, \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 4 & 3 \end{pmatrix}$$

Higher-Order Linear Differential Equations

We can show that the following vector functions

$$u = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad v = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$$w = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

are

- solutions of the linear system;
- linearly independent.

Hence

$$x(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

is the general solution of the system.

Higher-Order Linear Differential Equations

The method presented above to convert a homogeneous second and third order linear differential equation into a system of linear differential equations of first order extends to homogeneous linear differential equation of any order.

Higher-Order Linear Differential Equations

The method presented above to convert a homogeneous second and third order linear differential equation into a system of linear differential equations of first order extends to homogeneous linear differential equation of any order.

Using the same idea, *a homogeneous linear differential equation of* n*-order can be transformed into a system of* n *linear differential equations of first order.*