Math 3321
Systems of Linear Differential Equations

University of Houston

Lecture 22
1. Systems of Linear Differential Equations

2. Solution of Higher-Order Linear Differential Equations
Systems of Linear Differential Equations

Let $a_{11}(t)$, $a_{12}(t)$, ..., $a_{nn}(t)$, and $b_1(t)$, $b_2(t)$, ..., $b_n(t)$ be continuous functions on the interval $I$. 
Let \( a_{11}(t), a_{12}(t), \ldots, a_{nn}(t), \) and \( b_1(t), b_2(t), \ldots, b_n(t) \) be continuous functions on the interval \( I \).

The system of \( n \) first-order linear differential equations

\[
\begin{align*}
    x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\
    x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\
    &\vdots \\
    x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t)
\end{align*}
\]

is called a \textbf{first-order linear differential system}. 
Systems of Linear Differential Equations

Let $a_{11}(t), a_{12}(t), \ldots, a_{nn}(t)$, and $b_1(t), b_2(t), \ldots, b_n(t)$ be continuous functions on the interval $I$.

The system of $n$ first-order linear differential equations

$$
x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t)
$$
$$
x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t)
$$
$$
\vdots
$$
$$
x'_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t)
$$

is called a **first-order linear differential system**.

The system is **homogeneous** if

$$
b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0 \quad \text{on} \quad I.
$$

It is **nonhomogeneous** if the functions $b_i(t)$ are not all identically zero on $I$. 

Systems of Linear Differential Equations

Set

\[ A(t) = \begin{pmatrix}
    a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
    a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{pmatrix} \]

and

\[ x = \begin{pmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{pmatrix}, \quad b(t) = \begin{pmatrix}
    b_1(t) \\
    b_2(t) \\
    \vdots \\
    b_n(t)
\end{pmatrix}. \]
Set

\[ A(t) = \begin{pmatrix}
  a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
  a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{pmatrix} \]

and

\[ x = \begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{pmatrix}, \quad b(t) = \begin{pmatrix}
  b_1(t) \\
  b_2(t) \\
  \vdots \\
  b_n(t)
\end{pmatrix}. \]

The first-order linear differential system can be written in the **vector-matrix form**

\[ x' = A(t) x + b(t). \]  \tag{S}
Systems of Linear Differential Equations

In the first-order linear differential system

\[ x' = A(t) \, x + b(t), \quad (S) \]

the matrix \( A(t) \) is called the **matrix of coefficients** or the **coefficient matrix**.

The vector \( b(t) \) is called the **non-homogeneous term**, or **forcing function**.
In the first-order linear differential system

\[ x' = A(t) x + b(t), \]  

(S)

the matrix \( A(t) \) is called the **matrix of coefficients** or the **coefficient matrix**.

The vector \( b(t) \) is called the **non-homogeneous term**, or **forcing function**.

If \( b = 0 \), we have a homogeneous first-order linear differential system

\[ x' = A(t) x \]
A solution of the linear differential system

\[ x' = A(t) x + b(t), \quad \text{(S)} \]

is a differentiable vector function

\[ x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \]

that satisfies (S) on the interval \( I \).
Example 1:

\[ x'_1 = x_1 + 2x_2 - 5e^{2t} \]
\[ x'_2 = 3x_1 + 2x_2 + 3e^{2t} \]
Systems of Linear Differential Equations

**Example 1:**

\[ x_1' = x_1 + 2x_2 - 5e^{2t} \]
\[ x_2' = 3x_1 + 2x_2 + 3e^{2t} \]

We can write the system in matrix form as

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}' = \begin{pmatrix}
  1 & 2 \\
  3 & 2
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} + \begin{pmatrix}
  -5e^{2t} \\
  3e^{2t}
\end{pmatrix}
\]

or

\[
x' = \begin{pmatrix}
  1 & 2 \\
  3 & 2
\end{pmatrix} x + \begin{pmatrix}
  -5e^{2t} \\
  3e^{2t}
\end{pmatrix}
\]
Example 1:

\[ x_1' = x_1 + 2x_2 - 5e^{2t} \]
\[ x_2' = 3x_1 + 2x_2 + 3e^{2t} \]

We can write the system in matrix form as

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\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
+ \begin{pmatrix}
  -5e^{2t} \\
  3e^{2t}
\end{pmatrix}
\]

or

\[
x' = \begin{pmatrix}
  1 & 2 \\
  3 & 2
\end{pmatrix}x + \begin{pmatrix}
  -5e^{2t} \\
  3e^{2t}
\end{pmatrix}
\]

We will show that \( x(t) = \begin{pmatrix}
  -e^{2t} \\
  2e^{2t}
\end{pmatrix} \) is a solution.
By taking the derivative of $x(t)$, we obtain

$$x'(t) = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$
By taking the derivative of $x(t)$, we obtain

$$x'(t) = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

Next, substituting $x(t)$ into the system we obtain

$$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{2t} + 4e^{2t} \\ -3e^{2t} + 2e^{2t} \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

This shows that $x(t)$ is a solution.
By taking the derivative of $x(t)$, we obtain

$$x'(t) = \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

Next, substituting $x(t)$ into the system we obtain

$$x' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{2t} + 4e^{2t} \\ -3e^{2t} + 2e^{2t} \end{pmatrix} + \begin{pmatrix} -5e^{2t} \\ 3e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

This shows that $x(t)$ is a solution.
Example 2:

\[
\begin{align*}
x_1' &= 3x_1 - x_2 - x_3 \\
x_2' &= -2x_1 + 3x_2 + 2x_3 \\
x_3' &= 4x_1 - x_2 - 2x_3
\end{align*}
\]
Systems of Linear Differential Equations

Example 2:

\[ x_1' = 3x_1 - x_2 - x_3 \]
\[ x_2' = -2x_1 + 3x_2 + 2x_3 \]
\[ x_3' = 4x_1 - x_2 - 2x_3 \]

We can write in matrix form as

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
' =
\begin{pmatrix}
  3 & -1 & -1 \\
  -2 & 3 & 2 \\
  4 & -1 & -2
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

or

\[ x' = \begin{pmatrix}
  3 & -1 & -1 \\
  -2 & 3 & 2 \\
  4 & -1 & -2
\end{pmatrix} x \]
Systems of Linear Differential Equations

We will show by substitution into the system that

\[ x(t) = \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} \]

is a solution.
Systems of Linear Differential Equations

We will show by substitution into the system that

\[ x(t) = \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} \]

is a solution.

By taking the derivative of \( x(t) \), we obtain

\[ x'(t) = \frac{d}{dt} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \\ 3e^{3t} \end{pmatrix} = 3x(t) \]
By substitution of $x$ into the system of equations, we obtain

$$x' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} (3 + 1 - 1)e^{3t} \\ (-2 - 3 + 2)e^{3t} \\ (4 + 1 - 2)e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \\ 3e^{3t} \end{pmatrix} = 3x(t)$$

Hence, $x$ is a solution of the system of linear differential equations.
By substitution of $x$ into the system of equations, we obtain

$$x' = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} (3 + 1 - 1)e^{3t} \\ (-2 - 3 + 2)e^{3t} \\ (4 + 1 - 2)e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ -3e^{3t} \\ 3e^{3t} \end{pmatrix} = 3x(t)$$

Hence, $x$ is a solution of the system of linear differential equations.
Systems of Linear Differential Equations

Systems of linear differential equations share most properties of regular linear differential equations.

**Theorem**

Let $x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ and $b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$ be differentiable vector functions, $A(t)$ be a $n \times n$ matrix of continuous functions and $c$ be a constant $n$-vector. The initial-value problem

$$x' = A(t)x + b(t), \quad x(t_0) = c$$

has a unique solution $x = x(t)$. 
One major reason for studying systems of linear differential equations is their application to solve higher order linear differential equations.

**Fact**

An $n^{th}$ order linear equation can be converted into a system of $n$ first order linear equations.
Consider the homogeneous second order differential equation

\[ y'' + p(t)y' + q(t)y = 0 \]

To convert to a system of 2 first order linear equations, we proceed as follows
Higher-Order Linear Differential Equations

Consider the homogeneous second order differential equation

\[ y'' + p(t)y' + q(t)y = 0 \]

To convert to a system of 2 first order linear equations, we proceed as follows

1. Solve for  \( y'' \)

\[ y'' = -q(t)y - p(t)y' \]
Higher-Order Linear Differential Equations

Consider the homogeneous second order differential equation

\[ y'' + p(t)y' + q(t)y = 0 \]

To convert to a system of 2 first order linear equations, we proceed as follows

1. Solve for \( y'' \)
   \[ y'' = -q(t)y - p(t)y' \]

2. Introduce new dependent variables \( x_1, x_2 \), as follows:
   \[
   x_1 = y \\
   x_2 = x_1' (= y')
   \]
3. Now we can write the second order differential equation as the system:

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}' = \begin{pmatrix}
  0 & 1 \\
  -q(t) & -p(t)
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]
3. Now we can write the second order differential equation as the system:

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}' = \begin{pmatrix}
  0 & 1 \\
  -q(t) & -p(t)
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

Note: this system is just a special case of the “general” homogeneous system of two, first-order differential equations:

\[
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}' = \begin{pmatrix}
  a_{11}(t) & a_{12}(t) \\
  a_{21}(t) & a_{22}(t)
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]

or

\[
x' = A(t)x
\]
3. Now we can write the second order differential equation as the system:

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\begin{pmatrix}
    0 & 1 \\
    -q(t) & -p(t)
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

Note: this system is just a special case of the “general” homogeneous system of two, first-order differential equations:

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\begin{pmatrix}
    a_{11}(t) & a_{12}(t) \\
    a_{21}(t) & a_{22}(t)
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

or

\[x' = A(t)x\]

**NOTE:** We will learn in the next lecture how to find the solution of homogeneous systems first-order differential equation.
Higher-Order Linear Differential Equations

**Example:** \( y'' - 5y' + 6y = 0 \)

Following the approach presented above, we introduce new dependent variables \( x_1, x_2 \), as follows:

\[
\begin{align*}
  x_1 &= y \\
  x_2 &= x_1' (= y')
\end{align*}
\]
Higher-Order Linear Differential Equations

**Example:** \( y'' - 5y' + 6y = 0 \)

Following the approach presented above, we introduce new dependent variables \( x_1, x_2 \), as follows:

\[
x_1 = y \\
x_2 = x_1' \quad (= y')
\]

Hence we obtain the system:

\[
\begin{pmatrix}
  x_1 \\
x_2
\end{pmatrix}' = \begin{pmatrix}
  0 & 1 \\
-6 & 5
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
x_2
\end{pmatrix}
\]
Example: \( y'' - 5y' + 6y = 0 \)

Following the approach presented above, we introduce new dependent variables \( x_1, x_2 \), as follows:

\[
x_1 = y \\
x_2 = x_1' (= y')
\]

Hence we obtain the system:

\[
\begin{pmatrix}
x_1 \\ x_2
\end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

We claim that

\[
z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}
\]

are solutions of the linear system.
This is verified by substitution

\[ z' = \frac{d}{dt} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = 2z \]
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\[ z' = \frac{d}{dt} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = 2z \]

Also

\[
\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -6e^{2t} + 10e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} = 2z
\]
Higher-Order Linear Differential Equations

This is verified by substitution

\[ z' = \frac{d}{dt} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = 2z \]

Also

\[
\begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ -6e^{2t} + 10e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} = 2z
\]

This shows that \( z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) is a solution.
Higher-Order Linear Differential Equations

This is verified by substitution

\[ z' = \frac{d}{dt} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = 2z \]

Also

\[
\begin{pmatrix}
0 & 1 \\
-6 & 5
\end{pmatrix}
\begin{pmatrix}
e^{2t} \\
2e^{2t}
\end{pmatrix}
= 
\begin{pmatrix}
2e^{2t} \\
-6e^{2t} + 10e^{2t}
\end{pmatrix}
= 
\begin{pmatrix}
2e^{2t} \\
4e^{2t}
\end{pmatrix}
= 2z
\]

This shows that \( z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) is a solution.

A very similar calculation shows that \( w \) is also a solution.
Higher-Order Linear Differential Equations

By computing the Wronskian of $z$ and $w$, we find

$$W(z, w) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \neq 0,$$

so $z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $w = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$ are linearly independent solutions of the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since $y = x_1$, then $y(t) = C_1 e^{2t} + C_2 e^{3t}$ is the fundamental solution of the original second order linear equation.
By computing the Wronskian of $z$ and $w$, we find

$$W(z, w) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \neq 0,$$

so $z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $w = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$ are linearly independent solutions of the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

It follows that the general solution of the system is

$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$$
By computing the Wronskian of $z$ and $w$, we find

$$W(z, w) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t} \neq 0,$$

so $z = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}$, $w = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$ are linearly independent solutions of the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

It follows that the general solution of the system is

$$x(t) = C_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + C_2 \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix}$$

Since $y = x_1$, then $y(t) = C_1 e^{2t} + C_2 e^{3t}$ is the fundamental solution of the original second order linear equation.
Higher-Order Linear Differential Equations

Consider the homogeneous third order differential equation

\[ y''' + p(t)y'' + q(t)y' + r(t)y = 0 \]

To convert to a system of 3 first order linear equations, we proceed as follows
Consider the homogeneous third order differential equation

\[ y''' + p(t)y'' + q(t)y' + r(t)y = 0 \]

To convert to a system of 3 first order linear equations, we proceed as follows

1. Solve for \( y'' \)

\[ y''' = -r(t)y - q(t)y' - p(t)y'' \]
Higher-Order Linear Differential Equations

Consider the homogeneous third order differential equation

\[ y''' + p(t)y'' + q(t)y' + r(t)y = 0 \]

To convert to a system of 3 first order linear equations, we proceed as follows

1. Solve for \( y'' \)
   \[ y''' = -r(t)y - q(t)y' - p(t)y'' \]

2. Introduce new dependent variables \( x_1, x_2, x_3 \), as follows:
   \[ x_1 = y \]
   \[ x_2 = x_1' \ ( = y') \]
   \[ x_3 = x_2' \ ( = y'') \]
3. We write the third order differential equation as the system

\[ x_1' = x_2 \]
\[ x_2' = x_3 \]
\[ x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3 \]

That is

\[ x_1' = 0x_1 + 1x_2 + 0x_3 \]
\[ x_2' = 0x_1 + 0x_2 + 1x_3 \]
\[ x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3 \]
3. We write the third order differential equation as the system

\[ \begin{align*}
    x_1' &= x_2 \\
    x_2' &= x_3 \\
    x_3' &= -r(t)x_1 - q(t)x_2 - p(t)x_3
\end{align*} \]

That is

\[ \begin{align*}
    x_1' &= 0x_1 + 1x_2 + 0x_3 \\
    x_2' &= 0x_1 + 0x_2 + 1x_3 \\
    x_3' &= -r(t)x_1 - q(t)x_2 - p(t)x_3
\end{align*} \]

In vector-matrix form:

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \]
Higher-Order Linear Differential Equations

As in the case of 2 equations and 2 unknowns, we note that the system in the last slide is a special case of the “general” system of three, first-order differential equations:

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}' =
\begin{pmatrix}
  a_{11}(t) & a_{12}(t) & a_{13}(t) \\
  a_{21}(t) & a_{22}(t) & a_{23}(t) \\
  a_{31}(t) & a_{32}(t) & a_{33}(t)
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

or in vector-matrix form:

\[
x' = A(t)x
\]
Higher-Order Linear Differential Equations

**Example**

\[ y''' - 3y'' - 4y' + 12y = 0. \]

After introducing variables \( x_1, x_2, x_3, \) it can be written as

\[
\begin{align*}
x_1 &= y \\
x_2 &= x_1' \ (= y') \\
x_3 &= x_2' \ (= y'')
\end{align*}
\]
Example

\[ y''' - 3y'' - 4y' + 12y = 0. \]

After introducing variables \( x_1, x_2, x_3, \) in can be written as

\[
\begin{align*}
  x_1 &= y \\
  x_2 &= x'_1 (= y') \\
  x_3 &= x'_2 (= y'')
\end{align*}
\]

In vector-matrix form:

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}' =
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -12 & 4 & 3
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
\]

or

\[ x' = Ax, \quad \text{where} \quad A =
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  -12 & 4 & 3
\end{pmatrix}\]
Higher-Order Linear Differential Equations

We can show that the following vector functions

\[
\begin{align*}
    u &= \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \\
    v &= \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \\
    w &= \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \\ 4e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}
\end{align*}
\]

are

- solutions of the linear system;
- linearly independent.

Hence

\[
x(t) = C_1e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + C_2e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + C_3e^{-2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}
\]

is the general solution of the system.
The method presented above to convert a homogeneous second and third order linear differential equation into a system of linear differential equations of first order extends to homogeneous linear differential equation of any order.
The method presented above to convert a homogeneous second and third order linear differential equation into a system of linear differential equations of first order extends to homogeneous linear differential equation of any order.

Using the same idea, a homogeneous linear differential equation of n-order can be transformed into a system of n linear differential equations of first order.